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THE APPLICATION OF FINITE DIFFERENCE TECHNIQUES TO
HELICOPTER ROTOR BLADE RESPONSE CALCULATIONS

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ABSTRACT

Numerical procedures that may be used to calculate helicopter rotor response are reviewed, with emphasis placed on the finite difference approach. The treatment is neither exhaustive nor rigorous, but is aimed at providing a general discussion to aid in the choice of a suitable method. Procedures are discussed which are appropriate to different formulations of the problem. A transfer matrix technique is identified which is thought to be most efficient for performing calculations for hovering and vertical flight. Two further methods are derived which are suited to calculations for manoeuvring flight. The first method utilises a modal representation of the rotor and so promotes the solution in time only, while the second integrates the equations of motion directly and progresses the solution in both space and time.

1 INTRODUCTION

The calculation of helicopter rotor response is a complex task. Much work has been done over a period of many years to develop comprehensive dynamic and aerodynamic models of the helicopter rotor which adequately describe its behaviour. Relatively little effort has been expended, however, in investigating the numerical methods that must be applied to ensure both accuracy and efficiency in the calculation. In Ref 1, Hawkings reviews methods that are appropriate to the timewise progression of a modal solution. The general space-time problem is not considered by Hawkings, however, nor are the relative merits of modal, finite element and finite difference techniques.

More recently, activity in the helicopter community has focussed on extending analysis procedures to cover helicopter manoeuvring flight. The task represents a considerable analytical challenge, but a serious danger now arises in that current computers, powerful as they are, may be unable to cope with the immensity of the problem. It is pertinent, therefore, to review numerical analysis options at an early date, for the savings later could be great and, indeed, could mean the difference between success and failure.

In Ref 2, Bauchau and Liu point out that a finite element model for a three-bladed rotor, utilising 25 radial steps (hence 150 radial degrees of freedom) and 64 time steps, requires the solution of a 28,800 degree of freedom problem. Studies at DRA Farnborough, on the other hand, indicate that if a modal approach is adopted for a four-bladed rotor coupled to a fuselage, again using 25 radial steps, around 1.5 megabytes of data storage are required simply to capture all the various coupled rotor-fuselage modal components. Furthermore, it is by no means certain that either of these discretisations will be adequate. Many of the nonlinearities of the problem, including shock waves, stall, vortex crossings and interference effects associated with the fuselage and other rotors, are initially of unknown location. To capture these in the calculation may well require a more detailed discretisation, both in space and time, than either of the above examples suggests.

The purpose of this paper, therefore, is simply to review numerical methods in the context of the helicopter problem. The number of methods

available in all their diversity precludes anything like an exhaustive treatment and no attempt is made here to provide one. Emphasis is placed, rather, on finite difference techniques. The techniques are compared with other methods available, but only in an 'engineering' sense and not with the utmost rigour. The use of numerical examples is also kept to a minimum. Such a treatment would appear to be a little superficial, but we shall see that the fundamental issues of accuracy, stability and convergence depend not only on the numerical method but on the problem to which the method is applied. It is preferred, therefore, as far as possible to assess the methods by simple analytical means. Following the initial evaluation, a more detailed numerical investigation will be warranted within the context of the problem in hand.

2 THE ROTOR IN HOVER

In hovering and vertical flight, all timewise behaviour of the helicopter rotor is confined to its transient response. Velocities and accelerations may thus be expressed in terms of displacements and a frequency parameter, leaving equations of motion that are dependent on the radial coordinate x only. For the purposes of this Section, we express these equations in the form

$$q' = Cq + D \quad , \quad (1)$$

where q denotes a vector of blade loads and displacements. Since the problem is nonlinear, the matrix C and the vector D will both be functions of q . The perturbation procedure used to solve such equations is not considered here and this dependence is ignored.

Rotor blade equations, of course, may be presented in a variety of ways and equation 1 is but one of them. Equation 1 remains appropriate whatever aspect of rotor hovering performance or stability we wish to examine, and adapts most readily to a transfer matrix solution method compared to corresponding representations of higher degree. However, an energy-based formulation would be preferred if a finite element method were to be used.

The matrix C and vector D contain details of the structural and aerodynamic characteristics of the blade. Fig 1 presents the mass distribution for a typical blade, drawn as a sequence of straight lines. Fig 1, however, begs the question of how the data should be interpolated between data points. If, for instance, the mass variation is indeed linear between radial points $x = a, b$, then it is extremely unlikely that the stiffness distribution will be, and vice versa, an important and often overlooked point. Thus, if a high order solution method is to be successful, even with a simple tapered blade, further data points internal to each lengthwise section must be provided in order that the necessary high order derivatives may be estimated. Simple interpolation is not good enough. Equally, with a low order scheme, it is pointless reducing the step size to ensure a fine degree of convergence if the data are not known accurately enough to warrant it. Convergence is no guarantee of accuracy and, if the data are not defined in detail, a low order scheme with a limited number of steps might as well be used, for a semi-converged solution is just as likely to be correct as a converged one.

One way around the problem for blades of simple construction is to define the geometry, density and elastic moduli of the blade instead of the more usual mass and stiffness distributions. However, this is unlikely to be practical for rotor blades of normal construction and, if answers are required to a high degree of accuracy, the question of defining the radial variation of the blade data in a user-friendly manner must be addressed.

The second point to note from Fig 1 is the density of data points required in certain regions of the blade. Thus, step size in the discretisation may be

dictated not by the numerical method but by the data capture requirement. It is pointless using a high order method that permits a large size step if such radial steps are inconsistent with the data distribution.

Finally, and most obviously, we note that enormous radial discontinuities may be present in the data. Single step methods thus hold considerable attraction. They eliminate the potential pitfall of making unjustifiable assumptions about the existence of derivatives across interfaces, and simplify the bookkeeping when the step size is necessarily variable.

2.1 Accuracy of Methods of Different Order

In order to get a feel for the relative merits of methods of different order, we compare them with the exact transfer matrix for an unforced, non-rotating, uniform beam in bending over the interval x to $x + h$. If we let

$$\mathbf{q} = (w, \beta, M, V)^T ,$$

where w is the flap displacement, β the flap slope, M the bending moment and V the shear force, then the transfer matrix takes the form

$$\frac{1}{2} \begin{pmatrix} \cosh \zeta h + \cos \zeta h & \frac{1}{\zeta}(\sinh \zeta h + \sin \zeta h) & \frac{1}{\zeta^2 EI}(\cosh \zeta h - \cos \zeta h) & \frac{1}{\zeta^3 EI}(\sinh \zeta h - \sin \zeta h) \\ \zeta(\sinh \zeta h - \sin \zeta h) & \cosh \zeta h + \cos \zeta h & \frac{1}{\zeta EI}(\sinh \zeta h + \sin \zeta h) & \frac{1}{\zeta^2 EI}(\cosh \zeta h - \cos \zeta h) \\ \zeta^2 EI(\cosh \zeta h - \cos \zeta h) & \zeta EI(\sinh \zeta h - \sin \zeta h) & \cosh \zeta h + \cos \zeta h & \frac{1}{\zeta}(\sinh \zeta h + \sin \zeta h) \\ \zeta^3 EI(\sinh \zeta h + \sin \zeta h) & \zeta^2 EI(\cosh \zeta h - \cos \zeta h) & \zeta(\sinh \zeta h - \sin \zeta h) & \cosh \zeta h + \cos \zeta h \end{pmatrix} .$$

