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USING A MIXED FORMULATION

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ABSTRACT

A mixed formulation for calculating static equilibrium and stability eigenvalues of nonuniform rotor blades in hover is presented. The static equilibrium equations are nonlinear and are solved by an accurate and efficient collocation method. The linearized perturbation equations are solved by a one-step, second-order integration scheme. The numerical results correlate very well with published results from a nearly identical stability analysis based on a displacement formulation. Slight differences in the results are traced to terms in the equations that relate moments to derivatives of rotations. With the present ordering scheme, in which terms of the order of squares of rotations are neglected with respect to unity, it is not possible to achieve completely equivalent models based on mixed and displacement formulations. A study of the one-step methods reveals that a second-order Taylor expansion is necessary to achieve good convergence for nonuniform rotating blades. Numerical results for a hypothetical nonuniform blade, including the nonlinear static equilibrium solution, were obtained with no more effort or computer time than that required for a uniform blade with the present analysis.

1. Introduction

It has been found that nonlinearities in rotor-blade equations affect blade stability [1-4] — especially stability of coupled flap, lead-lag, and torsion degrees of freedom [1]. In these references it was found that essential nonlinear effects could be retained by perturbing the nonlinear equations of motion about the static equilibrium condition and solving for the eigenvalues of the linearized perturbation equations. Coefficients of the linearized perturbation equations then depend on the solution of the nonlinear static equilibrium equations.

This paper presents a method for stability analysis of nonuniform rotating blades with aerodynamic loading that utilizes the solution for the nonlinear static equilibrium equations in the linearized eigenvalue problem. Methods described in the literature have been limited to solution of various restricted versions of this problem [4-9]. These include a modal approach [4], an integrating matrix method [5], a Myklestad method [6], a Ritz finite-element method [7], a modal approach based on a mixed variational principle [8], a transmission matrix method [9], and a Galerkin finite-element method [10].

In section 2 of this paper, as in [1], the governing equations of motion are written as two sets of equations. One is a set of nonlinear ordinary differential equations that governs the static equilibrium condition. The other is a set of linear ordinary differential equations with an unknown eigenvalue that governs the dynamic behavior of small perturbation motions about equilibrium. These differential equations are written in a mixed, spatial-derivative form, unlike the displacement formulation of [1]. Several differences between mixed and displacement formulations are then discussed in the context of the present analytical task.

Then, in section 3, solution of the equations is discussed. A different technique is used to solve for the nonlinear static equilibrium from that used to solve for the linear stability eigenvalues. The nonlinear static equilibrium is solved by a collocation method for a mixed-order system of boundary value equations [11]. The software for this method is available in a program called COLSYS [12]. For the linearized perturbation equations, several methods of solving for the eigenvalues are discussed. These include COLSYS [12], a generalized eigenvalue approach [13], a Block-Stodola technique [14], and one-step numerical integration techniques of various orders. For the present formulation the one-step methods, which appear to be the most promising, are used.

Finally, in section 4, numerical results are presented for comparison with published data and for study of convergence properties associated with the one-step methods. Numerical results are also presented for stability eigenvalues of a hypothetical nonuniform blade.

2. The Governing Equations

The governing equations used herein were essentially derived in [15], in which the Houbolt and Brooks [16] equations are extended into the range of geometric nonlinear behavior. The integro-partial differential equations are transformed into ordinary differential equations by first expressing the vector of unknowns, z, as

$$z(x,t) = \bar{z}(x) + \tilde{z}(x,t) \tag{1}$$

Here $\bar{z}$ is the static equilibrium part and $\tilde{z}$ is a small, perturbed part. The equations are linearized in $\tilde{z}$ and converted to an eigenvalue problem with the transformation

$$\tilde{z}(x,t) = \hat{z}(x)e^{\lambda t} \tag{2}$$

where

$$\lambda = \sigma + i\omega \tag{3}$$

The static equilibrium equations (with the sign conventions and nomenclature defined in [15] are given in appendix A. From Equations (A1) through (A10), and (A17), the state vector, $\bar{z}$, for the nonlinear, static equilibrium behavior is written as

