

# LINEAR TIME INVARIANT MODEL APPROXIMATIONS OF TIME PERIODIC SYSTEMS

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## ABSTRACT

Several methods for analysis of linear time periodic (LTP) systems have successfully been demonstrated using harmonic decompositions. One method recently examined is to create a linear time invariant (LTI) model approximation by expansion of the state and control vectors into harmonic components, and formulating a corresponding system of time invariant equations. This LTI system is often significantly larger than the original LTP system due to a large number of harmonic components included for sufficient accuracy. This paper develops a method for evaluating the fidelity of an LTI approximation to the original LTP system, and follows with a method for reducing the size of the LTI system by selecting only the relevant harmonic components which should be retained in the analysis in order to maintain a high fidelity model. This reduced LTI approximation can then be used in control design and analysis with existing LTI methodology. The developed methods are applied to example LTP models.

## 1. NOTATION

$[A(t)]$	Periodic Eigenvectors
$B$	Blade Tip Loss Factor
$c_n$	Complex Fourier Coefficients
$N_B$	Number of Rotor Blades
$N_H$	Maximum Number of Harmonic Terms
$N_M$	Number of LTP Modes
$N_S$	Number of LTP States
$x$	State
$u$	Input
$y$	Output
$T$	Period
$\beta$	Flap Angle
$\gamma$	Lock Number
$\eta_j$	System Eigenvalues
$\Lambda_j$	Floquet Eigenvalues
$\mu$	Advance Ratio
$\nu$	Non-dimensional Flapping Frequency
$[\Phi(t)]$	Transition Matrix
$\phi_n$	Modal Participation
$\psi$	Azimuth Angle or Non-dimensional Time
$\Omega$	Non-dimensional Rotor Speed
$\{ \}$	Vector
$[ \ ]$	Matrix
$[-m_j-]$	Diagonal Matrix with Elements $m_j$
$( )_0$	Average or 0 <sup>th</sup> Harmonic Term
$( )_n$	n <sup>th</sup> Complex Harmonic Term
$( )_{nc}$	n <sup>th</sup> Cosine Harmonic Component
$( )_{ns}$	n <sup>th</sup> Sine Harmonic Component

## 2. INTRODUCTION

The analysis of linear time periodic (LTP) systems is well understood using several methods. One such method is Floquet Theory, developed by Gaston Floquet [1]. This theory has been shown to provide a thorough analysis of LTP systems through the use of modal participation factors [2]. These modal participation factors describe the relative magnitude of each harmonic component for each state.

Other methods involve using a harmonic decomposition of the LTP system. One method recently examined is to create a linear time invariant model approximation by expansion of the LTP system states into various harmonic state representations and formulating corresponding linear time invariant models. Crimi and Piarulli explore the LTP system by harmonic decomposition of periodic states [3, 4]. Prasad and Olcer use the harmonic decomposition to formulate a corresponding linear time invariant (LTI) system [5, 6, 7, 8]. This methodology provides a convenient framework, as methods for LTI systems are extremely well developed and understood. However, previous work has indicated that up to  $2N_B$  harmonic terms are necessary for high fidelity. That is to say the average  $x_0$  term up to and including  $2N_B$  harmonic terms, where “ $2N_B$  harmonic terms” refers to the pair of harmonic components  $x_{2N_B C} \cos(2N_B t)$  and  $x_{2N_B S} \sin(2N_B t)$  from the harmonic decomposition of the periodic states, are necessary for high fidelity. Thus the inclusion of  $2N_B$

harmonic terms makes the system size larger by  $4N_B$  (2 states  $x_{ic}$  and  $x_{is}$  for every  $i$ th harmonic term, and  $2N_B$  harmonic terms). For high fidelity models, this can easily increase the number of states to be in the hundreds or thousands, drastically increasing the computational cost. However, many of those states will have very small influence and excitation, and can be excluded to reduce the system size while still maintaining high fidelity.

There are several methods previously examined for studying the fidelity of systems. One such method is comparison of eigenvalues. Although matching eigenvalues are necessary for high fidelity, they are not sufficient for closely matching responses [7, 8, 9]. One method to address this is comparison of response to specific inputs such as steps or sine sweeps. One problem associated with this method is that the response may not fully represent the richness in dynamics as inputs may not excite specific dynamics. Another method which avoids this problem of looking at specific inputs is instead using the frequency response. The problem with this method however is that frequency response is difficult to do for time varying systems, and therefore difficult to apply to LTP systems.

This paper will illustrate a method for: 1) determining the modal participation of an LTI system, 2) evaluating the fidelity of an LTI system to the LTP system, 3) and selecting specific harmonic states to be retained for a selected amount of fidelity.

### 3. METHODOLOGY

Prior to analysis of the LTI system, several preparation steps must be taken. First, the LTP model must be obtained. From a nonlinear model, the LTP can be obtained using a perturbation scheme [7] for numerical models, or using another linearization technique for analytical models. For examples considered in this study, analytical models are presented in LTP form, and numerical LTP models are obtained by the numerical perturbation scheme. Second, the modal participation from the LTP system is obtained using Floquet analysis [2]. Third, the LTI model is approximated from the LTP model [7].

