Tiltrotor Whirl-Flutter Stability Investigation using Lyapunov Characteristic Exponents and Multibody Dynamics

Gianni Cassoni[†], Alessandro Cocco[†,*], Aykut Tamer[⋆], Andrea Zanoni[†], Pierangelo Masarati[†]
[†]Politecnico di Milano, Milano - Italy
[⋆]Imperial College London, London - UK
[*] alessandro.cocco@polimi.it

Abstract
This work discusses the use of Lyapunov Characteristic Exponents to assess the stability of nonlinear, time-dependent mechanical systems. Specific attention is dedicated to methods capable of estimating the largest exponent without requiring the Jacobian matrix of the problem, which can be applied to time histories resulting from existing multibody solvers. Tiltrotor whirl-flutter stability is analyzed. With respect to the available literature, the proposed method does not require the system to be strictly periodic, no linearization is required about a reference steady solution, and characteristic nonlinear aspects of stationary solutions like limit cycle oscillations are correctly identified and pointed out. A limitation lies in the ability to correctly identify the stability but no information is inferred about the related characteristic frequencies/periods, if any.

Introduction

Stability assessment is a fundamental aspect of the analysis and design of dynamical systems. The foundations of modern stability theory lie in Aleksandr M. Lyapunov's work [1].

Although for nonlinear problems of the general form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

stability is a local property of a specific solution, $\mathbf{x}(t)$, resulting from a specific set of initial conditions, $\mathbf{x}(t_0) = \mathbf{x}_0$, for Linear, Time-Invariant (LTI) problems, like those resulting from linearization about a steady reference condition in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{2}$$

it becomes a characteristic of the entire system. It can be evaluated from the inspection of the real part of the eigenvalues of matrix $\mathbf{A}$.

In many applications associated with rotorcraft dynamics, and specifically in helicopter rotor aeromechanics, problems often need to be formulated as time-periodic, although often still linear, through linearization about a periodic orbit, resulting in Linear, Time-Periodic (LTP) problems. In this case, stability assessment may exploit the periodicity of the motion through the well-known Floquet-Lyapunov approach. Asymptotic stability is associated with contraction, over a period T, of the so-called "monodromy matrix", which corresponds to the State Transition Matrix (STM) of the problem over one period. This approach is relatively popular in the rotorcraft aeromechanics community [2–5].

In general, however, when problems are non-linear and subjected to non-(strictly) periodic time dependence, as may occur in many transient-related problems, the ability to evaluate the stability of reference trajectories can be extremely useful. The use of Lyapunov Characteristic Exponents (LCE) or, in short, Lyapunov Exponents (LE), has been recently proposed in the field of rotorcraft aeromechanics [6–8], also with a focus on their sensitivity study [9–12], and other aerospace-related applications [10].

Lyapunov Characteristic Exponents

The LCEs indicate the rate of expansion or contraction of perturbations of a generic solution of the nonlinear differential problem of Eq. (1) along independent directions in the state space. As such, they describe the stability of the reference solution with respect to such directions.

Consider a solution $\mathbf{x}(t)$ of Eq. (1) for $t \geq t_0$ (some authors refer to it as the 'fiducial trajectory'), and a solution ${}_i\mathbf{x}(t)$ of the problem

$${}_i\dot{\mathbf{x}} = \mathbf{f}_{/\mathbf{x}}\big|_{\mathbf{x}(t),t}\ {}_i\mathbf{x}, \qquad {}_i\mathbf{x}(t_0) = {}_i\mathbf{x}_0 \tag{3}$$

for a perturbation ${}_i\mathbf{x}_0$ of arbitrary magnitude and direction. LCEs are defined as

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \|{}_i\mathbf{x}(t)\|\,. \tag{4}$$

Each λ_i is calculated from one of n linearly independent ${}_i\mathbf{x}_0$, the equivalent of the principal directions of a LTI problem. Since Eq. (3) is linear time-dependent, its solution

Presented at the 48th European Rotorcraft Forum, Winterthur, Switzerland, 6-8 September, 2022

can be expressed through the State Transition Matrix (STM) $\mathbf{Y}(t,t_0)$,

(5) $${}_i\mathbf{x} = \mathbf{Y}(t,t_0)\,{}_i\mathbf{x}_0$$

The STM $\mathbf{Y}(t,t_0)$ is the solution of $\dot{\mathbf{Y}} = \mathbf{f}_{/\mathbf{x}}\big|_{\mathbf{x}(t),t}\mathbf{Y}$, with $\mathbf{Y}(t_0,t_0) = \mathbf{I}$, namely the matrix that represents the evolution of the problem's state from time t_0 to t. Then Eq. (4) is equivalent to