$$\bar{z} = \left\lfloor \bar{V}_y \;\; \bar{M}_z \;\; \bar{v} \;\; \bar{\zeta} \;\; \bar{V}_z \;\; \bar{M}_y \;\; \bar{w} \;\; \bar{\beta} \;\; \bar{M}_x \;\; \bar{\phi} \;\; \bar{\psi} \right\rfloor^T \tag{4}$$

and Equations (A13), (A14), and (A15) define the aerodynamic loading for the hovering flight condition.

The linearized equations governing small perturbation motions from the static equilibrium state defined by the solution for the equations in appendix A are given in appendix B. The state variables, $\hat{z}$, for the perturbed state are defined in Equations (B1) through (B10), and (B14) through (B19). The vector $\hat{z}$ is

$$\hat{z} = \left\lfloor \hat{V}_y \ \hat{M}_z \ \hat{v} \ \hat{\zeta} \ \hat{V}_z \ \hat{M}_y \ \hat{w} \ \hat{\beta} \ \hat{M}_x \ \hat{\phi} \ \hat{\psi} \ \hat{v}^* \ \hat{w}^* \ \hat{\phi}^* \ \hat{V}_x \ \hat{u} \right\rfloor^T \quad . \tag{5}$$

and the equations may be represented in matrix form as

$$\hat{z}' = (A + \lambda B)\hat{z} \tag{6}$$

In the derivation of the equations in appendices A and B, terms of $O(\varepsilon^2)$ have been dropped with respect to unity where ε is the order of bending and torsion rotations, ζ, β, and ϕ. The ordering scheme outlined in [15] has been followed as closely as possible. As in [15], an exception has been made to include $O(\varepsilon^2)$ terms in the torsion equations that are uncoupled from the bending equations but are known to influence the uncoupled torsion frequency. It is not possible, however, to be <u>completely</u> consistent in ordering schemes in either a mixed formulation for the differential equations or in a displacement formulation. Furthermore, as will be shown in the next section, it is not always possible to reach complete agreement between the mixed and displacement formulations in the decision of which terms to retain, even when trying to follow the same guidelines on neglecting higher-order terms.

An exception to the guidelines in [15] on the ordering of terms has been to retain complete trigonometric expressions involving the variable ϕ. In [15], $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$ were approximated by the appropriate expansions in terms of $\cos\theta$, $\sin\theta$, and ϕ, neglecting terms of second and higher powers of ϕ. In this paper it has been convenient to retain the complete trigonometric expression in all equations except the equations relating rotation derivatives and moments which are presented below. No significant differences in the results of [1] and this paper will occur as a consequence of using this convenient trigonometric form.

When the equations are formulated as a system of first-order differential equations (i.e., a mixed formulation) several attractive features become apparent. These include the simplicity of the form of the governing differential equations, the absence of the derivatives of the elastic characteristics, the simplicity of applying numerical integration techniques, and of handling boundary and interface conditions. The mixed formulation was particularly convenient in this application since the integro-differential terms could easily be rewritten as differential equations [note Eq. (A17), (B14), and (B19)].

Concerning retention of specific terms, the main difference between a mixed-order formulation and displacement formulations involves the equations relating derivatives of rotations to moments which are

$$\begin{Bmatrix} \zeta' \\ \beta' \end{Bmatrix} = \begin{bmatrix} \underline{a_1\zeta - a_2\beta} & -a_1 + \underline{\underline{a_4\phi}} & a_2 - \underline{\underline{2a_1\phi}} \\ \underline{a_3\zeta - a_1\beta} & -a_3 - \underline{\underline{2a_1\phi}} & a_1 - \underline{\underline{a_4\phi}} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} + (e_A V_x / Z_0) \begin{Bmatrix} \cos(\theta + \phi) \\ \sin(\theta + \phi) \end{Bmatrix} \tag{7}$$