#### 3.1. Modal Participation

Once the LTP system has been obtained, the modal participation can be evaluated by evaluation of the transition matrix  $[\Phi(t)]$  and the periodic eigenvectors  $[A(t)]$  as shown in [2] by evaluating the following at  $t = T$ , and noting  $[A(T)] = [A(0)]$

$$(1) \quad [-\exp(\eta_j t) -] = [A(t)]^{-1}[\Phi(t)][A(0)]$$

$$(2) \quad [-\Lambda_j -] = [-\exp(\eta_j T) -]$$

$$(3) \quad \eta_j = \frac{1}{T} \text{Log}(\Lambda_j)$$

The periodic eigenvectors can then be evaluated as follows

$$(4) \quad [A(t)] = [\Phi(t)][A(0)][-\exp(-\eta_j t) -]$$

The periodic eigenvectors are then expanded using a complex Fourier series

$$(5) \quad A_{jk}(t) = \sum_{n=-N_H}^{+N_H} c_n \exp(in\Omega t)$$

The modal participation  $\phi_n$  for each harmonic component in each element is then calculated as

$$(6) \quad \phi_n = |c_n| \left( \sum_{n=-N_H}^{+N_H} |c_n| \right)^{-1}$$

The modal participation for each element in the periodic eigenvectors is unique, and will be used as the parameter for evaluating the fidelity of the LTI approximations to the LTP system. There are bookkeeping issues with the calculation of  $\eta_j = \frac{1}{T} \text{Log}(\Lambda_j)$ , since the logarithm is a complex logarithm, with a multivalued arctangent component. The integer added is chosen based on the application, and these issues are discussed in depth later on in the LTI Fidelity section.

#### 3.2. LTI Approximation

The LTI approximation method utilized was developed in [5]. The method is based on a harmonic decomposition of the states where  $x(t)$  are the LTP states and  $x_0, \dots, x_{nc}$ , and  $x_{ns}$  are the LTI states.

$$(7) \quad x(t) = x_0 + \sum_{n=1}^{N_H} x_{nc} \cos(nt) + x_{ns} \sin(nt)$$

$$(8) \quad \{x_{LTI}\} = \{x_0^T \dots x_{ic}^T \ x_{is}^T \dots x_{jc}^T \dots x_0^T \dots \dot{x}_{ic}^T \ \dot{x}_{is}^T \dots \dot{x}_{jc}^T \ \dot{x}_{js}^T\}^T$$

A similar expansion is done for the inputs  $u(t)$  and the outputs  $y(t)$ . It should be noted that while the modal participation calculation is done using complex harmonic coefficients, the LTI states use the trigonometric harmonic coefficients. Also, for a true representation of the LTP system, the LTI approximation would require an infinite number of harmonic terms and therefore an infinite dimensional system. This infinite dimensional system however, can be approximated by a finite dimensional one, since most of the terms will be near zero. It is suggested that for sufficiently high accuracy for

vibration analysis, at least  $2N_B$  harmonic terms are necessary (i.e. for a 4 bladed rotor  $N_H = 2N_B = 8$ ) [7]. The LTI system can then be formed as follows

$$(9) \quad \{\dot{x}_{LTI}\} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \{x_{LTI}\} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \{u_{LTI}\}$$

$$(10) \quad \{y_{LTI}\} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \{x_{LTI}\} + [E] \{u_{LTI}\}$$

The formulations for each element for each matrix are presented in closed form in [7].

### 3.3. LTI Analysis

Once the previous steps have been completed, the LTI analysis can begin. First, various LTI reductions are formed based on the LTP modal participation. Reductions are formed by retaining LTI states corresponding to LTP harmonic components with significant modal participation. Next, the modal participations of the LTI systems are determined by calculating the eigenvectors which correspond to the trigonometric Fourier coefficients of the periodic eigenvectors. Then converting the eigenvectors from trigonometric Fourier coefficients to complex Fourier coefficients, the modal participation can be calculated in the same manner as the periodic eigenvectors using the same formulation, and taking any harmonic coefficients that were not included to be zero.

### 3.4. LTI Reduction

The LTI formed by the previous LTI approximation method utilizes the harmonic decomposition of the each LTP state, including each harmonic term from 0 to  $2N_B$ . For many cases however, the harmonic states with little excitation can be approximated as 0 and left out entirely from the LTI while still maintaining high fidelity. The choice of which harmonic states to retain is given by the modal participation of the LTP system. LTI states with the highest corresponding LTP modal participation are retained, and LTI reductions are formed by excluding LTI states with corresponding small LTP modal participation. In general, reductions are formed by either of two ways: by starting with the fully expanded LTI system and then excluding the corresponding lowest modal participation harmonic states to reduce the size of the model; or starting with the highest modal participation harmonic states and including the next higher modal participation harmonic states, thus progressively increasing the size of the model.

Frequently the harmonic decomposition is only necessary for the periodic states and the non-periodic states will only require the average or 0<sup>th</sup> harmonic term. For a helicopter this means that the

rotor states are fully decomposed while the body and inflow states require only 0<sup>th</sup> harmonic terms. Further reductions to the LTI are formed excluding specific rotor harmonic states beginning with those corresponding to the lowest LTP modal participation.

### 3.5. LTI Modal Participation

The modal participation for each state is unique to each system. Thus, if the modal participation for an LTI system can be calculated and shown to be the same as the modal participation for the original LTP system the LTI system can be considered a good approximation for the LTP system. The steps to calculate LTI modal participation follow similarly to the LTP modal participation, with a few bookkeeping changes to keep in mind.