(6) $$\lambda_i = \lim_{t\to\infty}\frac{1}{t}\mathrm{Re}\left(\log\left(\mathrm{eig}\left(\mathbf{Y}(t,t_0)\right)\right)\right).$$

When all LCEs are negative, the solution is exponentially stable. When at least one LCE is positive, the solution is unstable, or leads to a chaotic attractor. When the largest LCE is zero, or the largest LCEs are zero, a limit cycle oscillation (LCO) is expected; i.e., there exists one direction, or multiple independent directions, in the state space along which the solution neither expands nor contracts. In case of multiple largest LCEs equal to zero, a higher order periodic or quasi-periodic attractor exists, e.g. a torus.

Note the analogy with the LTI case, since $\mathbf{Y}(t,t_0) \overset{\text{LTI}}{\equiv} \mathrm{e}^{\mathbf{A}(t-t_0)}$ and thus $\lambda_i \overset{\text{LTI}}{=} \mathrm{Re}(\mathrm{eig}(\mathbf{A}))$, and with the LTP one, in which

(7) $$\lambda_i \overset{\text{LTP}}{=} \frac{1}{T}\mathrm{Re}\left(\log\left(\mathrm{eig}\left(\mathbf{Y}(t_0+T,t_0)\right)\right)\right).$$

In this sense, one may consider the LCEs as sort of the eigenvalues of matrix $\mathbf{f}_{/\mathbf{x}}$, averaged over time.

LCEs are often called the 'spectrum' of the associated problem, much like the spectrum of LTI problems is represented by the eigenvalues of matrix $\mathbf{A} = \mathbf{f}_{/\mathbf{x}}$.

In most cases, the definition of Eq. (6) cannot be used in practice, because usually at least some of the elements of the STM contract to zero, in case of asymptotic stability of the solution, or expand to infinity, in case of instability, resulting in either under- or overflowing, or a mix of both. As proposed in the work of Benettin *et al.* [13], to estimate LCEs in practice one needs to exploit the re-orthogonalization of the local directions of evolution of the solution. Alternative approaches based on well known orthogonal decompositions (the Singular Value Decomposition (SVD) and the QR factorization, respectively) have been proposed [13–15].

Jacobian-Less Methods: Max LCE from Time Series

The estimation of LCEs from problems of interest in the rotorcraft field using those methods has been recently discussed in [6,8]. However, those methods require the knowledge of the Jacobian matrix of the problem, and the ability to integrate the associated linear, time dependent problem. The largest LCE, or Maximum LCE (MLCE) can also be estimated directly from a time series. This makes the approach rather interesting because it can be used in conjunction with experimental data, or with the results of numerical simulations performed using complex multibody models. Although the full spectrum cannot be obtained, estimation of the MLCE allows to identify the LCE that belongs to the least damped principal direction of the system, which is the most critical stability indicator. Among the algorithms proposed in the literature (see for example [16]), in this work the one proposed by Rosenstein et al. [17], described below, is used.

The trajectory matrix, $\mathbf{X}$, is constructed from an N-point time series x_i, $i = 1,\dots,N$, using the time delay method. Each row of matrix $\mathbf{X}$ is a phase-space vector, namely

(8) $$\mathbf{X} = \left[\begin{array}{cccc} \mathbf{X}_1 & \mathbf{X}_2 & \dots & \mathbf{X}_m \end{array}\right]$$

with

(9) $$\mathbf{X}_k = \left[\begin{array}{cccc} x_{1+(k-1)J} & x_{2+(k-1)J} & \dots & x_{M+(k-1)J} \end{array}\right]^T$$

with $k = 1,\dots,m$. Thus, $\mathbf{X} \in \mathbb{R}^{M\times m}$, with m, M, J, and N related by

(10) $$M = N-(m-1)J$$

where m is the embedding dimension, N the length of the time series, and J represents the so-called reconstruction delay. The embedding dimension is usually estimated in accordance with Takens' theorem, i.e., $m > 2n$.