These equations lead to Equations (A4) and (A8) in appendix A and Equations (B4) and (B8) in appendix B. Reference [1], which is a displacement formulation neglects the M_x contribution (i.e., the single-underlined terms in Equation 7) in a consistent manner on the basis that torsional moments are at least one order of magnitude less than the bending moments. References [1] and [15] use a quasi-coordinate as the torsion variable as discussed in [17]. For that formulation, the torsion moments when used in the bending equations are written as integrals of applied and inertial loads and are thus of $O(\varepsilon^2)$. In the bending equations they are multiplied by quantities of $O(\varepsilon)$ and are thus negligible. References [18] and [19], which are also displacement formulations, use Lagrangian torsion variables and in a consistent manner retain the single-underlined terms. There, the torsion moments are written in terms of the first derivatives of ϕ and are $O(\varepsilon)$ quantities. When they are multiplied by terms of $O(\varepsilon)$ in the bending equations, they are not formally negligible and thus are retained. The single-underlined terms are also retained in the mixed formulation presented here, although they could be neglected on the basis of the arguments in [1]. Results are presented in a later section which document the effect of the single-underlined terms.

Another more important difference between mixed and displacement formulations is illustrated by the double-underlined terms of Equation (7). In [1], [15], [18], and [19] the double-underlined terms were retained in deriving the fourth-order governing equations for the lead-lag and flap-bending behavior, but they were neglected in deriving the second-order torsion equation. These terms are consistently retained or neglected in the above references on the basis of essentially the same ordering scheme, and the retained terms result in a symmetric elastic stiffness matrix. For the present mixed formulation, however, there is no reason, a priori, to neglect these terms. We may expect these terms to have some observable effect, especially when there is significant static equilibrium deformation, such as when θ is large. This effect will be illustrated in the next section.

Previously, we discussed some advantages of the mixed formulation. There are also some drawbacks to use of mixed formulations. The matrix operator for the structural terms in the equations no longer appears in symmetric form, and the number of variables in the state vector is several times that of displacement formulations. For some applications, however, the convergence is more rapid for mixed formulations than for displacement formulations [20 and 21], thereby offsetting somewhat the latter disadvantage. Further, force resultants are obtained with the same degree of accuracy as the displacements in the mixed formulation. The disadvantage of the nonsymmetric form of the structural operater can be somewhat offset by selection of solution techniques that do not make use of matrix symmetry or positive definiteness. Basically, the choices considered here were whether to use collocation methods, generalized eigensystem approaches, or one-step integration methods.

3. Solution of the Systems of Equations

For obtaining the nonlinear, static equilibrium solution, the collocation method [11] was used since the software [12] was readily available and since the method ensures a high accuracy. The program is capable of treating general systems of nonlinear multipoint boundary value problems up to order four with a variety of options available to the user. COLSYS produces an approximate solution to a user-specified accuracy using a polynomial spline which can be evaluated at any point within the domain of the problem. The inertial, geometric, and elastic properties of the blade can be expressed as functions of the axial coordinate in this approach, and the high-order, spline-fit method leads to almost as high a degree of accuracy at noncollocation points as at the collocation points. Further, trigonometric expressions involving state variables, such as $\sin(\theta + \phi)$, can be expressed exactly without the small angle assumptions on ϕ that are often made.

The collocation software package, COLSYS, was written to solve a set of nonlinear multipoint boundary value problems. As such, it can also be applied to eigenvalue problems using the approach outlined in [22]. After investigation of this approach for the eigenvalue problem, it was discarded for various reasons. In particular, since the complex roots doubled the number of required governing equations, the existing code could no longer accommodate these cases without modification. A further limitation is that a priori estimates of the desired complex eigenvectors and eigenvalues must be provided.

