Harmonic Response of A Elastomeric Lag Damper*

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Abstract Harmonic response analysis of the hyper-viscoelastical structures is dealt with in this paper. an element harmonic balance method based on Hamilton variational principal and the properties of intrinsic derivates of tensor and trace function is proposed. A set of new formulas using finite element stress incremental theory and harmonic balance principal is derived. This method is applied to analysis the frequency response analysis of the elastomeric lag damper successfully.

Notation

a^e	harmonic coefficient vector of the element node displacement
a_i^e	i-th sub-block of a^e
A, B	matrix variables
b^e	harmonic coefficient vector of the element pressure
b_i^e	i-th component of b^e
C	deformation tensor
d	differential operator $d(\,)$
e_{17}	identity vector, of which the 17-th component is 1
E	strain tensor
f^e, q	external force vector acted on the element
F	deformation gradient
$g_i(a^e, b^e)$	nonlinear function vector defined by (3.21)
$h_i(a^e)$	constrain function vector of the element
H_i	interpolation function gradient of the element
I	3×3 identity matrix
K	function matrix defined by (3.18)
K_0	function matrix defined by (3.12)
K_v	function matrix defined by (3.17)
$\overline{K}_{ij}$	matrix defined by (3.19)
$\overline{L}_{ij}$	matrix defined by (3.26)
m	number of the harmonic functions
M	mass matrix
n_h	damp tuning coefficient
n_f	external force tuning coefficient
N	interpolation function matrix
N_i	i-th component of N
p, p^e	the pressure acted on the element
q_i^e	i-th harmonic component of the element external force
r	number of nodes of the element
$R_{ij}(A,B)$	matrix function
S	Kirchhoff stress tensor
t, τ, t_1, t_2	time
t_r	trace operator
T	vibration period, or matrix transpose operator
u	displacement vector of the element
u_i	displacement vector of i-th node of the element
$W_{ij}(A)$	matrix function
X	coordinate vector of the element
X_i	coordinate vector of i-th node of the element
δ	variation operator
μ_0	material constant
$\mu_1(t)$	material function
ρ	material mass density
σ	Cauch stress tensor
$\psi_i(t)$	harmonic function
ω	vibration frequency
Ω^e	element volume

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$(\ )^{\bullet}$	d/dt
$(\ )^{\bullet\bullet}$	d^2/dt^2

1. Introduction

The elastomeric lag damper is an advanced damped set applied to the rotor systems of helicopter. The material properties of the elastomer have explicit nonlinearity, incompressibility, and irreversible thermodynamic and fading memory characteristic. So it plays the role of dissipating energy and reducing vibration in the rotor systems of helicopter. The mathematical model of the damper is described by nonlinear functional differential equations with infinite delays and incompressible constraints. It is difficult to solve this vibration problem using ordinary analytical methods such as finite element method, etc. The harmonic balance method is a useful tool to solve vibration problems of nonlinear system with a lot of degrees of freedom. But the incremental harmonic balance method[1] combined the harmonic balance method with Newton-Raphsen method is unsuitable for solve these problems. For engineering nonlinear vibration problems having high dimensions and complex nonlinear properties, there is no a strongly useful method in the reference we know by now.

In order to solve these kinds of the problems, we proposed an element harmonic balance method[2], based on Hamilton variational principal and the properties of intrinsic derivates of tensor and trace function. In the numerical calculation, the curve continuation method is used to trace the unstable branches of the solution to replace Newton-Raphson method .A set of new formulas using finite element stress increment theory and the harmonic balance principal is derived to analysis frequency response of an elastomeric lag damper successfully.