After constructing the trajectory matrix, the algorithm locates the nearest neighbor, $\mathbf{X}_{\hat{j}}$, of each point on the trajectory. It is found by searching for the point that minimizes the distance from each particular reference point, $\mathbf{X}_j$. The distance is expressed as

(11) $$d_j(0) = \min_{\mathbf{X}_{\hat{j}}}\left\|\mathbf{X}_j - \mathbf{X}_{\hat{j}}\right\|$$

where $d_j(0)$ is the initial distance from the jth point to its nearest neighbor, and $\|\cdot\|$ denotes the Euclidean norm.

An additional constraint is that nearest neighbors have a temporal separation greater than the mean period ($\bar{T}$, the reciprocal of the mean frequency of the power spectrum, although it can be expected that any comparable estimate, e.g., using the median frequency of the magnitude spectrum, yields equivalent results) of the time series,

(12) $$\left|j-\hat{j}\right| > \bar{T}$$

Thanks to this, each pair of neighbors can be considered as nearby initial conditions for different trajectories.

The largest Lyapunov exponent is then estimated as the mean rate of separation of the nearest neighbors. The jth pair of nearest neighbors diverge approximately at a rate given by the largest Lyapunov exponent:

(13) $$d_j(i) \approx C_j\mathrm{e}^{\lambda_1(i\Delta t)}$$

where C_j is the initial separation. By taking the logarithm of both sides

(14) $$\log d_j(i) \approx \log C_j + \lambda_1(i\Delta t)$$

which represents a set of approximately parallel lines, for $j = 1, \dots, M$, each with a slope roughly proportional to λ_1. The largest Lyapunov exponent is calculated using a least-squares fit to the “average” line defined by

$$y(i) = \frac{1}{\Delta t} \langle \log d_j(i) \rangle \tag{15}$$

where $\langle \cdot \rangle$ denotes the average over all values of j.

Numerical Results

The use of the Jacobian-less method by Rosenstein et al. was proposed in [18] to address the rather important rotorcraft-related problem of ground resonance with relatively simple models, for validation purposes, including a model formulated using MBDyn[1], a free general-purpose multibody solver [19]. In such case, this method is of particular interest, since it does not require the capability of the solver to expose the Jacobian matrix of the problem, and can thus be applied to results obtained from any solver.

The problem analyzed in the following is the aeroelastic simulation of a tiltrotor semi-span model (Fig. 1) in proximity of whirl-flutter conditions, using a rather sophisticated aeroservoelastic model [20].

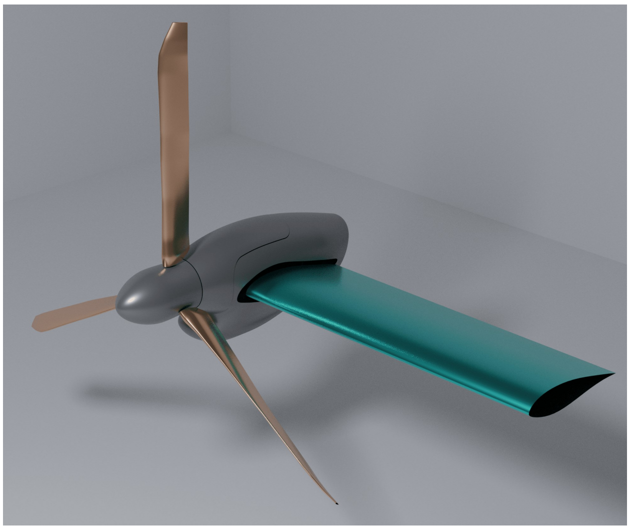


Figure 1: Tiltrotor semispan model.

The entire model can be divided into two main sub-components: the wing-pylon assembly and the proprotor. The wing-pylon model has been developed using a geometrically exact, composite ready beam finite element model [21, 22] to reproduce the fundamental frequencies and mode shapes of an equivalent finite element stick model, which was tuned to match the full-scale aircraft dynamics at the rotor. The proprotor is a three-bladed stiff-in-plane rotor with gimballed hub (Fig. 2). It consists of the pitch control chain, three blades, and the yoke, all modeled using the previously mentioned beam elements.