Next, the generalized eigenvalue approach was investigated. In this approach, Equation (6) is discretized using a finite-difference approximation. The one chosen here involved the central difference approximation given by:

$$z_{i+1} - z_i = (h/2)(A + \lambda B)_{i+1/2}(z_i + z_{i+1}) \tag{8}$$

where the subscripts denote the locations at which the variables are evaluated.

This results in a generalized eigenvalue problem of the form

$$(\bar{A} + \lambda\bar{B})Z = 0 \ . \tag{9}$$

The dimensions of the matrices $\bar{A}$ and $\bar{B}$ are $16k \times 16k$, where 16 is the dimension of the matrices A and B in Equation (6), and k is the number of segments in the discretization. In preliminary investigations the subroutine EIGZF [13] was used to obtain all the eigenvalues for this generalized eigenvalue problem. It was found that as many as 100 segments would be required to obtain 3-place accuracy for the second eigenvalue. Unfortunately, the lack of symmetry prevents the banded structure of the matrices from being exploited; thus, a storage problem is created.

One alternate approach is to find selected eigenvalues. The Block-Stodola method [14] uses a block inverse-iteration algorithm to find the first few eigenvalues. In those cases which were tested for this method, no more than the first eigenvalue could be computed accurately. Here, because of the lack of symmetry, there is no underlying variational principle that can be employed as in [14]. Although it is possible to symmetrize one of the two matrices and obtain the hypotheses in [14], this results in a significant degree of fill-in. As a consequence, this approach was not pursued.

The one-step methods using transfer matrices similar to the Myklestad method seem to be the most promising. The term one-step method refers to those methods which depend only upon station i to obtain solutions at station i + 1. Higher-order terms of a Taylor expansion can be used with the one-step method to speed convergence. The method can be described as follows. A Taylor expansion of $z(x_{i+1})$ about x_i is

$$z(x_{i+1}) = z(x_i) + z'(x_i)h + z''(x_i)h^2/2 + z'''(x_i)h^3/6 + \ . \ . \ . \tag{10}$$

where

$$h = x_{i+1} - x_i \tag{11}$$

Now, in Equation (6), let

$$C = A + \lambda B \tag{12}$$

A first-order method can be obtained from the first two terms of Equation (10), yielding

$$z_{i+1} = z_i + hC_i z_i \tag{13}$$

where Equation (6) is used to eliminate $z'(x_i)$. Equation (13) has $O(h^2)$ local truncation error. A second-order method can be similarly obtained from the first three terms in Equation (10) yielding

$$z_{i+1} = z_i + hC_i z_i + (h^2/2)(C_i' z_i + C_i^2 z_i) \tag{14}$$

which has $O(h^3)$ local truncation error. Here Equation (6) is used to eliminate $z'(x_i)$ and $z''(x_i)$. Both Equations (13) and (14) may be rewritten as $z_{i+1} = T_i z_i$, where T_i is a suitably chosen matrix evaluated at x_i, so that

$$z_{i+1} = (I + hC_i)z_i \tag{15}$$

$$z_{i+1} = [I + hC_i + (h^2/2)(C_i' + C_i^2)]z_i \tag{16}$$

The form of T_i depends on the order of the method. Hence, $z_n = Tz_0$ where $T = T_{n-1}T_{n-2}\cdots T_0$. The homogenous boundary conditions at x_o and x_n are then used to reduce matrix T to a smaller (6 × 6) matrix $\bar{T}$ whose determinant det $(\bar{T})$ is a polynomial in λ. The desired eigenvalues correspond to those λ's that produce zero determinant for $\bar{T}$. In our implementation of the one-step methods, subroutine ZANLYT [13] is used to find one or more of the complex roots of the real polynomial det$(\bar{T})$. Subroutine LEQTIC [13] is used to decompose matrix $\bar{T}$ into L and U factors. The product of the pivots produced from this factorization gives the determinant of $\bar{T}$. Finally, whenever C_i' is used, it is approximated by the backward difference

$$C_i' \cong (C_i - C_{i-1})/h \tag{17}$$

and C_0' is assumed to be zero. In any one-step method, care must be taken to define the C' terms properly at those points where cusps or discontinuities in the axially varying properties occur.