EI denotes the bending stiffness, m the mass per unit length, and ζ is defined in terms of the frequency ω as

$$\zeta = (m\omega^2/EI)^{1/4} .$$

Expanding the transfer matrix as a power series in h yields the matrix S , given by

$$S = \begin{pmatrix} 1 + \frac{m\omega^2 h^4}{24EI} & h & \frac{h^2}{2EI} & \frac{h^3}{6EI} \\ \frac{m\omega^2 h^3}{6EI} & 1 + \frac{m\omega^2 h^4}{24EI} & \frac{h}{EI} & \frac{h^2}{2EI} \\ \frac{m\omega^2 h^2}{2} & \frac{m\omega^2 h^3}{6} & 1 + \frac{m\omega^2 h^4}{24EI} & h \\ m\omega^2 h & \frac{m\omega^2 h^2}{2} & \frac{m\omega^2 h^3}{6EI} & 1 + \frac{m\omega^2 h^4}{24EI} \end{pmatrix} + o(h^5) . \quad (2)$$

We wish to ascertain how many integration steps n are required as each succeeding power of h is taken into account. Let us consider the calculation of the fourth flap mode. If r denotes the length of the blade, then

$$\zeta r \approx 7\pi/2 \approx 11 .$$

Thus

$$\frac{m\omega^2 r^4}{EI} \approx 14641 .$$

Replacing h in the transfer matrix by r/n and non-dimensionalising transforms S to the form

$$S = \begin{pmatrix} 1 + \frac{14641}{24n^4} & \frac{1}{n} & \frac{14641}{2n^2} & \frac{14641}{6n^3} \\ \frac{14641}{6n^3} & 1 + \frac{14641}{24n^4} & \frac{14641}{n} & \frac{14641}{2n^2} \\ \frac{1}{2n^2} & \frac{1}{6n^3} & 1 + \frac{14641}{24n^4} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{2n^2} & \frac{14641}{6n^3} & 1 + \frac{14641}{24n^4} \end{pmatrix} .$$

We may scale this matrix as we wish, by operating on a row and its corresponding column, to make say each n^2 error equal. Over n steps, the total error in each element of q arising from the n^2 terms may thus be estimated as

$$\varepsilon = \frac{14641^{1/2}}{2n} .$$

ε then denotes the error that might be expected from a first order integration process. For a 1% error, ie $\varepsilon = 0.01$, approximately 6050 steps are indicated as being necessary. The result parallels that of earlier RAE studies in which it was found that, even with 950 integration steps, a first order method is seriously in error in its calculation of low order mode loading.

The errors in the fourth bending mode associated with second and third order methods can likewise be estimated respectively as

$$\varepsilon = \frac{14641^{3/4}}{6n^2} , \quad \varepsilon = \frac{14641}{24n^3} .$$

For a 1% error, 150 and 40 steps appear to be required. The contribution to the overall response progressively gets smaller with each succeeding mode, however, and some relaxation of this accuracy requirement may be acceptable.

The superiority of a second order method over a first order one is clear. The added complexity of the method is more than compensated for by the reduction in discretisation steps required. Beyond second order, the law of diminishing returns applies rapidly. With a third order method, it must be remembered that function evaluations will be required at one internal point of each section, effectively doubling the number of computational points. Furthermore, each calculation will be more complex, while the discretisation of the data input may be too fine to capitalise on the accuracy of the method. In the DRA blade stability program, a second order transfer matrix method has been adopted. It is felt that this offers the best compromise between accuracy and efficiency.

2.2 Second Order Methods

Choices have to be made even within the narrow field of second order methods. The unforced version of equation 1, when solved by the modified Euler method, for example, is re-cast as

$$q(x + h) = \left(I - \frac{h}{2}C(x + h) \right)^{-1} \left(I + \frac{h}{2}C(x) \right) q(x) ,$$

whereas, with second order Runge-Kutta, it becomes

$$q(x + h) = \left(I + \frac{h}{2} (C(x) + C(x + h)) \right) + \frac{h^2}{2} C(x)C(x + h) q(x) .$$

The first of these relationships solves the equation

$$q' = q/x$$

exactly, while the second solves

$$q' = -q/x .$$

Thus, the error is dependent not only on the numerical method but on the problem to which it is applied.

Although it may be tempting to seek a second order method which offers the best accuracy for helicopter applications, experience at DRA suggests that the range of helicopter problems is sufficiently great that such an optimum cannot be identified. For this reason, the second order method chosen for the DRA blade stability program is the simplest possible. It is derived as follows

$$\begin{aligned} q(x + h) &= q(x) + hq'(x) + \frac{h^2}{2} q''(x) + o(3) \\ &= \left(I + hC + \frac{h^2}{2}(C' + C^2) \right) q(x) + hD + \frac{h^2}{2} (CD + D') + o(3) , \end{aligned}$$

on using equation 1. However,

$$C'(x) = \frac{1}{2} (C(x) + C(x + h)) + o(1) .$$

Thus

$$q(x + h) = \left(I + h\bar{C} + \frac{h^2}{2} \bar{C}^2 \right) q(x) + h\bar{D} + \frac{h^2}{2} \bar{C}\bar{D} + o(3) ,$$

where

$$\bar{C} = \frac{1}{2} (C(x) + C(x + h)) , \text{ etc} .$$

For the uniform beam, this yields a transfer matrix which contains the second order terms of S and no more. The method is marginally more efficient than second order Runge-Kutta in situations where \bar{C} can be stored, since one less matrix addition is required. It has performed successfully in the DRA blade stability program for several years. Around 150 integration steps are used if accurate results are sought, but around 30 give reasonable estimates.

It will become apparent in the next Section that, being 'explicit', the method is not unconditionally stable. Our problem, however, is bounded by the ends of the rotor blade, and errors cannot grow indefinitely. By treating the 'worst case' in the accumulated error estimate of Section 2.1, in which the component errors are all additive, numerical stability considerations are effectively accommodated in the accuracy assessment. Indeed, to seek unconditional stability is unlikely to prove successful, since blade bending components contain a positive exponential element.

Difficulties may still arise, however, if the matrix C contains elements that are numerically too large for the given step length h (ie we depart significantly from the uniform blade of Section 2.1). Such difficulties are likely to be local and have yet to present a practical problem. Hinges positioned within the blade offer the greatest challenge. In the DRA program, these are represented correctly, by introducing further degrees of freedom rather than small regions of low stiffness. In this way, the most likely cause

of numerical instability is avoided.

2.3 Comparison with Finite Elements

Finally in relation to the hover case, it is noted that the standard finite element formulation for a uniform beam in bending with a consistent mass representation can be reconfigured to transfer matrix form. The finite element equations express the loads at either end of an element in terms of the displacements at those ends. Simple algebraic manipulation can re-express these equations such that the loads and displacements at one end of the element are given in terms of the loads and displacements at the other. The transfer matrix that results, when compared with equation 2 above, is found to be correct to order four. Thus, we may expect a finite element solution to converge significantly more rapidly for a uniform beam than any of the methods described so far. As we have seen, however, this accuracy will not carry over to non-uniform beams unless some account is taken of the radial data variation across each element. This will not add to the number of nodal degrees of freedom, but it will add to the computation and computational advantage will be lost.

Finite elements can nevertheless offer several other benefits. They are particularly suited to sub-structured systems. They eliminate loads from the representation, expressing the problem in terms of displacements only, and they typically result in a mass and stiffness matrix for the overall system from which all the modes and natural frequencies can be extracted by conventional techniques. None of these advantages, however, is particularly relevant to the analysis of helicopter rotor blades. Helicopter rotor blades have few sub-structures, the loads we wish to calculate anyway and, once the full aeroelastic equations are accommodated, there is little need to examine modes in detail, aside from those that potentially can go unstable. On the debit side, a finite element method requires substantial pre-processing of the energy expressions to get them into a solvable form. This would be a significant drawback were it not for the algebraic manipulation packages now available on computers. On balance, the finite difference approach has been preferred in the DRA blade stability program because of the ease with which modifications to the blade representation can be incorporated. The choice, however, is not clear cut.