[1] https://www.mbdyn.org/

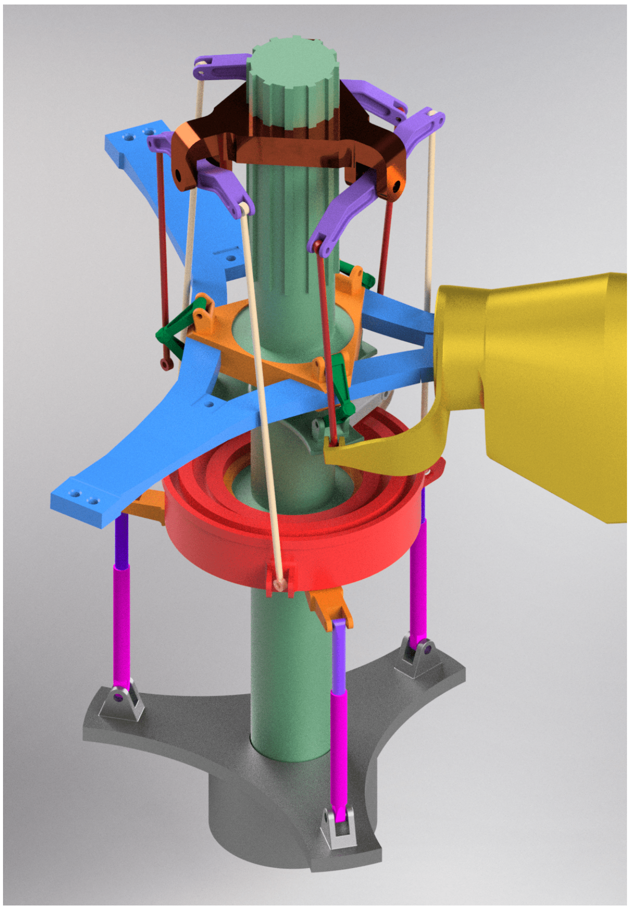
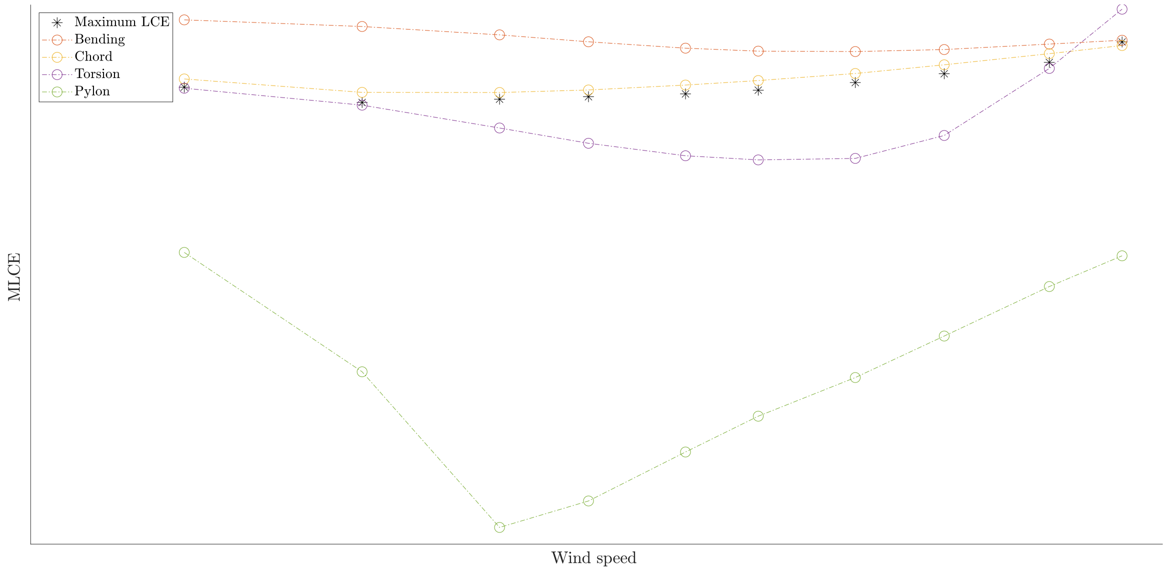


Figure 2: Proprotor multibody model.