The modal participation for each harmonic component for a given LTP state is given by the magnitude of the harmonic (Fourier) coefficient divided by the sum of the magnitudes for each harmonic coefficient for the given state. The Fourier coefficients for the LTI system are found by solving for the eigenvalues and eigenvectors of the LTI system. The Fourier coefficients for each state are then directly given by the corresponding harmonic states in the LTI eigenvectors. It is important to note however, that the LTP modal participation relies on a complex Fourier series expansion, thus the LTP harmonic terms and harmonic coefficients are in complex form. The LTI expansion uses trigonometric harmonic terms and harmonic coefficients, and will first need to be transformed into complex form. The transformation between complex and trigonometric harmonic coefficients using the previous decomposition nomenclature is as follows

From complex to trigonometric

$$(11) \quad x_0 = c_{+0}$$

$$(12) \quad x_{nc} = c_{+n} + c_{-n}$$

$$(13) \quad x_{ns} = i(c_{+n} - c_{-n}) = \frac{c_{-n} - c_{+n}}{i}$$

From trigonometric to complex

$$(14) \quad c_{+0} = x_0$$

$$(15) \quad c_{+n} = \frac{x_{nc} - ix_{ns}}{2}$$

$$(16) \quad c_{-n} = \frac{x_{nc} + ix_{ns}}{2}$$

Once the LTI eigenvalue problem has been solved and the eigenvectors have been transformed from

trigonometric to complex, the modal participation for the LTI system can be computed in exactly the same manner as the LTP system. It is worth noting that for any harmonic states that have been reduced, their harmonic coefficients are considered to be zero and thus have a corresponding zero modal participation. It is also worth noting that similar to the LTP modal participation each LTI harmonic coefficient magnitude is divided by the sum of all magnitudes for each LTI harmonic state. Thus the LTI harmonic coefficients are normalized by the sum of LTI harmonic states, not the sum of LTP harmonic coefficients. Also, similar to the LTP modal participation, the sum of all harmonic LTI modal participations is 1.0 for each original (LTP) state.

### 3.6. LTI Fidelity

The fidelity of each LTI reduction is assessed by comparing the LTI modal participation with the LTP modal participation. However, there can be confusion as to which LTI mode should be compared to which LTP mode. As the LTP has  $N_S$  states, it also has  $N_S$  corresponding modes, however the eigenvalues associated with these modes are not unique. Although Peters addresses this issue by showing that the choice of integer added to the Floquet exponent is arbitrary [2], the integers chosen for this analysis and the examples presented later on are first constrained by the LTI system, as the LTI system will only have a set number of corresponding eigenvalues, i.e., the LTI system does not include the infinitely many eigenvalues given in the LTP analysis, since there is no bookkeeping integer choice in the LTI eigen-analysis. The choice of integers is made to follow conventional rotorcraft analysis, where body and inflow modes are kept at 0/rev (the exponent added is 0), while rotor modes have eigenvalues at various frequencies. Specifically eigenvalue integers are chosen to correspond to the conventional coning, nutation, precession, and differential modes for a 4 bladed rotor in the examples given. Once the LTP eigenvalues and modes have been decided, the only LTI modes to be compared are the ones with eigenvalues matching the LTP eigenvalues.

The error for each LTI system is calculated by taking the weighted sum of the absolute value of the difference of modal participations for each harmonic for a given state, and taking the average of that value for each rotor displacement state and each mode. The fidelity for each LTI system is then calculated as 1 minus the Error:

$$Error = \sum_{j=1}^{N_S} \frac{1}{N_S} \sum_{i=1}^{N_M} \frac{1}{N_M} \sum_{n=1}^{N_H} |\phi_{LTP,n} - \phi_{LTI,n}| \phi_{LTP,n} \quad (17)$$

$$(18) \quad Fidelity = 1 - Error$$

There are several things worth noting at this point about the calculation. First, the error only includes rotor displacement states. Hence it does not include body, inflow, and rotor velocity states. This is done because as noted previously, in many cases the body and inflow states are not decomposed under the assumption that they exhibit very negligible periodicity and thus would have negligible effects on the error. Furthermore, the motivation for study of LTP and LTI approximations is often for vibration analysis and reduction of the rotor displacement states, thus the body, inflow, and rotor velocity states are excluded.

The error calculated has already been normalized since the modal participation is a normalized parameter, i.e. error of 0 corresponds to an exact match and 1 corresponds to 100% error. The LTP modal participation factor outside of the absolute value factor is a weighting term which puts the heaviest weight on highest participating harmonic components.

It is also worth noting that the comparison between the two can be done since both have been calculated using harmonic coefficients in complex form. The comparison can also be done in trigonometric form, however complex form was chosen because complex form is used in previously explored modal participation studies and is often the preferred form in Fourier analysis.

### 3.7. LTI Reduction Selection

At this point the analysis of various LTI reductions has been completed, and the only remaining step is selection of a specific LTI reduction. This choice is done by selecting the lowest fidelity LTI reduction which is higher than the chosen desired fidelity over the range of advance ratios considered. If the LTI reductions were created properly, the lowest LTI meeting the desired fidelity criteria will be the smallest sized system with the least number of LTI states.

Graphically, the process is to create a plot of each LTI reduction fidelity for the range of advance ratios considered. The LTI reduction selected is then chosen as the LTI reduction which lies above and closest to the desired fidelity. Examples are given in the next section.

## 4. ANALYTICAL MODELS

### 4.1. Flapping Equation

The first model examined is the flapping equation for a single rotor blade in rotating coordinates [10].

$$(19) \quad \beta(t) = \beta_0 + \sum_{i=1}^{N_H} \beta_{ic} \cos(i\Omega t) + \beta_{is} \sin(i\Omega t)$$

$$(20) \quad \ddot{\beta}(t) + C(t)\dot{\beta}(t) + K(t)\beta(t) = 0$$

$$(21) \quad C(t) = \frac{\gamma}{8} \left[ 1 + \frac{4\mu}{3} \sin t \right]$$

$$(22) \quad K(t) = v^3 + \frac{\gamma}{8} \left[ \frac{4\mu}{3} \cos t + \mu^3 \sin(2t) \right]$$

Where  $\beta(t)$  is the flapping angle, with periodic coefficients  $C(t)$  and  $K(t)$  and parameters Lock number, advance ratio, and non-dimensional flapping frequency as  $\gamma, \mu$ , and  $v$  respectively.