2. The matrix decomposition of the deformation gradient

For an isoparameter finite element which has r nodes, interpolation function matrix N may be written as

$$N = [N_1 I, N_2 I, \cdots, N_r I] \tag{2.1}$$

where N_i is the interpolation function of i-th node, I is a 3×3 identity matrix. The coordinate X and displacement u of a physical point of the element can be denoted as

$$X = \sum_i^r N_i X_i\ ,\quad u = \sum_i^r N_i u_i \tag{2.2}$$

where $X_i\ and\ u_i$ denote the coordinate and displacement of i-th node of the element respectively. Define the interpolation function gradient of the element

$$H_i = \left[\frac{\partial N_i}{\partial X}\right]^T,\quad i = 1,\cdots,r \tag{2.3}$$

then deformation gradient F, deformation tensor C and strain tensor E can be denoted as

$$\begin{cases} F = I + \dfrac{\partial u}{\partial X} \\ \quad = I + \sum_i^r u_i H_i^T \\ C = F^T \cdot F \\ \quad = I + \sum_i^r [H_i u_i^T + u_i H_i^T] + \sum_{i,j}^r (u_i^T u_j) H_i H_j^T \\ E = (C - I)/2 \\ \quad = \sum_i^r [H_i u_i^T + u_i H_i^t]/2 + \sum_{i,j}^r (u_i^T u_j) H_i H_j^T /2 \end{cases} \tag{2.4}$$

The following variation of F can be denoted by

$$\delta F = \sum_i^r \delta u_i H_i^T\ ,\quad \delta F^T = \sum_i^r H_i \delta u_i^T \tag{2.5}$$

and we have

$$\delta E = (F^T \delta F + \delta F^T F)/2 \tag{2.6}$$

especially

$$A : \delta E = A : (F^T \delta F) = A : (\delta F^T F) \tag{2.7}$$

where A is any 3×3 symmetric matrix. For writing conveniently, we have dropped the superscript "e " of u_i^e in the equation (2.4) and (2.5).

3. Harmonic coefficient equation and its tangent stiffness matrix

Based on hamilton variational principal, the virtue power equation of an element can be written as

$$\begin{aligned} \int_{t_1}^{t_2} [\int_{\Omega^e} S : \delta E d\Omega + \int_{\Omega^e} \rho N^T N d\Omega \ddot{u}^e \cdot \delta u^e \\ - f^e \cdot \delta u^e] dt = 0 \end{aligned} \tag{3.1}$$

where S is Kirchhoff stress tensor, f^e is the generalized external force acted on the element, Ω^e is the volume of the element, t is time. If t_1 *and* t_2 are supposed to have arbitrary, then Hamilton variational principal will be changed as virtue displacement principal, that is

$$\int_{\Omega^e} S:\delta E d\Omega + \int_{\Omega^e} \rho N^T N d\Omega \ddot{u}^e \cdot \delta u^e - f^e \cdot \delta u^e = 0 \tag{3.2}$$

If take $T = t_1 - t_2$, and T is the period of the periodic solution of node displacemant, then Hamilton variational principal will be changed as harmonic balance principal, that is

$$\int_0^T [\int_{\Omega^e} S:\delta E d\Omega + \int_{\Omega^e} \rho N^T N d\Omega \ddot{u}^e \cdot \delta u^e - f^e \cdot \delta u^e] dt = 0 \tag{3.3}$$

$$\delta u^e(0) = \delta u^e(T), \quad \delta \dot{u}^e(0) = \delta \dot{u}^e(T)$$

Suppose node displacement u^e has following harmonic form

$$u^e = \sum_{i=1}^{m} a_i^e \psi_i(t) \tag{3.4}$$

where dimension of a_i^e is same as u^e, $\{\psi_1, \psi_2, \cdots, \psi_m\}$ is a normal trigonometry function group , that is ,following formulates are satisfied

$$\frac{1}{\pi}\int_0^{2\pi} \psi_i(t)\psi_j(t)dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{3.5}$$

Suppose ω is the vibration frequency , then corresponding to i-th harmonic component in the equation (3.3), i-th harmonic balance equation is

$$\begin{aligned} &\frac{1}{\pi}\int_0^{2\pi} (\int_{\Omega^e} S:\delta E d\Omega) dt \\ &- \omega^2 (\int_{\Omega^e} \rho N^T N d\Omega) a_i^e \cdot \delta a_i^e \\ &= \frac{1}{\pi}\int_0^{2\pi} f^e(t) \cdot \psi_i(t) dt \cdot \delta a_i^e \\ &i = 1, \cdots, m \end{aligned} \tag{3.6}$$