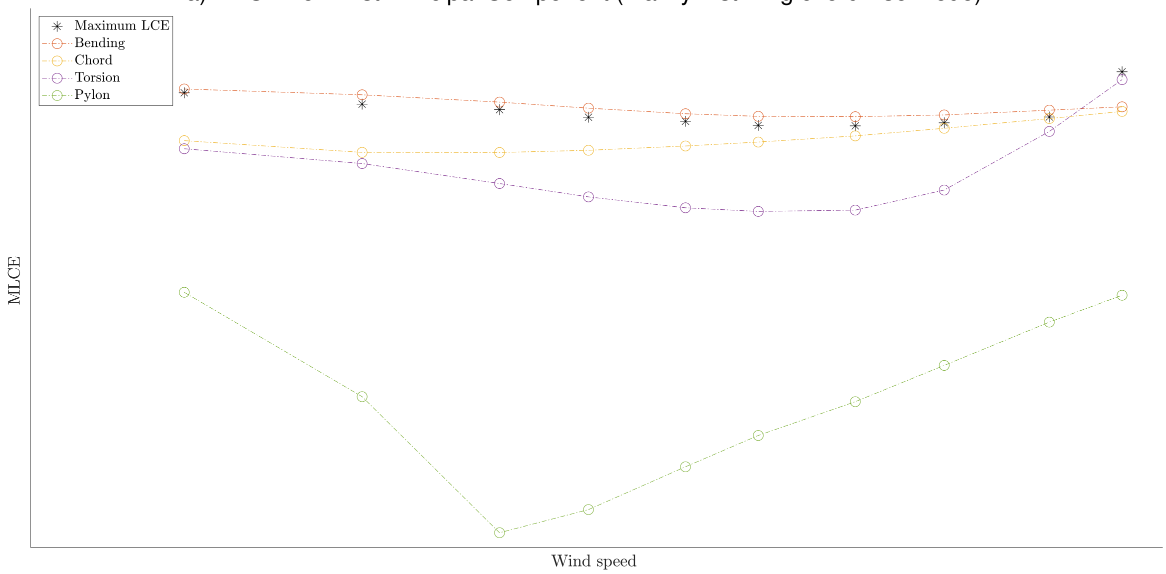
In the original formulation, the algorithm proposed by Rosenstein et al. is applied to a single time series, namely a scalar signal. In complex multivariable problems, the choice of a suitable signal can be critical, along with some parameters of the method, related to the duration of the signal and other factors discussed in the original paper. To overcome this critical aspect, a Proper Orthogonal Decomposition of the multiple signals resulting from the multibody dynamics simulation is performed, and the transformed signals corresponding to the largest singular values are considered. Figure 3 compares the results obtained with the proposed analysis to that resulting from the Periodic Operational Modal Analysis (POMA) proposed in [23–25]. The MLCEs estimated from the first three principal components (PC) are considered, starting from the top.

The first PC (top plot) shows results compatible with those of the first wing chordwise mode. This occurs since the excitation that caused the transient was dominated by a collective pitch perturbation, which directly excites the wing chordwise mode.

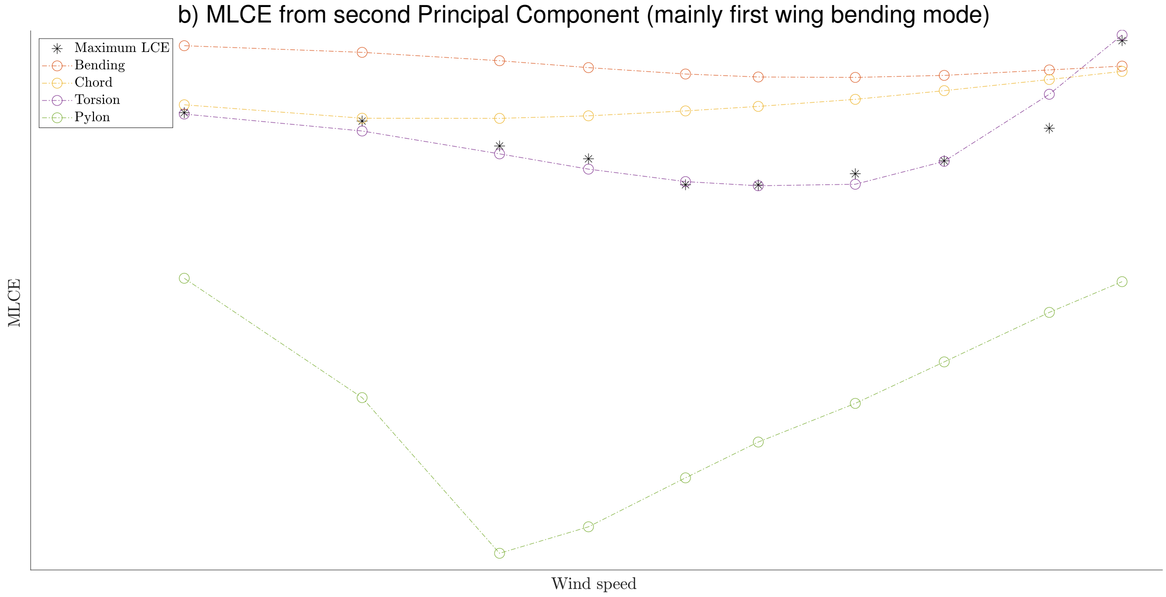
The second PC (mid plot) shows results compatible with those of the first wing bending mode, the most lightly damped one, except for the analysis at the highest speed, which is dominated by whirl-flutter, with a substantial participation of wing torsion.