The parameters are chosen with Lock number of 12, blade flap frequency of 1.0, and advance ratio varying from 0.2 to 2.0. The LTP modal participation and full LTI approximation includes up to the 8th harmonic term, corresponding to a total of 34 LTI states, i.e.  $(2N_H + 1) * N_S$  states. The modal participation for each system is calculated and compared and the LTI model is determined to have accuracy above 99.5% for the entire range of advance ratios considered. Reduced

$$(23) \quad \{\ddot{\beta}(\psi)\} + [C(\psi)]\{\dot{\beta}(\psi)\} + [K(\psi)]\{\beta(\psi)\} = \{0\}$$

$$(23) \quad [C(\psi)] = \begin{bmatrix} \frac{\gamma}{8} B^4 & 0 & \frac{\mu\gamma}{12} B^3 & 0 \\ 0 & \frac{\gamma}{8} B^4 & 2 & \frac{\mu\gamma}{6} B^3 \sin(2\psi) \\ \frac{\mu\gamma}{6} B^3 & -2 & \frac{\gamma}{8} B^4 & -\frac{\mu\gamma}{6} B^3 \cos(2\psi) \\ 0 & \frac{\mu\gamma}{12} B^3 \sin(2\psi) & -\frac{\mu\gamma}{12} B^2 \cos(2\psi) & \frac{\gamma}{8} B^4 \end{bmatrix}$$

$$(25) \quad [K(\psi)] = \begin{bmatrix} v^2 & 0 & 0 & \frac{\mu^2\gamma}{8} B^2 \sin(2\psi) \\ \frac{\mu\gamma}{6} B^3 & v^2 - 1 + \frac{\mu^2\gamma}{16} B^2 \sin(4\psi) & \frac{\gamma}{8} (B^4 + \frac{1}{2}\mu^2 B^2 - \frac{1}{2}B^2\mu^2 \cos(4\psi)) & \frac{\mu\gamma}{6} B^3 \cos(2\psi) \\ 0 & -\frac{\gamma}{8} (B^4 - \frac{1}{2}\mu^2 B^2 + \frac{1}{2}v^2 B^2 \cos(4\psi)) & v^2 - 1 - \frac{\mu^2\gamma}{16} B^2 \sin(4\psi) & \frac{\mu\gamma}{6} B^3 \sin(2\psi) \\ \frac{\mu^2\gamma}{8} B^2 \sin(2\psi) & \frac{\mu\gamma}{6} B^3 \cos(2\psi) & \frac{\mu\gamma}{6} B^3 \sin(2\psi) & v^2 \end{bmatrix}$$

Where  $\{\beta(t)\}$  is the flapping angle states in multi-blade coordinates, with periodic matrix coefficients  $[C(t)]$  and  $[K(t)]$  and parameters Lock number, advance ratio, non-dimensional flapping frequency, and blade tip loss factor as  $\gamma, \mu, p$ , and  $B$  respectively. The parameters chosen are blade flap frequency of 1.0, blade tip loss factor of 1.0, and advance ratio varying from 0.3 to 2.0. Two different Lock number cases are considered.

The first case is for a Lock number of 12. The LTP modal participation and full LTI approximation includes up to the 8<sup>th</sup> harmonic term, corresponding to a total of 136 LTI states. After calculating modal participation, the fidelity of the LTI approximation is found to be above 96.5% for the range of advance ratios considered. Reduced LTI

LTI approximations are then formed using up to 0 to 8 harmonic terms: 0; 0 and 1; 0,1 and 2, et cetera. The fidelity of each LTI reduction is then calculated and is shown in Figure 1 which provides a clear method for selecting an LTI reduction for a given level of accuracy. For a minimum desired accuracy of 95%, the necessary harmonic terms to be included are 0, 1 and 2. The result is 10 LTI states versus the original 34 state full LTI approximation. For a minimum desired accuracy of 97%, the necessary harmonic terms are 0, 1, 2, and 3, resulting in 14 LTI states compared to the original 34.

## 4.2. Isolated Rotor Flapping

The next model examined is for an isolated rotor considering the flapping motion only. The model considers a four bladed rotor with identical blades in multi-blade coordinates [11].

approximations are then formed using up to 0 to 8 harmonic terms for each degree of freedom. The fidelity for each LTI reduction is then calculated and is shown in Figure 2, again providing a clear method for selecting an LTI model reduction for a given level of accuracy. For a minimum desired accuracy of 93%, necessary harmonic terms to include are 0, 1, 2, 3, and 4 for each degree of freedom. The result is 72 LTI states versus the original 136 in the full approximation. For a minimum desired accuracy of 96%, necessary harmonic terms to include are 0, 1, 2, 3, 4, and 5 for each degree of freedom, resulting in 88 LTI states compared to the original 136.

The second case is for a Lock number of 9.6. As the first and second cases are identical except for Lock number, the following analysis and

results are similar. As seen in Figure 3, for a minimum accuracy of 94.5%, necessary harmonic terms to include are 0, 1, 2, 3, and 4, again resulting in 72 LTI states. For a minimum accuracy of 96%, necessary harmonic terms to include are 0, 1, 2, 3, 4, and 5, again totaling 88 LTI states compared to 136.