3 MODAL SOLUTION IN FORWARD FLIGHT

In forward flight, the helicopter rotor blade is subject to an oscillatory aerodynamic loading and its response is a function of both space and time. One way of determining the response is to use the blade normal modes calculated from the hovering condition. This allows the radial dependence of the equations of motion to be eliminated, leaving equations in time only, ie

$$\ddot{q} + 2\nu\omega\dot{q} + \omega^2q = F(t) \quad . \quad (3)$$

We are thus considering in this Section equations specifically of second degree.

The attraction of the modal method is that the modal degrees of freedom represented on the left-hand side of the equations are decoupled (allowing a scalar notation to be used). However, marked coupling may still be present through the forcing function $F(t)$. The coupling may arise not only from the aerodynamic loading, but from structural parameters, like blade root pitch, if these are altered from their values when the modes were calculated.

A large variety of methods are available for solving equation 3. The discussion here focusses on the Euler, Z-transform and central difference methods. The classifications are used generically in this investigation, although strictly they apply to individual techniques. Comments on other methods are made where appropriate.

Whilst emphasis is placed on solving equation 3, consideration is also given to the problems of extending the methods to cover the coupled case, when the left-hand side of the equation incorporates non-diagonal matrices.

3.1 Euler's Method and Aspects of Numerical Stability

Numerical instability was not found to be a concern with the integration methods of Section 2 due to the bounded nature of the problem to which the methods were applied. The timewise solution in steady forward flight is likewise bounded, by a single rotor cycle (for single rotor configurations), and accuracy considerations alone may again suffice to define a suitable numerical method. In manoeuvring flight, however, the timewise progression is potentially unbounded and care must be taken to ensure that errors do not build up to such an extent that they ultimately swamp the solution. (The 'steady' forward flight of multiple rotor systems may effectively be included in this category, whether or not the rotors operate at speeds that are rational multiples of one another.) We seek, therefore, a numerical method that will tackle the more demanding manoeuvring flight problem, in the knowledge that the steady case will follow as a matter of course.

Consider first, the simplest manoeuvre, ie steady forward flight itself, treated not as a bounded problem but as a continuous integration in time. The condition for stability of both the numerical method and the physical system is that the eigenvalues of the Floquet transition matrix for a single rotor cycle each have modulus less than or equal to one. In this way, any error in the solution (or perturbation of the physical system) is prevented from growing as the Floquet matrix is repeatedly applied. The requirement is additional to that of a converged forced response and is necessary if even the most basic manoeuvring flight calculation is to be made. When the physical system is well damped, the condition may be met by a variety of numerical methods. When it is only marginally so, some methods may struggle and require a substantially smaller step size than is necessitated from steady forced response considerations alone.

Numerical methods that offer 'unconditional' stability thus hold attraction. Such methods generally sacrifice accuracy over a single step in order to generate errors which are oscillatory and bounded in total. Ultimately, this is preferable to smaller stepwise errors which are all additive.

Perhaps the best known and most widely used 'unconditionally' stable algorithm is that due to Euler. The modified Euler equations take the form

$$\begin{aligned} q^+ &= q^0 + \frac{k}{2} (\dot{q}^+ + \dot{q}^0) \\ \dot{q}^+ &= \dot{q}^0 + \frac{k}{2} (\ddot{q}^+ + \ddot{q}^0) \end{aligned} \quad (4)$$

where superfixes 0, + identify variables at times t , $t + k$. (These are a particular form of Newmark's equations, corresponding to Newmark parameters β , γ taking values $\frac{1}{4}$ and $\frac{1}{2}$). Application of equations 4 to the simple equation

$$\ddot{q} + pq = 0 \quad (5)$$

yields the recurrence relation

$$\begin{aligned} \begin{pmatrix} q^+ \\ \dot{q}^+ \end{pmatrix} &= \left(1 + \frac{pk^2}{4}\right)^{-1} \begin{pmatrix} 1 - \frac{pk^2}{4} & k \\ -pk & 1 - \frac{pk^2}{4} \end{pmatrix} \begin{pmatrix} q^0 \\ \dot{q}^0 \end{pmatrix} \\ &= Sq^0 \quad \text{say,} \end{aligned} \quad (6)$$

when \ddot{q} is eliminated.

This is the general form for all one-step methods, and stepwise numerical stability is assured if the eigenvalues of S , like those of the Floquet matrix, each have modulus less than or equal to one. It is readily shown that Euler's method is stable for all k and all positive p . The result is ensured by approximating the exact elements of S , which are trigonometric functions in k , by one polynomial divided by another. Implicitness of the integration scheme is thus essential.

Although the example is simple, it serves to illustrate a number of important points. The first to note is that, although the Euler method is referred to as 'unconditionally' stable, it is not strictly so, for p must be positive. Thus the stability of the method, like its accuracy, depends on the problem to which it is applied. The attraction of the Euler method is that its stability characteristics reflect those of the known solution to equation 4.

What if p is not constant, however, but varies with time? This situation is particularly relevant to helicopter rotor blades, for which the equations of motion in forward flight contain periodic coefficients. The stability of the physical system is unknown, so it is impossible to ensure that the numerical method exactly reflects it. Indeed, one of the major reasons for performing a numerical calculation in the first place is in order that physical stability may be assessed. We require a numerical method that neither adds damping nor removes it. Yet, how can we be sure that the Euler method complies in this respect if our assessment of its stability is based on the constant coefficient case?

To illustrate the problem further, we note that the \ddot{q} substitution into equations 4 can be made with equal accuracy in two distinct ways, ie

$$\dot{q}^+ = \dot{q}^0 - \frac{k}{2}(p^+q^+ + p^0q^0) \quad (7)$$

$$\dot{q}^+ = \dot{q}^0 - \frac{\bar{p}k}{2}(q^+ + q^0) \quad , \quad (8)$$

where
$$\bar{p} = \frac{1}{2}(p^+ + p^0) \quad .$$

Each preserves the spirit of averaging within the Euler method. The first, however, is stable when

$$p^+ \geq p^- \quad ,$$

the second when

$$\bar{p} \geq 0 \quad .$$

These requirements are both independent of k . Thus numerical stability is not conditional upon reducing the step size to a small enough value, as it is with other common discretisations. More importantly, we cannot say anything about stability overall based on stability considerations for a single step, since S is now varying and, with it, its eigenvectors. One of the most important features of numerical stability is that errors are oscillatory from step to step and to some extent self-cancelling. Clearly, the way p varies is now important, and general conclusions about stability for arbitrarily varying p cannot be drawn. All that can be done in steady forward flight is to seek convergence of the Floquet matrix, and this we have to do for all numerical methods whether they purport to be unconditionally stable or not.

Of the two representations 7 and 8, 8 is to be preferred even though it is perhaps the less natural to apply. It generates a transfer matrix which has a

unit determinant and which captures the stability characteristics of the physical system to a higher power of k . The condition for its stability is more basic, being dependent on the first term in the Taylor series for p , rather than the second. Error is more likely to be minimised, furthermore, since p must originate from a positive value in hover for stable flight in that mode.

The second point to note from the simple example concerns the local behaviour of p . Although a problem may be well-behaved overall, p may still exhibit awkward local characteristics, in a region of stall flutter for example. The physical system itself may be locally unstable even though, overall, it is not. In such regions, physical 'errors' will grow from random perturbatory forces, or turbulence, and these may be averaged out using measurements taken over a number of cycles. Numerical error, on the other hand, will grow from an identical rounding error each cycle, assuming that a fully converged solution has been obtained, and no amount of averaging will remove it. Further tests for numerical convergence are therefore required in these more demanding computational areas.

The third point to note is that, in many formulations of the equations of motion, the pq term is not identified explicitly but is contained implicitly within the forcing function F . This will arise, for instance, if the aerodynamic load is derived from look-up tables. Some iteration on F^+ will be required. More importantly, all such forcing is potentially destabilising, whatever numerical method is used, whether p varies or not. It is advisable, therefore, to locate as much as possible of the q dependence on the left-hand side of the equations, even if only an estimate can be made. This may be undertaken either at the modes calculation or at the forced response calculation stage.