For the i-th equation, δE would be supposed only including i-th variation δa_i^e, that is, $\delta E = \frac{\partial E}{\partial u^e} : \delta a_i^e \psi_i(t)$,Above formulates also can be supposed the projection of the equation (3.3) to periodic function space spanned by $\{\psi_1, \psi_2, \cdots, \psi_m\}$. According to arbitrariness of variation, the equation (3.6) also can be written as

$$\begin{aligned} &\frac{1}{\pi}\int_0^{2\pi} (\int_{\Omega^e} S:\frac{\partial E}{\partial u^e} d\Omega) \psi_i(t) dt \\ &- \omega^2 (\int_{\Omega^e} \rho N^T N d\Omega) a_i^e \\ &= \frac{1}{\pi}\int_0^{2\pi} f^e(t) \cdot \psi_i(t) dt \\ &i = 1, \cdots, m \end{aligned} \tag{3.7}$$

The equation (3.7) is the element harmonic balance equation based on harmonic balance principal.

In order to obtain tangent stiffness matrix, consider the increment problems of $\frac{1}{\pi}\int_0^{2\pi} (\int_{\Omega^e} S:\delta E d\Omega) dt$. From the equation (3.7), we first have

$$S:\delta E = S:F^T \delta F = FS:\delta F \tag{3.8}$$

Because there are no harmonic coefficient variables in δF, we have

$$\begin{aligned} &\mathrm{d}(\frac{1}{\pi}\int_0^{2\pi} (\int_{\Omega^e} S:\delta E d\Omega) dt) \\ &= \frac{1}{\pi}\int_0^{2\pi} (\int_{\Omega^e} d(S:\delta E) d\Omega) dt \end{aligned} \tag{3.9}$$

and

$$d(S:\delta E) = dS:\delta E + (dFS):\delta F \tag{3.10}$$

The second item of the right side of the equation (3.10) can be simplified as

$$(dFS):\delta F = \sum_{i,j}^{r} \delta u_i^T [(H_i^T S H_j) I] du_j \tag{3.11}$$

Define following block matrix

$$[K_0]_{ij} = (H_i^T S H_j) I \tag{3.12}$$

that is

$$(dFS):\delta F = \delta u^e \cdot K_0 \cdot du^e \tag{3.13}$$

Obviously, every 3×3 sub-block in the matrix K_0 is a identity matrix multiplied by a number, this result is more compact than the classical result in the statics problem, in which, S was expanded from 3×3 to 9×9 matrix.

The stress increments of the first item of the right side of the equation (3.10) may be very complex when stress is complex. for the real calculation, the useful intermediate results are given below[2].

$$
\begin{cases}
t_r(\delta EAdE) = \sum_{i,j}^{r} \delta u_i^T [FW_{ij}(A)F^T] du_j \\
t_r(A\delta E)t_r(BdE) = \sum_{i,j}^{r} \delta u_i^T [FR_{ij}(A,B)F^T] du_j \\
t_r(A\delta F)t_r(BdF) = \sum_{i,j}^{r} \delta u_i^T [A^T H_i H_j^T B] du_j \\
t_r(\delta FAdF) = \sum_{i,j}^{r} \delta u_i^T [H_j H_i^T A] du_j \\
t_r(dFA\delta F) = \sum_{i,j}^{r} \delta u_i^T [A^T H_j H_i^T] du_j \\
t_r(\delta FAdF) = \sum_{i,j}^{r} \delta u_i^T [(H_i^T A H_j) I] du_j
\end{cases}
\tag{3.14}
$$

where both A and B are 3×3 matrix, and the matrix function $W_{ij}(A)$ *and* $R_{ij}(A,B)$ are defined as

$$
\begin{aligned}
W_{ij}(A) &= [H_j H_i^T A + A H_j H_i^T \\
&+ (H_i^T A H_j) I + (H_i^T H_j) A] / 4 \\
R_{ij}(A,B) &= (A + A^T) H_i H_j^T (B + B^T) / 4 \\
& i,j = 1,\cdots,r
\end{aligned}
\tag{3.15}
$$