a) MLCE from first Principal Component (mainly first wing chordwise mode)



b) MLCE from second Principal Component (mainly first wing bending mode)



c) MLCE from third Principal Component (mainly first wing torsion mode)

Figure 3: MLCE compared with real part of eigenvalues from POMA.

Indeed, the wing torsional mode dominates the third PC (bottom plot), which shows greater damping that the other modes in the speed range below whirl-flutter.

Conclusions

This paper discussed the use of estimated Lyapunov Characteristic Exponents to evaluate the stability of a complex mechanical system, namely a tiltrotor semi-span model close to whirl-flutter conditions. The problem was formulated without *a priori* linearization or coordinate transformation to reduce the azimuthal dependence of the equations. Nonlinear phenomena, like limit-cycle oscillations, can be detected. By resorting to Jacobian-less methods, the Maximum LCE, i.e., the most critical one regarding stability, can be estimated from time series computed using general-purpose multibody formulations, without the need to modify existing solvers.

Copyright Statement

References

[1] Aleksandr Mikhailovich Lyapunov. *The General Problem of the Stability of Motion*. Taylor & Francis, London - Washington, DC, 1992. Translated and edited by A. T. Fuller.

[2] David A. Peters and Kurt H. Hohenemser. Application of the Floquet transition matrix to problems of lifting rotor stability. *Journal of the American Helicopter Society*, 16(2):25–33, 1971. doi:10.4050/JAHS.16.25.

[3] James C. Biggers. Some approximations to the flapping stability of helicopter rotors. *Journal of the American Helicopter Society*, 19(4):24–33, 1974. doi:10.4050/JAHS.19.24.

[4] P. Friedmann and J. Silverthorn. Aeroelastic stability of periodic systems with application to rotor blade flutter. In *AIAA/ASME/SAE 15th Structures, Structural Dynamics and Materials Conference*, Las Vegas, Nevada, April 17–19 1974.

[5] C. Eugene Hammond. An application of Floquet theory to prediction of mechanical instability. *Journal of the American Helicopter Society*, 19(4):14–23, 1974. doi:10.4050/JAHS.19.14.

[6] Aykut Tamer and Pierangelo Masarati. Helicopter rotor aeroelastic stability evaluation using Lyapunov exponents. In *40th European Rotorcraft Forum*, Southampton, UK, September 2–5 2014.

[7] Aykut Tamer and Pierangelo Masarati. Do we really need to study rotorcraft as linear periodic systems? In *AHS 71st Annual Forum*, Virginia Beach, VA, USA, May 5–7 2015.

[8] Aykut Tamer and Pierangelo Masarati. Stability of nonlinear, time-dependent rotorcraft systems using Lyapunov characteristic exponents. *Journal of the American Helicopter Society*, 61(2), April 2016. doi:10.4050/JAHS.61.022003.

[9] Pierangelo Masarati and Aykut Tamer. Sensitivity of trajectory stability estimated by Lyapunov characteristic exponents. *Aerospace Science and Technology*, 47:501–510, November 2015. doi:10.1016/j.ast.2015.10.015.