Although the general method of analysis does not change from the single rotor blade flapping equation to a four bladed isolated rotor flapping, there are a few differences in the details of the calculations. First is the effect on the LTI eigenvalues and vectors. For the four bladed rotor considered, each blade is assumed to be identical. The result is that at low speeds (advance ratios less than 0.3) eigenvalues are repeated. For the LTP system this is not a problem since the integer added to the Floquet exponent can be chosen to create distinct eigenvalues where the real parts are the same, but the imaginary parts differ by  $k\Omega$ . For the LTI system, although the eigenvalues utilized are chosen to match the LTP eigenvalues, the repeated LTI eigenvalues are still present. Although the corresponding eigenvectors are distinct, certain elements can be changed as long as the eigenvectors are linearly independent. The result is that the LTI eigenvector elements corresponding to the LTP Fourier coefficients can be scrambled across multiple eigenvectors, with no way to recover the eigenvectors that match the LTP Fourier coefficients. Thus, to avoid this issue of repeated eigenvalues, the advance ratio range is chosen such that there are no repeated eigenvalues. It should be noted however, that the advance ratio range which gives distinct eigenvalues is outside of the range for which the model remains a valid approximation for an isolated rotor. Thus, although the model comes from an isolated rotor, it should be considered as mathematical only and its analysis used for understanding other models.

Next and perhaps most obvious is how multiple degrees of freedom affect the harmonics chosen for reductions. The harmonic terms for each state are chosen in order to maintain similar fidelity compared to all other states for a given overall system fidelity. In other words, each state has a similar contribution to overall system fidelity. For the analytical models, it is straightforward in that the lowest harmonic terms have the highest modal participation and should be included first followed by higher harmonic terms, i.e., for the first reduction all degrees of freedom start with the 0th harmonic term, for the next all degrees of freedom have 0 and 1 harmonic terms, for the next reduction all degrees of freedom have 0, 1, and 2 harmonic terms, et cetera. For the following numerical models, it will be shown that for a four bladed rotor only a few specific harmonic terms are really necessary providing most

of the fidelity, with all other harmonic terms having very small contributions to fidelity.

## 5. NUMERICAL MODELS

### 5.1. Isolated Rotor Flapping

The first numerical model considered is an isolated rotor in flapping. It is important to note that although this model is similar to the analytical isolated rotor flapping, the resulting LTI reductions are different due to behavior not captured by the analytical model. The specific model used is a generic four bladed isolated rotor model in FLIGHTLAB [12], with wind speed 64.7 knots, rotor speed of 27 radians per second, and rotor radius of 26.83 feet. This corresponds to a single advance ratio of 0.15 which represents a normal operating speed for a generic model. The LTP modal participation and full LTI approximation includes up to the 8<sup>th</sup> harmonic term, corresponding to 136 LTI states.

The analysis proceeds similarly to previous examples. After calculating modal participation for the LTP system which can be seen in Figure 4, it is clear that for the coning angle, only harmonic terms 0 and 4 have significant modal participation. Similarly, for lateral and longitudinal tip path plane angles, only harmonic terms 0 and 4 have significant modal participation. Also, only the 2<sup>nd</sup> harmonic term has significant modal participation for the differential flapping. Thus, it is clear that harmonic terms 0 and 4 are necessary for coning and tip path plane tilt angles, and only harmonic term 2 is necessary for the differential flapping angle. Including only these harmonic terms produces fidelity of 95.31% compared to 95.51% for the full LTI fidelity. The resulting LTI size is 22 states for the reduced LTI, which is significantly smaller than the 136 states for the full LTI approximation. It is clear that including other harmonic terms will increase the fidelity only by 0.20%, however the tradeoff is a small increase in fidelity for a much larger number of LTI states, which may be necessary for some applications, however the 95.31% will usually be sufficient. Thus, in most cases it is recommended to use only the aforementioned 0 and 4 harmonic terms for coning and tip path plane tilt angles, and 2<sup>nd</sup> harmonic term for differential flapping. This behavior of a few isolated harmonic terms (usually 1 or 2) for each degree of freedom containing nearly all of the fidelity will occur even in more inclusive models, as shown in the following examples.

### 5.2 Generic Helicopter Model

This example utilizes the FLIGHTLAB generic helicopter model. This model includes 27

LTP states comprised of: 8 body states; Peters-He 3 state inflow with 1 harmonic and maximum radial variation power of 1; and 16 rotor states including 8 flapping, and 8 lead-lag. The rotor radius is 26.833 feet, the rotor speed is 27 radians per second, and the model is linearized about a wind speed of 64.7 knots corresponding to an advance ratio of 0.15.

The addition of states including both rotor and non-rotor states (body and inflow) warrants consideration of how these states are handled. The lead-lag states are treated exactly as the flapping states and are expanded and included in the LTI states. The lead lag modes and states are included in the eigen and Floquet analysis for LTI and LTP systems respectively, and the lead lag displacement states are included in the modal participation, error, and fidelity. The body and inflow states are treated differently. As described in the LTI Fidelity section, body and inflow states are included in all of the analysis except for the model error and fidelity calculations. This means that the body and inflow modes are included in the eigen analysis and subsequent modal participation, error, and fidelity calculations; However, the body and inflow modal participations themselves are not included in error and fidelity calculations. The body and inflow modes are retained to keep the richness in rotor dynamics associated with those modes, but the body and inflow states themselves are not expanded (only the averages are considered) in the LTI and are subsequently not included in the error and fidelity. The states that are decomposed include up to 8 harmonic terms for the full LTI approximation, resulting in a size of 283 LTI states (272 rotor, 8 body, and 3 inflow states).

The Floquet analysis (including up to the 8th harmonic term for modal participation) of the LTP system clearly indicates that isolated harmonic terms for each degree of freedom contain nearly all of the dynamics for the system. Similar to the isolated rotor, the flapping differential states and lead lag differential states have modal participations which primarily include only the 2<sup>nd</sup> harmonic term. The remaining rotor states include primarily only the 0<sup>th</sup> and 4<sup>th</sup> harmonic terms. Thus, the primary LTI reduction considered has only the 2<sup>nd</sup> harmonic term for differential rotor states, and 0<sup>th</sup> and 4<sup>th</sup> harmonic terms for each of the other rotor states for a total of 55 reduced LTI states. The resulting fidelity of this reduced LTI model is 98.39%. This compares to the full LTI model fidelity of 98.95%, with the difference of the two models being only 0.56%. The reduced LTI size is also significantly smaller at 55 states compared to the 283 full LTI approximation states. From this reduction, further increases in fidelity can be achieved by including the harmonic terms with the highest subsequent modal participations; however the 98.39% fidelity should be sufficient for most applications. Thus, the reduction of including

only harmonic term 2 for differential states, and only 0 and 4 for the remaining rotor states is recommended for most applications.