If there are axial variations in inertial, geometric, and elastic properties or in the tension, the term $C_i' z_i$ in Equation (16) can have significant impact on the speed of convergence. Rotor blades have all of these axial variations. Therefore, the method illustrated by Equation (16) was used in this paper. As shown in the next section the number of segments required to produce accurate results increases substantially if the $C_i' z_i$ term is omitted from Equation (16).

4. Numerical Results and Discussion

Numerical results were obtained using an IBM 360/67 computer and are presented in this section for several rotor-blade configurations. These results are intended to serve primarily three purposes: (1) to compare with published results and thus, at least partially, validate the present computer program; (2) to study the convergence properties of one-step methods; and (3) to present some new results for a nonuniform blade which may serve as a reference problem for future analytical studies. The numerical values of the various inertial, elastic, and geometric properties are presented in Table 1 for configurations to be compared with results from [1] and [9] and for the hypothetical nonuniform blade.

4.1. Comparison with published results

In this section numerical results are presented to compare with some published data. No attempt will be made to compare with all of the many available numerical results in the literature. Instead, we will focus on results from an in vacuo configuration in [9] and the aeroelastic stability results of [1]. Because of the very good agreement between the present results and those published, it was felt that a presentation of the results in tabular form would facilitate comparison with published data and understanding of the effects of the underlined terms in Equation (7) on the aeroelastic stability results of [1]. For serving the latter purpose we designate two constants k1 and k2. For k1 (or k2) = 1, the single-(or double-)underlined terms in Equation (7) are included. For k1 (or k2) = 0, the single-(or double-)underlined terms in Equation (7) are deleted.

As pointed out above, consistent application of the ordering scheme in a displacement formulation, such as in [1], may lead to discarding terms that may not necessarily be negligible in an analogous ordering scheme for a mixed formulation as in this paper. In order to match results of [1], it was necessary to experiment with deleting and retaining the underlined terms in Equation (7). One example of the dilemmas faced in trying to be consistent is apparent when the torsion moment M_x is written as an integral of applied and inertial loads. In this case it is clearly $O(\varepsilon^2)$, as noted in [15] and explained in [17]. However, when M_x is written in terms of the derivatives of rotations, it contains one term, which is $O(\varepsilon)$. If M_x is regarded as $O(\varepsilon)$, the single-underlined terms must remain (k1 = 1); if M_x is regarded as $O(\varepsilon^2)$, these terms should be neglected for the sake of consistency (k1 = 0). A second example involves the double-underlined terms. When the moment components M_y and M_z appear in the bending equations in a displacement formulation, they appear multiplied by $O(1)$ quantities. On the other hand, in the torsion equation they are multiplied by $O(\varepsilon)$ quantities. Because terms up through $O(\varepsilon^2)$ are retained in all equations in [1], it is consistent to take only the dominant terms in expressions for M_y and M_z in the torsion equation but retain the more accurate expression, including double-underlined terms, for M_y and M_z in the bending equations. In the present mixed formulation, however, dropping of the double-underlined terms (k2 = 0) in Equation (7) would result in matching the torsion equation of a displacement formulation equation but result in an oversimplified set of bending equations. Retention of the double-underlined term (k2 = 1) results in matching the bending equations of a displacement formulation, but adds additional $O(\varepsilon^3)$ terms that are inconsistent in the torsion equation of the displacement formulation. Thus, a consistent set of equations in a mixed formulation may, when transformed to a displacement form, result in neglect or retention of terms that would be differently treated in a set of equations written consistently in

TABLE 1. VALUES OF INERTIAL, ELASTIC, AND GEOMETRIC PARAMETERS FOR COMPARISON OF RESULTS WITH [1] AND [9].