Finally, we note that, in general manoeuvring flight, p is no longer periodic but is an arbitrary function of t . Even if our calculations in steady forward flight are totally satisfactory, this is no guarantee that they will remain so in manoeuvring flight and, in some situations, the calculated transient response may exhibit significant, if not divergent, error. More convergence testing is required.

In summary, therefore, the initial apparent advantage of the Euler method, and other 'unconditionally' stable algorithms, can be seriously eroded by the practicalities of the problem. Periodic coefficients, local instability, implicit forcing and manoeuvring flight can all create special numerical circumstances for which convergence of the solution must be tested. Despite this, the modified Euler method remains a robust numerical technique and combines good stability characteristics with efficient stepwise computation.

3.2 The Z-transform Method

An alternative approach to producing numerical stability is to capitalise on the particular form of equation 3, for which the complementary function is known, and derive difference equations which follow the complementary function exactly. This effectively is what the Z-transform method does. A derivation of the Z-transform difference equations is provided in Ref 3 and precised in Ref 4. A lot of complicated talk of Laplace transforms, hold circuits and sampling pulses of data is to be found. Let us instead simply seek a solution to equation 3 over the interval $t - k$ to $t + k$ of the form

$$q^+ = (X \cos \alpha \omega k + Y \sin \alpha \omega k) e^{-\nu \omega k} + \sum q_n k^n, \quad (9)$$

where $\alpha^2 = 1 - \nu^2$. Note that X , Y and q_n may depend on t . We may readily eliminate X and Y from corresponding expressions for q_0^+ and q_0^- , to obtain

$$q^+ - 2q_0^+ e^{-\nu \omega k} \cos \alpha \omega k + q^- e^{-2\nu \omega k} = \sum q_n (k^n + (-k)^n e^{-2\nu \omega k}) - 2q_0^- e^{-\nu \omega k} \cos \alpha \omega k. \quad (10)$$

However, from equation 3,

$$F^0 = q_2 + 2\nu\omega q_1 + \omega^2 q_0 \quad ,$$

and equation 10 can be reduced, for instance, to

$$\text{LHS} = F^0 k^2 e^{-\nu\omega k} + o(4) \quad (11)$$

or
$$\text{LHS} = \frac{F^0 k}{\alpha\omega} e^{-\nu\omega k} \sin \alpha\omega k + o(4) \quad (12)$$

or
$$\text{LHS} = \frac{F^0}{\omega^2} \left(1 - 2e^{-\nu\omega k} \cos \alpha\omega k + e^{-2\nu\omega k} \right) + o(4) \quad . \quad (13)$$

Equations 10 - 13 differ significantly from those quoted in Ref 4. The Ref 4 equations do not progress the transient response exactly, in that they omit α from the left-hand side, nor do they progress the forced response as accurately. The zero order hold equations of Ref 4 appear to have a third order error, whilst the first order hold equations, which should be better, admit a second order error. This includes a term $2q_0\omega^2 k^2/3$, which is present even for constant forcing.

The formulations presented here effectively combine the Z-transform progression of the complementary function with a Taylor series progression of the particular integral. As such, they combine the stability characteristics of the Z-transform method with an accurate and simple progression of the forced response, whilst requiring the evaluation of F at only the current point.

Each of the above expressions is equivalent to

$$\text{LHS} = F^0 k^2 (1 - \nu\omega k) + o(4) \quad ,$$

and clearly an infinite number of trigonometric expressions can be devised to approximate this to the given accuracy in k. Those quoted have been chosen for their particular attributes. Equation 11 offers simplicity, equation 12 integrates exactly the response to the forcing function

$$F(t) = (X \cos \alpha\omega t + Y \sin \alpha\omega t) e^{-\nu\omega t}$$

and equation 13 integrates exactly the response to a constant forcing function. The performance of each method will thus depend on the problem to which it is applied.

If values of the forcing function are available at other than the current point, further improvements in accuracy can be made. Again, a variety of procedures can be devised. The following offers particular simplicity,

$$\text{LHS} = \frac{1}{6\alpha^2 \omega^2} \left(F^+ + 10F^0 e^{-\nu\omega k} + F^- e^{-2\nu\omega k} \right) \left(1 - \cos \alpha\omega k \right) + o(6) \quad ,$$

whilst
$$\text{LHS} = \frac{1}{\omega^4 k^2} \left[\left(\omega^2 k^2 F^0 - \nu\omega k F_1 + (4\nu^2 - 1)F_2 \right) \left(1 - 2e^{-\nu\omega k} \cos \alpha\omega k + e^{-2\nu\omega k} \right) \right. \\ \left. + \left(\frac{\omega^2 k^2}{2} F_1 - 2\nu\omega k F_2 \right) \left(1 - e^{-2\nu\omega k} \right) + \frac{\omega^2 k^2}{2} F_2 \left(1 + e^{-2\nu\omega k} \right) \right] + o(6), \quad (14)$$

where

$$F_1 = F^+ - F^- \\ F_2 = F^+ - 2F^0 + F^- \quad ,$$

integrates exactly the response to a quadratic forcing function. Method 14 (likewise, method 13) has the advantage that the power of ω is minimised in the error term. It is thus most suited to 'stiff' equations. It is to be noted that the above two methods admit only a sixth order error. They are thus particularly powerful.

It is apparent from equation 13 that accuracy will be lost when $v = 0$ and $\omega k = 2\pi$, for the forcing term then reduces to zero. This condition remains a problem for all the Z-transform methods derived in this Section and effectively provides an upper bound for k if accuracy is to be assured. With equation 14, however, the condition has to be satisfied almost exactly for inaccuracies to accrue and, even then, the errors are not intolerable. Away from $\omega k = 2\pi$, equation 14 provides excellent accuracy, even with very high frequency modes.

In cases where F is dependent on q or \dot{q} , equation 14 no longer remains explicit. A matrix inversion then becomes necessary, or else an iterative procedure must be devised. Iteration is unavoidable with equation 14, of course, if the dependence of F on q or \dot{q} is not explicitly defined. Equation 13 must be applied initially. The estimates of q and \dot{q} so found are then substituted into equation 14 and the process repeated until a suitably accurate solution is obtained.

Expressions for velocity progression are required. These can take the form

$$\begin{aligned} \dot{q}^+ + v\omega q^+ + 2\alpha\omega q^0 e^{-v\omega k} \sin \alpha\omega k - (\dot{q}^- + v\omega q^-) e^{-2v\omega k} \\ = \frac{1}{\omega^3 k^2} \left(\left(\omega^2 k^2 F^0 - v\omega k F_1 + (4v^2 - 1) F_2 \right) \left(v(1 - e^{-2v\omega k}) + 2\alpha e^{-v\omega k} \sin \alpha\omega k \right) \right. \\ \left. + \left(\frac{\omega k}{2} F_1 - 2v F_2 \right) \left(1 + v\omega k - (1 - v\omega k) e^{-2v\omega k} \right) \right. \\ \left. + \frac{\omega k}{2} F_2 \left(2 + v\omega k + (2 - v\omega k) e^{-2v\omega k} \right) \right) + o(5) \end{aligned} \quad (15)$$

$$= \frac{1}{3\alpha\omega} \left(F^+ + 4F^0 e^{-v\omega k} + F^- e^{-2v\omega k} \right) \sin \alpha\omega k + o(5) \quad (16)$$

$$= \frac{F^0}{\omega} \left(v(1 - e^{-2v\omega k}) + 2\alpha e^{-v\omega k} \sin \alpha\omega k \right) + o(3) \quad (17)$$

$$= 2F^0 k e^{-v\omega k} + o(3) \quad (18)$$

Again, the transient response is followed exactly when F is independent of q and \dot{q} . Methods 15 and 17 progress the forced response with minimal ω error, whilst methods 16 and 18 offer simplicity.