The numerical examples presented so far have only required 1 or 2 harmonic terms for each rotor state and only the average for body and inflow states for retaining high fidelity in the LTI reductions. The following example however will illustrate the need for significant changes and considerations for models which are more inclusive and contain richer dynamics.

### 5.3. Generic Helicopter with Finite State Inflow

The model used in this example is the FLIGHTLAB generic helicopter model, except with a 6 state inflow model for a total of 30 LTP states (16 rotor, 8 body, 6 inflow). This 6 state inflow model is a Peters-He Finite State inflow model with 2 inflow harmonics and a maximum radial variation power of 2. The use of a higher fidelity inflow model follows from the well-known excitation of high frequency responses from more inclusive inflow models.

Previous examples assumed that body and inflow harmonic dynamics were minimal, and thus only utilized the average of these states. Proceeding with analysis identical to the last example with no added considerations, the maximum fidelity for the full LTI model is 92.92%, with 286 LTI states. Following the previous 2 harmonic term for differential rotor states 0 and 4 harmonic terms for all others for the LTI reduction, the resulting model fidelity is a much lower 81.46% with 50 reduced LTI states. Although these fidelities may be sufficient, they are far from the 98% range of previous approximations. The reason for this distinct change is clearly the change in inflow models. The inflow model has a strong influence on the resulting LTI eigenvalues, and the result is a significant shift away from the LTP eigenvalues as seen in figure 5 for the indicated eigenvalues. This result is explored by Olcer [7], and is the result of  $N_B/2$  vibrations for even bladed rotors. The solution proposed by Olcer is the use of an extended LTI (eLTI), which includes the harmonic decomposition for body and inflow states in addition to the rotor states.

An extended LTI is formed by expansion of the body and inflow states in addition to the rotor states. The analysis for the extended LTI\* has no additional considerations, thus the error and fidelity still only includes the rotor displacements. The resulting fidelity for a full extended LTI\* (each LTP state is expanded up to the 8<sup>th</sup> harmonic term) is 98.98%. The resulting system however, is significantly larger at 510 LTI states than the unextended full LTI at 286 states. Thus, it is desirable to reduce not only the eLTI rotor harmonic states, but also the body and inflow as well. After

calculation of the LTP modal participation, it is clear that 0, 2, and 4 harmonics have the highest influence for each state, including body and inflow states. Thus eLTI reduction should contain harmonic terms 0, 2, and 4 for each state totaling 150 LTI states. This eLTI reduction has a resulting fidelity of 98.47%, only 0.51% from the maximum fidelity while being significantly smaller at 150 states compared to 510 states. Thus, for most cases, this eLTI reduction should have adequate fidelity, and should be used.

## 6. SUMMARY AND CONCLUSIONS

It has been shown how one can compute the fidelity of an LTI approximation for an LTP system. Using this fidelity, one can then select a reduced LTI system which will contain far fewer LTI states than the full LTI expansion while still maintaining a similar level of fidelity. For analytical 4 bladed rotor flapping models, the LTI states retained should begin with the 0<sup>th</sup> harmonic term and add subsequently higher harmonic terms until the desired level of fidelity has been reached. For numerical four bladed generic helicopter models, the LTI states to be retained are isolated harmonic terms which are indicated by the modal participations. Although only four bladed generic helicopter numerical models are presented as examples, three bladed numerical isolated rotor models were also studied, and similar results were found. For systems with less complex inflow models these isolated harmonic terms are generally the 0<sup>th</sup> and  $N_B$  terms (with  $N_B/2$  for differential modes). For systems with more complex inflow models, an eLTI is necessary which includes more harmonics for both rotor and non-rotor states.

The methodology discussed presents methods for reduction of the system matrices for an LTI model and is recommended for applications where the system matrices are of particular interest. However, for applications in which input output behavior is of interest, it is recommended that a new reduction method be developed by using comparisons of time and frequency responses between LTI and LTP systems.