They have following simple property

$$
\begin{aligned}
W_{ij}(A)^T &= W_{ji}(A^T), \\
R_{ij}(A,B)^T &= R_{ji}(B,A) \\
i,j &= 1,\cdots,r
\end{aligned}
\tag{3.16}
$$

Now suppose K_v is the matrix which are determined by the first item of the right side of the equation (3.10), corresponding to the equation (3.13), that is

$$
dS:\delta E = \delta u^e \cdot K_v \cdot du^e \tag{3.17}
$$

So, the matrix K is defined as,

$$
K = K_0 + K_v \tag{3.18}
$$

Then, according to the equation (3.13),(3.18),(3.9) and (3.10), j-th tangent stiffness matrix $\overline{K}_{ij}$ corresponding to i-th harmonic can be written as

$$
\begin{aligned}
\overline{K}_{ij} &= \frac{1}{\pi}\int_0^{2\pi} \left(\int_{\Omega^e} K d\Omega\right) \psi_i(t) \psi_j(t) dt \\
& i,j = 1,\cdots,m
\end{aligned}
\tag{3.19}
$$

Now consider the incompressibility of the material. The pressure p^e acted on the element has following harmonic form

$$
p^e = \sum_{i=1}^{m} b_i^e \psi_i(t) \tag{3.20}
$$

where b_i^e is a scalar. In the equation (3.7), if define

$$
\begin{aligned}
M &= \int_{\Omega^e} \rho N^T N d\Omega \\
g_i(a^e,b^e) &= \frac{1}{\pi}\int_0^{2\pi} \left(\int_{\Omega^e} S:\frac{\partial E}{\partial u^e} d\Omega\right) \psi_i(t) dt \\
q_i^e &= \frac{1}{\pi}\int_0^{2\pi} f^e(t)\cdot \psi_i(t) dt \\
i &= 1,\cdots,m
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
a^e &= [(a_1^e)^T, (a_2^e)^T, \cdots, (a_m^e)^T]^T \\
b^e &= [b_1^e, b_2^e, \cdots, b_m^e]^T
\end{aligned}
\tag{3.22}
$$

and using the equation

$$
h_i(a^e) = 0 \tag{3.23}
$$

to express the constraints condition resulted in by the incompressibility, then the element harmonic coefficient equation can be simple written as

$$
\begin{aligned}
-\omega^2 M a_i^e + g_i(a^e,b^e) &= q_i^e \\
h_i(a^e) &= 0 \\
i &= 1,\cdots,m
\end{aligned}
\tag{3.24}
$$

Considered the definition (3.19),we have

$$
\begin{aligned}
\overline{K}_{ij} &= \frac{\partial g_i(a^e,b^e)}{\partial a_j^e} \\
i,j &= 1,\cdots,m
\end{aligned}
\tag{3.25}
$$

and the following relation is proved easily[2]

$$
\begin{aligned}
\overline{L}_{ij} &= \frac{\partial g_i(a^e,b^e)}{\partial b_j^e} = \left[\frac{\partial h_i(a^e,b^e)}{\partial a_j^e}\right]^T \\
i,j &= 1,\cdots,m
\end{aligned}
\tag{3.26}
$$

So, the j-th expanded tangent stiffness matrix corresponding to i-th harmonic is

$$
\begin{bmatrix} \overline{K}_{ij} & \overline{L}_{ij} \\ [\overline{L}_{ij}]^T & 0 \end{bmatrix} \tag{3.27}
$$