It might be tempting to apply the higher accuracy expressions iteratively until convergence is obtained. The convergence tolerances imposed, however, may not be justified by the accuracy of the stepwise integration. It is readily found that one iteration will suffice to restore the accuracy level of the higher order method if F depends only on q , but that two iterations are required if F depends on \dot{q} . If further iterations appear to be necessary to generate a converged solution across one step, then the solution, converged or otherwise, is inaccurate and a step length reduction must be made.

This iterative procedure clearly adds strength to the method. If F is dependent on q or \dot{q} , the transient response is no longer progressed exactly but it is still progressed as accurately as the forced response. The prospect now exists of applying the method to multiple degree of freedom problems, in which

the stiffness and damping matrices are no longer diagonal. The left-hand side of the equations can be diagonalised in some simple, but appropriate fashion, accompanied by an equivalent modification to the right-hand side.

The decision as to whether iteration is the preferred solution method, or whether matrix inversion or indeed the simpler method with a reduced step size is used, should be made in the light of the specific application. Convergence tests in the problem areas defined in Section 3.1 will be required.

In summary, the Z-transform offers the best combination of accuracy and stability characteristics for a limited (but common) class of problem. It is not easy to apply in a way that retains these advantages in the general case, however, and its suitability will depend largely on how far the equations of motion stray from the form of equation 3. In theory, the Z-transform equations can be generalised to multi-degree of freedom systems, but exponential and trigonometric functions of matrices result which are unlikely to be computed efficiently. The degree of modal coupling is crucial and will depend on what is included in the preliminary modes calculation. Damping is likely to present the biggest problem, for to uncouple the damping matrix requires the introduction of complex modes. Equation 3, as it stands, merely allows modal damping to be added as something of an afterthought, in the way that structural damping is commonly represented.

3.3 Central Differences

Central difference formulae may be derived readily from the foregoing Z-transform expressions simply by transferring all the stiffness and damping terms to the right-hand side of the equations. There results

$$\begin{aligned} q^+ - 2q^0 + q^- &= \frac{k^2}{12} \left(F^+ + 10F^0 + F^- - \omega^2(q^+ + 10q^0 + q^-) - 2v\omega(\dot{q}^+ + 10\dot{q}^0 + \dot{q}^-) \right) + o(6) \\ &= k^2 \left(F^0 - \omega^2 q^0 - 2v\omega \dot{q}^0 \right) + o(4) \quad . \end{aligned}$$

Unlike the standard central difference expressions, the above make use of \dot{q} . The progression of \dot{q} also derives from the equivalent Z-transform expressions, and takes the form

$$\begin{aligned} \dot{q}^+ - \dot{q}^- &= \frac{k}{3} \left(F^+ + 4F^0 + F^- - \omega^2(q^+ + 4q^0 + q^-) - 2v\omega(\dot{q}^+ + 4\dot{q}^0 + \dot{q}^-) \right) + o(5) \\ &= 2k \left(F^0 - \omega^2 q^0 - 2v\omega \dot{q}^0 \right) + o(3) \quad . \end{aligned}$$

By retaining the \dot{q} dependence, the low order expressions permit the solution to progress in an explicit manner, without a matrix inversion, even when the damping matrix is non-diagonal. A non-diagonal damping matrix is virtually inevitable with helicopter rotor blades, since aerodynamic and Coriolis forces are both present. Although the calculation of \dot{q} as well as q appears to double the computation, \dot{q} will be required if aerodynamic tables are to be used.

The factors governing the choice of central difference method are identical to those pertaining to the Z-transform. Thus iteration, matrix inversion and the low order method with reduced step size are all variations to be considered in the light of the application. It should be emphasised, however, particularly with central differences, that if the step size is dictated by stability requirements, the higher order method offers no advantage over the low order method for an equal computational effort. Thus, the stability condition for an undamped system using the higher order method improves from

$$\omega k \leq 2$$

to only

$$\omega k \leq \sqrt{6} \quad .$$

The central differences method offers the best combination of accuracy and stepwise efficiency. It does not guarantee stability, however, and should not be used in situations where high frequency modes are present but contribute nothing to the response.

3.4 One-step Formulations

The methods outlined above are based on the Taylor series expansion of the forced response. If the forcing function contains non-linearities, such polynomial expansions become invalid and accuracy may be lost. In such circumstances, it may be better to opt for a one-step method, in an effort to contain the effect of the non-linearity to the time interval in which it occurs. Accuracy may be lost, but the step length will be dictated by non-linearity capture requirements anyway.

Utilising similar techniques to those of Section 3.2, we can derive the following one-step Z-transform methods

$$\begin{aligned} q^+ &= \left(\omega q^0 (\alpha \cos \alpha \omega k + \nu \sin \alpha \omega k) + \dot{q}^0 \sin \alpha \omega k \right) \frac{e^{-\nu \omega k}}{\alpha \omega} \\ &= \frac{1}{\alpha \omega^3 k} \left(\left(\omega k F^0 - 2\nu(F^+ - F^0) \right) \left(\alpha - e^{-\nu \omega k} (\alpha \cos \alpha \omega k + \nu \sin \alpha \omega k) \right) \right. \\ &\quad \left. + (F^+ - F^0) \left(\alpha \omega k - e^{-\nu \omega k} \sin \alpha \omega k \right) \right) + o(4) \\ &= \frac{k^2}{6} \left(F^+ + 2F^0 e^{-\nu \omega k} \right) + o(4) \\ &= \frac{F^0}{\alpha \omega^2} \left(\alpha - e^{-\nu \omega k} (\alpha \cos \alpha \omega k + \nu \sin \alpha \omega k) \right) + o(3) \\ &= \frac{k^2}{2} F^0 e^{-\nu \omega k} + o(3) \quad , \end{aligned}$$

together with

$$\begin{aligned} \dot{q}^+ &= \left(\dot{q}^0 (\alpha \cos \alpha \omega k - \nu \sin \alpha \omega k) - \omega q^0 \sin \alpha \omega k \right) \frac{e^{-\nu \omega k}}{\alpha} \\ &= \frac{1}{\alpha \omega^2 k} \left(\left(\omega k F^0 - 2\nu(F^+ - F^0) \right) e^{-\nu \omega k} \sin \alpha \omega k \right. \\ &\quad \left. + (F^+ - F^0) \left(\alpha - e^{-\nu \omega k} (\alpha \cos \alpha \omega k - \nu \sin \alpha \omega k) \right) \right) + o(3) \\ &= \frac{k}{2} \left(F^+ + F^0 e^{-\nu \omega k} \right) + o(3) \\ &= \frac{F^0}{\alpha \omega} e^{-\nu \omega k} \sin \alpha \omega k + o(2) \\ &= k F^0 e^{-\nu \omega k} + o(2) \quad . \end{aligned}$$

Simplified relationships which are equivalent to the two-step central difference formulae can be deduced in similar fashion. They are

$$q^+ - q^0 - k\dot{q}^0 = \frac{k^2}{6} \left(F^+ - \omega^2 q^+ - 2\nu\omega\dot{q}^+ + 2(F^0 - \omega^2 q^0 - 2\nu\omega\dot{q}^0) \right) + o(4)$$

$$= \frac{k^2}{2} \left(F^0 - \omega^2 q^0 - 2\nu\omega\dot{q}^0 \right) + o(3) \quad ,$$

$$\dot{q}^+ - \dot{q}^0 = \frac{k}{2} \left(F^+ - \omega^2 q^+ - 2\nu\omega\dot{q}^+ + F^0 - \omega^2 q^0 - 2\nu\omega\dot{q}^0 \right) + o(3)$$

$$= k \left(F^0 - \omega^2 q^0 - 2\nu\omega\dot{q}^0 \right) + o(2) \quad .$$

The Euler equations also fall into the category of one-step procedures but have been discussed already in Section 3.1. The more accurate of the above simplified equations is, in fact, equivalent to the Euler formulation with one term added to the displacement progression, ie

$$q^+ = q^0 + \frac{k}{2} (\dot{q}^+ + \dot{q}^0) - \frac{k^2}{12} (\ddot{q}^+ - \ddot{q}^0) \quad .$$

Stability for an undamped system using this expression is conditional upon

$$\omega k \leq \sqrt{12} \quad .$$

Application of the one-step formulae twice allows the velocity to be eliminated from the displacement expressions and permits equivalent two-step formulae to be retrieved. The less accurate formulae yield identically their less accurate two-step counterparts. Thus the better accuracy of a two-step method is an illusion and is insufficient reason, in itself, for adopting a two-step approach. (There equally seems little point in identifying a two-step equivalent to the Euler method.) The more accurate two-step formulae are not retrieved in this manner, however, and offer a superior accuracy than can be achieved by repeated application of one-step formulae.