## 7. ACKNOWLEDGEMENTS

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## 9. FIGURES

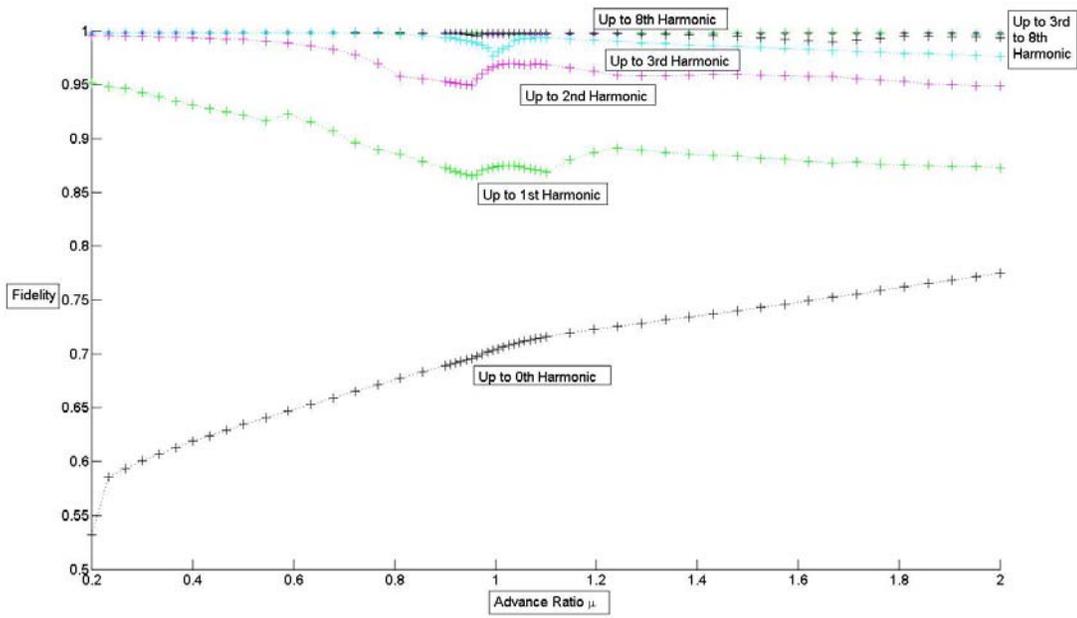


Figure 1. Fidelity, Single Rotor Blade Flapping, Lock Number = 12.0, Blade Frequency = 1.0

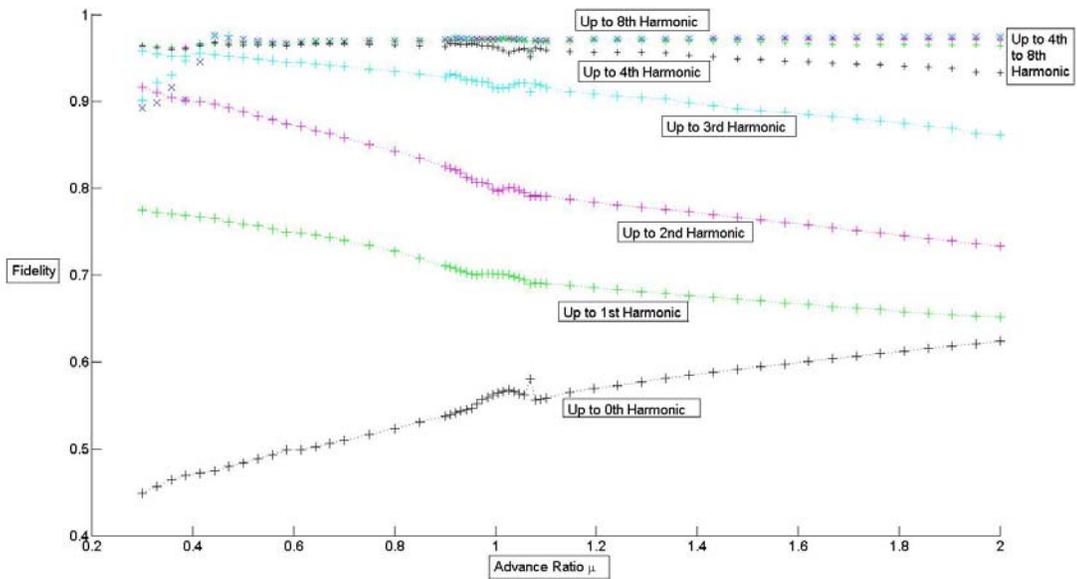


Figure 2. Fidelity, Isolated Four-Bladed-Rotor Flapping, Lock Number = 12.0, Blade Frequency = 1.0

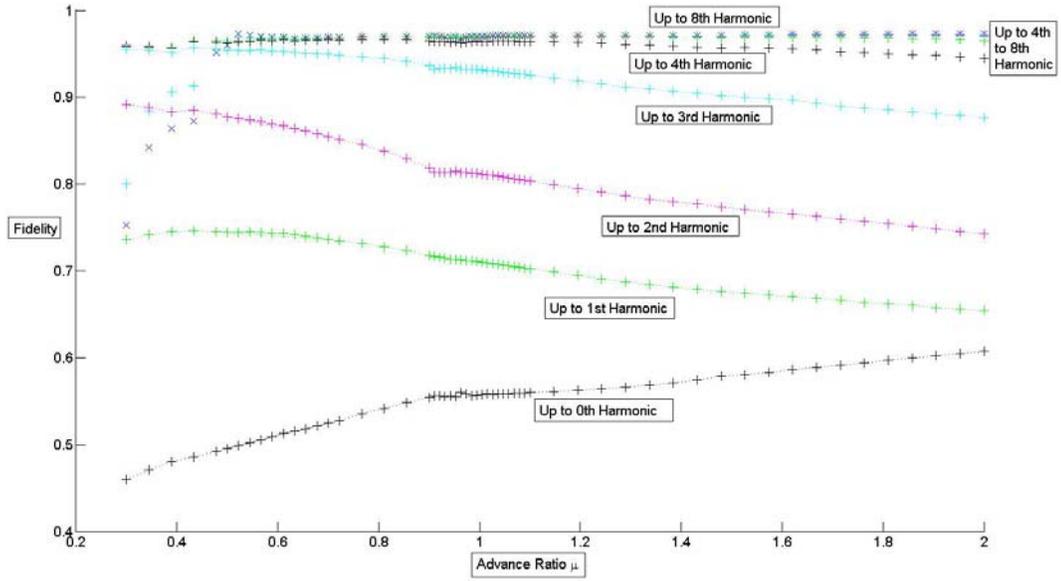


Figure 3. Fidelity, Isolated Four-Bladed-Rotor Flapping, Lock Number = 9.6, Blade Frequency = 1.0