4. Application

In this section, as an application of the theory, we consider the harmonic response of an elastomeric lag damper which is consisted of rustless steel, alloy aluminium and elastomer. The rustless steel and alloy aluminium supposed to have no deformation, and the material property of the elastomer is supposed near to

ZN-1 viscoelastic material. Considered the real structure of the damper, the deformation state was supposed to be the plane strain state. The following constitute relation is supposed[3]

$$\sigma = -pI + F \cdot [\mu_0 I + \int_0^t \mu_1 (t-\tau)\dot{E}(\tau)d\tau] \cdot F^T \tag{4.1}$$

where σ is the Cauch stress tensor, p is the pressure, μ_0 *and* $\mu_1(t)$ is the material constant and material function respectively. And the material function is taken the following form[4]

$$\mu_1 = \sum_i^n a_i e^{-b_i t} \tag{4.2}$$

According to the test data of ZN-1 viscoelastic material, the basic data of the material are follows[5]

$$\begin{cases} \rho = 78.95 \times 10^{-7} Ns^2 / cm^4 \\ \mu_0 = 3.955N \\ a_1 = 40.43499N \quad a_2 = 1327.9885N \\ b_1 = -1626.705 \quad b_2 = -157222.4 \end{cases} \tag{4.3}$$

and taking

$$\begin{aligned} n_h &= a_1 + a_2 \\ q(t) &= n_f \, Sin(\omega t) e_{17} \end{aligned} \tag{4.4}$$

where e_{17} denote identity vector, of which the 17-th component is 1.

In the numerical calculation, the curve continuation method is used to trace the stable and unstable branches of the solutions. The partial results are showed in the figure 1 to figure 5.

In figure 1, we take n_f = 0.3N, and n_h =2, 0.5, 0.125 respectively. It is shown that the damping effect is very clear, because n_h denotes the damp coefficient actually. In order to study the effect of the exponent items in material function, we simply take $a_1 = 2,\ a_2 = 0, n_f = 0.3N$, and calculate three cases of $b_1 = 1,\ 6,\ 20$ respectively, the results are shown in figure 2. The little b_1 , the little the damp, because b_1 has function of both damp and delay. The figure 3 show the results in the linear and nonlinear cases, the nonlinear property of the structure present hard spring characteristic in generally. The figure 4 and figure 5 show the relation between the pressure amplitude and the structure displacement amplitude. They are almost linear in the linear and little deformation conditions, and have rapid increase tendency being similar to the response curve in the finite displacement.

Reference

1. Al, S. I., Chewing, Y. K., Amplitude Incremental Variation Principal for Nonlinear Vibration of Elastic System, SAME J. Apple. Mech. Vol. 48,1981.
2. Hen. S. H., teal., On Perturbation Procedure for Limit Cycle Analysis, Int. J. Non-Linear Mech., 26, 1(1991).
3. Hanna Jingling, On the Analytical Methods of Nonlinear vibration Systems and the Dynamic Analysis of a Kind of Super-Viscoelastic Structures, PH.D. Dissertation, Department of Aircraft, Nanjing University of Aeronautics and Astronautics, (In Chinese),1994.
4. Christensen, R.M., A Nonlinear Theory of Viscoelasticity for Application to Elatomers. J. Appl. Mech., 47(1980), pp. 295-310.
5. Chen Qian, Dynamic Analysis of Elastic-Viscoelastic Complex Structures, PH.D. Dissertation, Department of Aircraft, Nanjing University of Aeronautics and Astronautics, (In Chinese), 1987.
6. Gandhi, F., Chopra, I., Analysis of Bearingless Main Rotor Dynamics with the Inclusion of an Improved Time Domain Nonlinear Elastomeric Damper Model, Presented at the AHS 51st Annual Forum. fort Worth, Texas, May 9-1, 1995.