3.5 Experience with the Z-transform Method

The danger of drawing general conclusions about numerical methods from experience with particular examples has been stressed. Nevertheless, some indication of how well the Z-transform method works is desirable, since the formulations derived here are quite different from those cited in earlier References. The equation

$$\ddot{q} + 2\nu\omega\dot{q} + \omega^2 q = \sin \Omega t \quad ,$$

subject to the initial conditions $q(0) = \dot{q}(0) = 0$ is readily solved exactly. It may be re-cast in the form

$$\ddot{q} + 2(1 - \beta)\nu\omega\dot{q} + (1 - \gamma)\omega^2 q = \sin \Omega t - 2\beta\nu\omega\dot{q} - \gamma\omega^2 q \quad ,$$

which allows an iterative scheme based on equation 14 to be applied for various values of ν , ω , β , γ and step length k , treating the right-hand side as an implicitly defined forcing function.

Table 1 summarises the results for a variety of parameter combinations, the error being rounded to the nearest one percent of the maximum response amplitude. An 'x' denotes severe inaccuracy and a 'u' numerical instability. It may be seen that the method performs extremely well in situations where other methods would fail. In particular, if $\Omega = 1$ denotes the rotational frequency, the response of the mode at 11.5Ω to a forcing at 12Ω is predicted with

excellent accuracy with only a 5° azimuth step and with half the stiffness and damping located within the forcing function.

3.6 General Discussion and Comparison with Other Methods

The length of a computation will be determined by three factors:

- 1 The number of steps dictated by numerical stability
- 2 The number of steps dictated by overall accuracy
- 3 The efficiency of the computation over a single step.

The Euler method would appear to be best on counts 1 and 3, the Z-transform method on counts 1 and 2, and the central differences method on counts 2 and 3.

All methods must be tested for convergence in the context of periodic coefficients, local instability, implicit loading and manoeuvring flight. Further aspects of the problem which will determine the choice will be the frequency span of the response, the frequencies of the input modes, the degree of modal coupling and the degree of non-linearity. The latter will influence whether an iterative, matrix inversion or simple explicit version of the chosen method is required.

The methods considered have generally made use of the second degree form of equation 3. Equation 3 can of course be re-cast as two equations of first degree, by introducing the velocity as a dependent variable, and many methods are available which tackle this general form of problem. Fourth order Runge-Kutta and the Adams and Milne methods¹ are examples. These methods have rightful place in Section 2 and would have been considered there had the specific problem of the radial variation of blade data not proved critical. What such methods gain in their general applicability, however, they tend to lose in their accuracy. Thus, fourth order Runge-Kutta offers an accuracy one order of magnitude less than the method of equation 14, yet requires one force evaluation per step more. Furthermore, it is only conditionally stable at best.

Finite elements in time have received some recent attention in the literature. The theory is developed in Ref 5. Problems of numerical stability prompt modification of the element using a 'relaxed integration' technique, by which accuracy is again traded for stability. The technique seems to be something of a black art, and it is not clear how it may be generalised to cope with the practical problems discussed here. If the method is applied to equation 3, for instance, it would appear that the complementary function is progressed in identical fashion to the Euler method. The forced response, on the other hand, is progressed differently, with an error that has greater dependence on natural frequency and so is amplified with 'stiff' equations. The present message would appear to be that the direct application of the Euler method offers greater accuracy and greater simplicity.

4 DIRECT INTEGRATION OF THE FORWARD FLIGHT EQUATIONS

The modal method has been applied successfully for many years to the calculation of individual rotor blade response in forward flight. The method has the advantage of expressing the problem in terms of a few modal coordinates, thereby minimising the computation and promoting an understanding of the underlying physics. It has the disadvantage that the modal series may not converge rapidly in certain situations. Poor convergence may be encountered either in regions of nonlinear forcing, such as vortex crossings or lag damper attachments, or when system parameters, like blade pitch, have a value significantly different from that for which the modes were calculated. The modal method, in addition, is essentially a two step process. It is thus labour

intensive and is unsuitable for casting within a formal aeroelastic optimisation routine.

For the more general problem of predicting helicopter manoeuvring flight, the need for substantially more coupled rotor-fuselage modes has been noted. It is estimated that around 50 are required. Furthermore, these modes are more complex than their individual blade counterparts. They have components in each blade and in the fuselage, each of these components has collective, reactionless and two cyclic sub-components (for a four-bladed rotor), and each of these has a real and imaginary part. Computations of vehicle response are thus significantly more involved (and grow with the square of the number of blades) and physical insight is likely to be impaired.

It is pertinent, therefore, to examine the merits of direct integration. In such an approach, the equations of motion are promoted directly in space and time, without recourse to an intermediate modes calculation. The key to the success of the approach is the numerical method employed. The various demands placed on numerical methods for tackling the individual space and time problems have been discussed. These must now be met by one overall technique, which captures all the characteristics of the system yet still progresses the solution with accuracy, efficiency and stability. If achieved, the solution may well progress as fast as a coupled rotor-fuselage modes solution but without the attendant limitations.

The problem appears daunting but, as a result, many methods can be identified as unsuitable and can be eliminated immediately, to the extent that only one will be considered here. We recall from previous Sections the attributes of the methods considered so far:

- 1 One-step radial integration, necessitated by the massive discontinuities in blade data that may occur.
- 2 Minimal (preferably one) step timewise integration, necessitated by the requirement to capture aerodynamic nonlinearities like shock, stall and vortex crossings.
- 3 Numerical stability.

This last requirement precludes methods based on the Z-transform and central difference methods of Section 3. We do not now have available suitable frequency estimates to apply the Z-transform method successfully, and the conditional stability requirement for central differences is almost certain to be violated with a representation that effectively permits unlimited modal frequencies. Since a minimal timewise discretisation is also required, the modified Euler method appears to remain the sole option.

We recall from Section 3.1 that the 'data averaging' interpretation of the Euler method, equation 8, is preferred on stability grounds to the 'force averaging' interpretation of equation 7 when time varying coefficients are present. This leads to a fourth requirement for the space-time method, that of:

- 4 Data averaging.

Consider the general equation of motion for a helicopter in manoeuvring flight expressed in the form

$$\dot{q}' = A\ddot{q} + B\dot{q} + Cq + D \quad .$$

As in Section 2, we assume that some perturbation procedure has already been applied to the equations to remove the nonlinearity. We satisfy this equation at the central point of the interval bounded by $x, x + h$ and $t, t + k$ using

parameters evaluated at the corners of the interval only. These are identified in Fig 2. Thus

$$\frac{1}{h} \left(q^{++} + q^{+0} - q^{0+} - q^{00} \right) = \frac{\bar{A}}{k^2} \left(q^{++} + q^{+0} - q^{+0} - q^{00} - k(\dot{q}^{+0} + \dot{q}^{00}) \right) + \frac{\bar{B}}{k} \left(q^{++} + q^{+0} - q^{+0} - q^{00} \right) + \frac{\bar{C}}{2} \left(q^{++} + q^{+0} + q^{0+} + q^{00} \right) + 2\bar{D} + o(2) \quad (19)$$

$$\dot{q}^{++} = \frac{2}{k} \left(q^{++} - q^{+0} \right) - \dot{q}^{+0} + o(2) \quad , \quad (20)$$

where
$$\bar{A} = \frac{1}{4} \left(A^{++} + A^{+0} + A^{0+} + A^{00} \right) + o(2) \quad , \text{ etc.}$$

Some variations in these equations are admissible. The expression

$$\dot{q}^{++} = \frac{2}{k} \left(q^{++} + q^{00} - q^{+0} - q^{0+} \right) - \dot{q}^{+0} + \dot{q}^{0+} + \dot{q}^{00} + o(3)$$

might appear preferable to equation 20 on accuracy grounds, for instance, but is simply the two-step equivalent of equation 20 found by combining it with the corresponding expression for \dot{q}^{0+} . We saw in Section 3.4 that such combinations can appear to improve accuracy although, in fact, they do not.