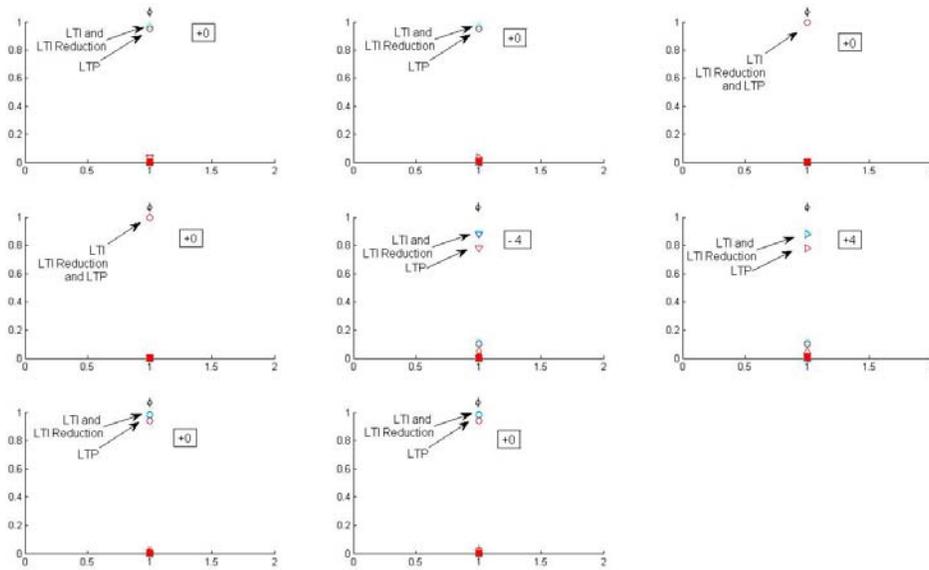


Figure 4. Modal Participation, Isolated Four-Bladed-Rotor Flapping,  $\beta_0$

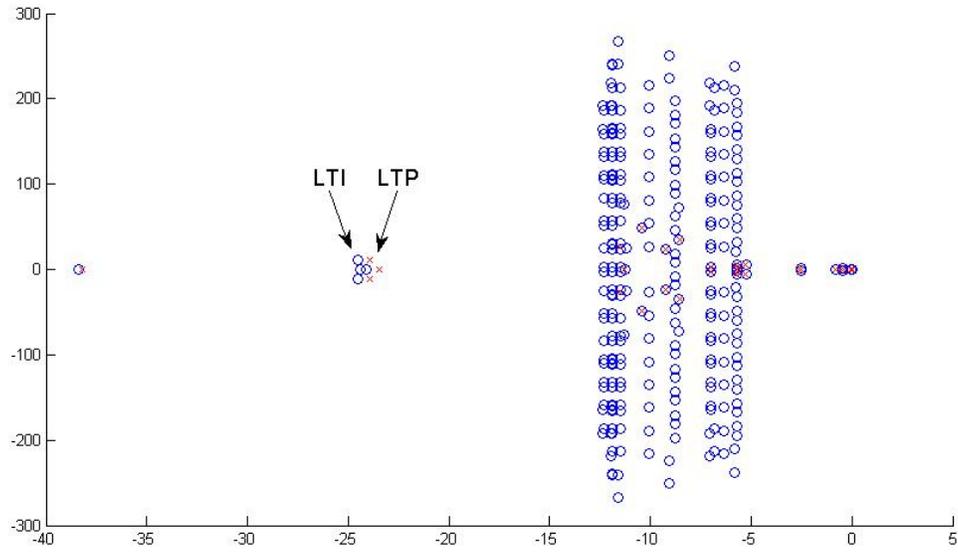


Figure 5. Eigenvalues, Generic Helicopter Model with 6 State Inflow, LTP and full LTI with 0<sup>th</sup> body and inflow harmonic term

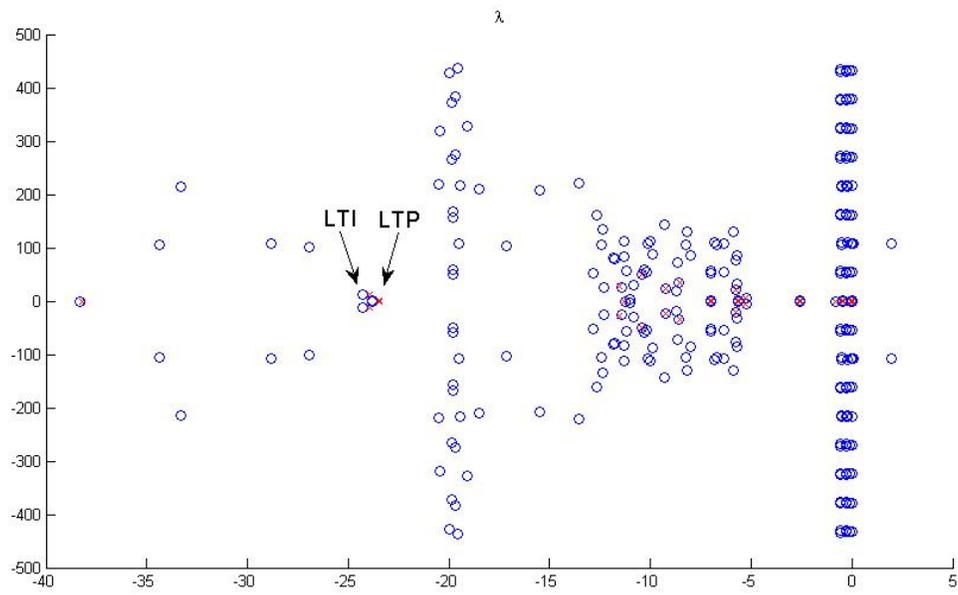


Figure 6. Eigenvalues, Generic Helicopter Model with 6 State Inflow, LTP and full extended LTI with 0<sup>th</sup>, 2<sup>nd</sup>, and 4<sup>th</sup> body and inflow harmonic terms