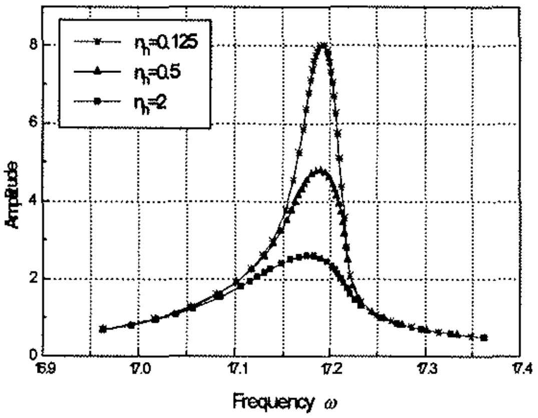

Fig. 1 effect of different damps (n_f=0.3N)

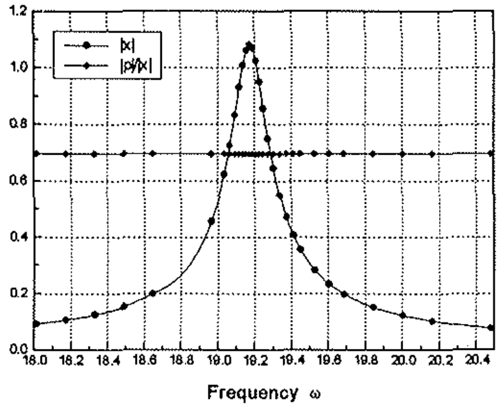

Fig. 4 pressure characteristics in the linear case
b_f=6, n_f=0.3N

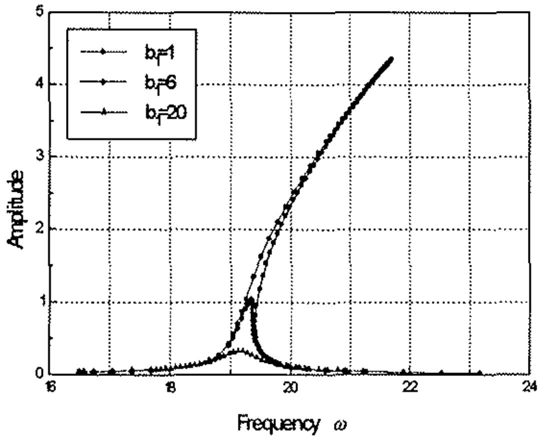

Fig. 2 effect of exponent items in material function
a_f=2, a_2=0, n_f=0.3N

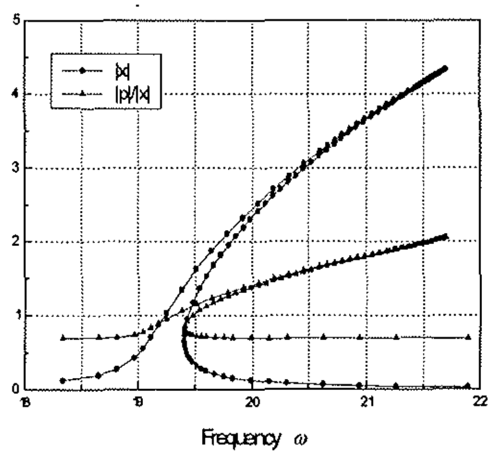

Fig. 5 pressure characteristics in the nonliear case
b_f=6, n_f=0.3N

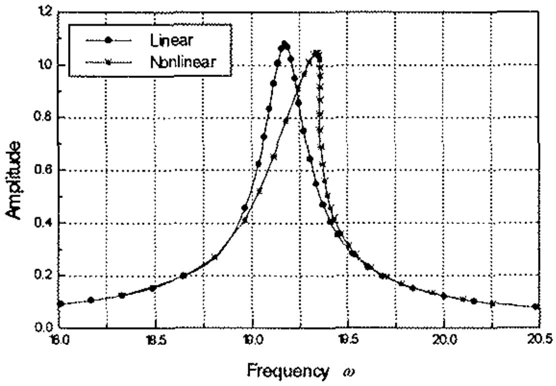

Fig. 3 hard spring characteristic of the structure
b_f=6, n_f=0.3N