Alternative discretisations of the velocity and acceleration are also possible in equation 19. These are retrieved by eliminating parameters via equation 20. The form of equation 19 is considered to be best for numerical computation, however, for it allows the radial integration at any given time to be performed in terms of q only. The velocity can be determined subsequently, after which the integration can move on to the next time step.

Equation 19 can be expressed in the form

$$Rq^{++} = Sq^{0+} + T \quad . \quad (21)$$

A conventional transfer matrix approach applied to equation 21 involves repeated application of the matrix $R^{-1}S$. This matrix has eigenvalues with modulus greater than one, with the result that the transfer matrix formulation is ill-conditioned. The expression for q^{++} remains accurate, but it becomes the difference of larger and larger quantities. Eventually machine rounding errors take over. The problem can be traced to the $1/k^2$ terms in equation 19 and is aggravated as the time step is reduced.

To avoid the problem, we simply consider the blade representation at each time step to be one large matrix equation to be solved by some alternative, more appropriate means. At each radial step, we have a further equation

$$E^{0+} q^{0+} = F^{0+} \quad , \text{ say,} \quad (22)$$

which originates from the prevailing boundary conditions at the end of the blade from which the integration started. Equations 21 and 22 can be combined to give

$$q^{0+} = S^{-1} (Rq^{++} - T) \quad (23)$$

and

$$E^{0+} S^{-1} Rq^{++} = E^{0+} S^{-1} T + F^{0+}$$

or

$$E^{++} q^{++} = F^{++} \quad . \quad (24)$$

At the other end of the blade, the remaining boundary conditions are combined

with equation 24 and solved for q^{++} . Back substitution into equation 23 then yields the complete solution. This is in contrast to the transfer matrix approach in which the unknowns are determined at the end of the blade from which the integration started and a forward substitution is made.

Equations 23 and 24 as they stand remain ill-conditioned, but it is an easy matter to scale equation 24 as we wish and substitute it into equation 23 in a way that removes the ill-conditioning of $S^{-1}R$. The back substitution for q^{0+} can then be made without hindrance. Such a refinement is impossible with the transfer matrix approach, since the counterpart to equation 24 also contains unknowns at the blade end.

In practice, the equations will not be solved precisely as above, using matrix inversion, but by using Gaussian elimination. Gaussian elimination is computationally less efficient than the transfer matrix methods of Section 2, but it is not greatly so and, if the ill-conditioning is removed, some computational inefficiency may be regarded as a small price to pay. The above remarks are now equivalent to saying that some pivoting of the matrix rows is required at each step. Although such pivoting may be done formally, we note that the troublesome terms are inertial and all appear in expressions for the loads. As a first attempt, therefore, we seek numerical integrity by pivoting equation 22 directly with the load expressions in equation 21.

The above method is applied to the simple torsion equation for a cantilever beam

$$\phi'' = a\ddot{\phi} + b(1 - \beta)\dot{\phi} + c(1 - \gamma)\phi + \sin \Omega t + b\beta\dot{\phi} + c\gamma\phi \quad .$$

The parameters β , γ are again used to convert loading from explicit to implicit form, (although direct integration does permit the implicit loading to be kept to a minimum, through estimating the explicit content where necessary). The solution is compared to the 'exact' solution, as found by summing the modal series to a high degree of accuracy.

Results for various parameter combinations are presented in Table 2. In all cases, $\beta = \gamma = 1$, so some iteration is required. It is generally found that one iteration will suffice if the implicit loading depends only on ϕ but that two iterations are necessary if it depends on $\dot{\phi}$. If more iterations appear to be required, the solution is inaccurate and some reduction in step size is called for.

Parameter 'a' is chosen to provide fundamental modes with frequency ω at roughly $\frac{1}{2}$, 1 and 5 times that of the fundamental forcing, $\Omega = 0.3$, whilst parameters 'b, c' are chosen to introduce around 50% critical damping and to increase the fundamental frequency by about 20%. The quoted error is the maximum encountered, expressed as a percentage of the maximum response amplitude to the nearest one percent. Generally, the error is less than this. An 'x' denotes severe inaccuracy, not numerical instability.

With $\Omega = 0.3$, and with the number of radial integration steps $n = 150$ and $\frac{1}{2}^\circ$ azimuth step, there is no perceptible error between the solutions. (Results with $n = 30$ are virtually identical.) Even with 5 radial steps and 20° azimuth step, the error is not large. This bodes well not only for making quick performance assessments, but for embedding the process within an optimisation routine.

If the forcing frequency is increased to $\Omega = 3.6$, we can effectively monitor the blade response to a 12 per rev input. This is a severe vibration prediction problem, but the 150 radial steps and $\frac{1}{2}^\circ$ azimuth step still produce accurate answers. With $a = 100$, however, the predicted response now exhibits a

slight phase error of around $\frac{1}{2}^\circ$.

Clearly, the most sensitive problem occurs at high forcing frequency with the lowest blade fundamental frequency. This is because higher order modes feature most prominently in the response, and the variability of the solution in both space and time is at its greatest. The most severe example investigated, however, is likely far to exceed that encountered in practice. When $a = 100$, it is the 12th mode that has frequency closest to 3.6. This arises because the fundamental mode has frequency around 0.5Ω . Such a mode is unlikely to occur in the torsional sense. It might well be found in the lag sense, but the lag modes then are more widely spaced in frequency. Either way, an influential 12th mode is unlikely in practice.

Success with a particular example continues to be no guarantee of success in the general case. Nevertheless, it is apparent that the method described performs well for a comprehensive range of practical parameter values. The pivoting scheme indeed cures the ill-conditioning problem, and the solution is progressed with accuracy, robustness, efficiency and stability.

It is difficult to imagine how any other difference scheme can combine more successfully the four attributes identified at the start of this Section. The possibility remains, however, that finite elements can do so. With finite elements there are two options, either the element varies with time or it does not. If it varies with time, a new element representation must be derived at each radial and timewise step, a task that is likely to be computationally inefficient. If the finite element does not vary with time, all the timewise variance must be incorporated within the forcing function and the possibility of numerical instability will be exacerbated. Either way, direct integration seems preferable.

5 CONCLUDING DISCUSSION

The satisfactory integration of the equations of motion of a helicopter in manoeuvring flight represents a challenging computational problem. A number of options exist which, once exercised, determine the pattern of subsequent dynamic, aerodynamic and computational development. It is important that the right choice be made at the outset, otherwise much precious effort may be wasted pursuing methods that are either inefficient or, worse still, not up to the task.

The equations of motion are of fourth degree in blade flap and lag, but of second degree in torsion and tension. They behave much like second degree hyperbolic equations, however, and care must be taken to ensure that the solution is progressed with numerical stability. The degrees of freedom can be highly coupled, and these couplings are perhaps best identified by introducing further dependent variables which reduce the equations to first degree in the spatial coordinate. Twelve coupled partial differential equations result.