[10] Aykut Tamer and Pierangelo Masarati. Aeroelastic stability estimation of control surfaces with freeplay nonlinearity. *l'Aerotecnica Missili e Spazio*, 96(3):154–164, 2017. doi:10.1007/BF03404750.

[11] Aykut Tamer and Pierangelo Masarati. Sensitivity of Lyapunov exponents in design optimization of nonlinear dampers. *J. of Computational and Nonlinear Dynamics*, 14(2):021002, February 2019. doi:10.1115/1.4041827.

[12] Aykut Tamer and Pierangelo Masarati. Generalized quantitative stability analysis of time-dependent comprehensive rotorcraft systems. *Aerospace*, 9(1), 2022.

[13] Giancarlo Benettin, Luigi Galgani, Antonio Giorgilli, and Jean-Marie Strelcyn. Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. part 1: Theory. *Meccanica*, 15(1):9–20, March 1980. doi:10.1007/BF02128236.

[14] Karlheinz Geist, Ulrich Parlitz, and Werner Lauterborn. Comparison of different methods for computing Lyapunov exponents. *Progress of Theoretical Physics*, 83(5):875–893, May 1990. doi:10.1143/PTP.83.875.

[15] Luca Dieci and Erik S. Van Vleck. Lyapunov spectral intervals: Theory and computation. *SIAM Journal on Numerical Analysis*, 40(2):516–542, 2002. doi:10.1137/S0036142901392304.

[16] Ch. Skokos. *The Lyapunov Characteristic Exponents and Their Computation*, pages 63–135. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010. doi:10.1007/978-3-642-04458-8_2.

[17] Michael T. Rosenstein, James J. Collins, and Carlo J. De Luca. A practical method for calculating largest Lyapunov exponents from small data sets. *Physica D: Nonlinear Phenomena*, 65(1–2):117–134, May 1993. doi:10.1016/0167-2789(93)90009-P.

[18] Gianni Cassoni, Andrea Zanoni, Aykut Tamer, and P. Masarati. Stability of rotorcraft ground resonance by estimating Lyapunov characteristic exponents from multibody dynamics. In *ASME IDETC/CIE 2022*, St. Louis, Missouri, USA, August 14–17 2022. doi:10.1115/DETC2022-88995.

[19] Pierangelo Masarati, Marco Morandini, and Paolo Mantegazza. An efficient formulation for general-purpose multibody/multiphysics analysis. *J. of Computational and Nonlinear Dynamics*, 9(4):041001, 2014. doi:10.1115/1.4025628.

[20] Alessandro Cocco, Alberto Savino, and Pierangelo Masarati. Flexible multibody model of a complete tiltrotor for aeroservoelastic analysis. In *ASME IDETC/CIE 2022*, St. Louis, Missouri, USA, August 14–17 2022. doi:10.1115/DETC2022-88973.

[21] Gian Luca Ghiringhelli, Pierangelo Masarati, and Paolo Mantegazza. A multi-body implementation of finite volume C^0 beams. *AIAA Journal*, 38(1):131–138, January 2000. doi:10.2514/2.933.

[22] Olivier A. Bauchau, Peter Betsch, Alberto Cardona, Johannes Gerstmayr, Ben Jonker, Pierangelo Masarati, and Valentin Sonneville. Validation of flexible multibody dynamics beam formulations using benchmark problems. *Multibody System Dynamics*, 37(1):29–48, 2016. doi:10.1007/s11044-016-9514-y.

[23] Alessandro Cocco, Pierangelo Masarati, Stefan van 't Hoff, and Bart Timmerman. Tiltrotor whirl flutter analysis in support of NGCTR aeroelastic wind tunnel model design. In *47th European Rotorcraft Forum*, Glasgow, UK (virtual), September 7–10 2021.

[24] Alessandro Cocco, Alberto Savino, Andrea Zanoni, and Pierangelo Masarati. Comprehensive simulation of a complete tiltrotor with pilot-in-the-loop for whirl-flutter stability analysis. In *48th European Rotorcraft Forum*, Winterthur, Switzerland, September 6–8 2022.

[25] Alessandro Cocco, Pierangelo Masarati, Stefan van 't Hoff, and Bart Timmerman. Numerical whirl-flutter analysis of a tiltrotor semi-span wind tunnel model. *CEAS Aeronautical Journal*, available online September 2022. doi:10.1007/s13272-022-00605-2.