Potential troublesome features of the equations are their nonlinearity, their periodic coefficients, local physical instability, massive radial discontinuities in blade data, nonlinear aerodynamic forcing at unknown locations, implicitly defined loading, and coupled rotating and non-rotating systems. The procedure used to solve the equations must be accurate, efficient, robust and stable.

Methods are described for performing calculations for the steady hovering and vertical flight states and a preferred method is identified. Methods for promoting a timewise solution using a modal discretisation in the spacewise sense are also discussed, and Z-transform equations are developed which offer a substantial improvement in accuracy over existing Z-transform methods.

The considerable increase in complexity of a modal formulation for manoeuvring flight compared to that required for level flight is considered to detract from the advantages of modal methods for the manoeuvring case. The direct integration of the equations of motion thus holds increased attraction. A numerical method is presented which offers perhaps the best chance of overcoming all the above-mentioned problems, by combining data averaging with an implicit integration scheme and applying it over the simplest of space-time intervals.

The use of numerical examples has deliberately been kept to a minimum in order to avoid drawing conclusions that are case specific. What examples there are, although simple, give a clear indication of the power of the methods described. In particular, the direct integration approach for the space-time equations appears to be worthy of serious consideration for further computational development. It appears to be well-suited to simple performance estimation work and is also, therefore, ideal for incorporation into a formal aeroelastic optimisation routine.

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Table 1

Z-TRANSFORM INTEGRATION OF A SINGLE MODE EQUATION

Ω	ω	ν	Az step deg	Error % $\beta, \gamma=0$	Error % $\beta, \gamma=1/2$		
1	0.5	0.1	5.0	0	0		
	1.0	0.5		0	0		
	11.5	0.1		0	0		
	12	0.5	0.1	20.0	0	0	
		1.0	0.5		0	0	
		11.5	0.1		0	u	
		12	0.5	0.1	60.0	1	1
			1.0	0.5		0	0
			11.5	0.1		2	u
12	0.5	0.1	5.0	0	0		
	1.0	0.5		0	0		
	11.5	0.1		1	2		
	0.5	0.1		20.0	x	x	
1.0	0.5	x	x				
11.5	0.1	x	u				

Table 2

DIRECT INTEGRATION OF BLADE TORSION EQUATION

Ω	a	b	c	ω	Az step deg	Error % n=150	Error % n=5
0.3	1	0	0	1.57	0.5	0	0
	1	1	1	1.86	0.5	0	0
	1	0	0	1.57	20	0	0
	1	1	1	1.86	20	2	2
	25	10	0	0.31	0.5	0	2
	25	10	0	0.31	20	1	1
	100	0	0	0.16	0.5	0	4
	100	10	1	0.19	0.5	0	4
	100	0	0	0.16	20	5	3
	100	10	1	0.19	20	3	1
3.6	1	1	1	1.86	0.5	0	5
	25	10	0	0.31	0.5	1	x
	100	10	0	0.16	0.5	2	x

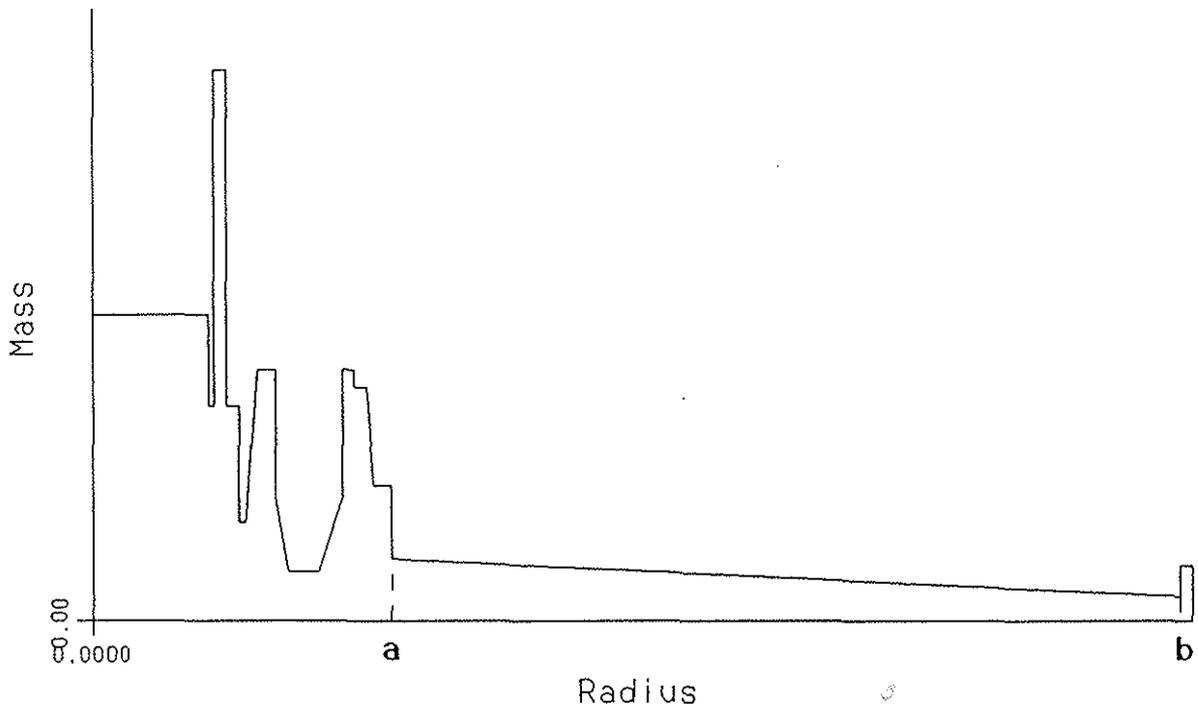


Fig 1 Typical rotor blade mass distribution

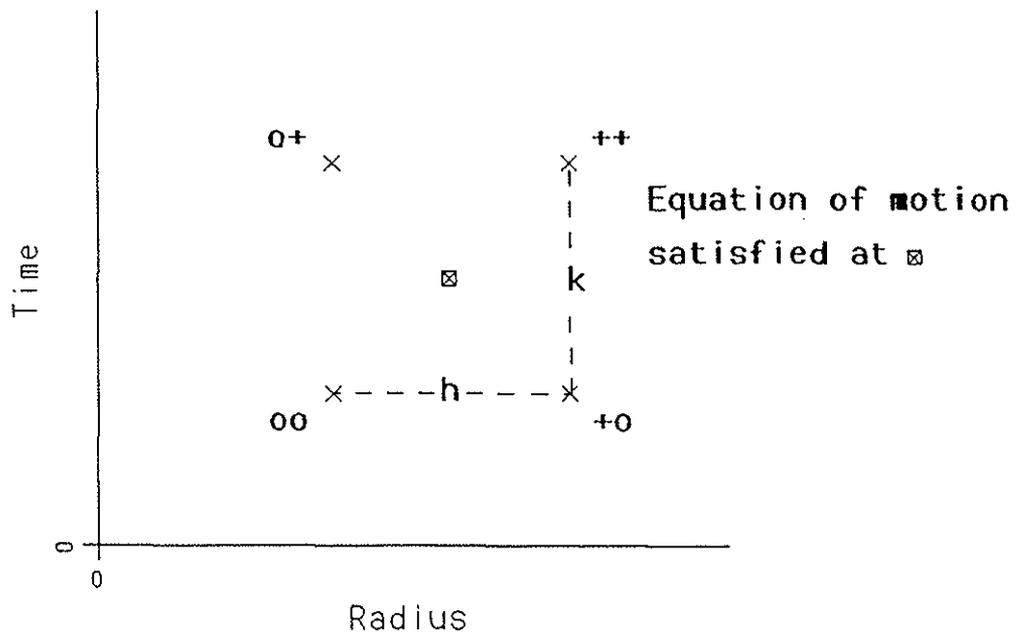


Fig 2 Solution interval for space/time integration