(a) Reference [1] Configuration

$\rho_\infty cR/m = 5/(6\pi)$

$bc/(\pi R) = 0.1$

$c/R = \pi/40$

$a = 2\pi$

$c_{d_0}/a = 0.01$

$EI_y/(m\Omega^2R^4) = 0.014605 \ (\omega_w = 1.15\Omega)$

$$EI_z/m\Omega^2R^4) = \begin{cases} 0.026787 \ (\omega_v = 0.7\Omega, \text{ soft in-plane case}) \\ 0.16696 \ (\omega_v = 1.5\Omega, \text{ stiff in-plane case}) \end{cases}$$

$GJ/(m\Omega^2R^4) = 0.0056732 \ (\omega_\phi = 5\Omega)$

$k_m/R = k_{m_2}/R = 0.025$

$k_{m_1} = e_A = e = 0$

$(k_A/k_m)^2 = 1.5$

$\theta = 0, \ 0.3$ rad

(b) Reference [9] Configuration

$\Omega = 0$

$\rho_\infty = 0$

$R = 40$ in.

$m = 0.000125$ lb sec^2 in.$^{-2}$

$EI_y = 25000$ lb in.^{2}

$EI_z = 7500$ lb in.^{2}

$\theta = \pi/4$ rad

$GJ = 9000$ lb in.^{2}

$k_{m_1}^2 = k_{m_2}^2 = 1$ in.^{2}

$e_A = k_A = 0$

$e = \sqrt{2}$ in.

(c) Tapered Case

All properties at blade root identical to those listed above in part (a) for the soft-in-plane case. All properties are constant except EI_y, EI_z, GJ, and m, which are multiplied by the factor $(1 - 0.1x)$.

displacement form. We expect differences such as these to be small. Otherwise, we must conclude that whatever has been taken as an ordering scheme must itself be somehow inconsistent.

A sample of results generated for the simplified configuration from [9] is presented in Table 2. The rotor speed is zero in this case, and therefore there is no aerodynamic loading and no static equilibrium deformation. There is nevertheless, an offset between the blade mass center and shear center and nonzero pitch angle (the properties are given in Table 1). Thus, all degrees of freedom are coupled. The number of segments used by the example from [9] is unknown. However, it is clear that the present results based on a second-order, one-step method ($C' = 0$ for this case, since the beam is uniform with no tension) are tending towards those of [9] as the number of segments is increased. With only 24 segments, the first mode is within 0.1% of Murthy's result.

TABLE 2. COMPARISON WITH RESULTS FROM [9] FOR FREE VIBRATION FREQUENCIES OF BLADE WITH MASS CENTER OFFSET.

	ω(rad/sec)		
Segments	Mode 1	Mode 2	Mode 3
16	30.7552	53.6968	179.8088
24	30.7962	53.7691	182.3980
32	30.8107	53.7947	183.3519
40	30.8174	53.8065	183.8023
Ref. [9]	30.8295	53.8277	184.6175

When the rotor speed is nonzero, the presence of steady aerodynamic loads and, in some cases, inertial loads, causes significant static deformation which must be taken into account properly in order to obtain correct stability eigenvalues [1]. Results from the present analysis (COLSYS) are presented in Table 3 along with those of [1] for both soft-in-plane ($\omega_v = 0.7$) and stiff-in-plane ($\omega_v = 1.5$) cases with $\theta = 0$ and 0.3. The quantities tabulated are tip deflections of $\bar{v}$, $\bar{w}$, $\bar{\phi}$, and the accuracy is specified to be four significant figures. The solution is neither difficult to obtain nor particularly time consuming. The agreement is quite good regardless of the choice of k1 and k2. We conclude from this that the static equilibrium solution is not strongly affected by the presence of the underlined terms in Equation (7) for the limited range of values presented here. Such is not the case for the stability eigenvalues, however.

Table 4 presents the stability eigenvalues based on a second-order, one-step method with C' terms included and 16 segments (converged to 3 significant figures). Again, the agreement is very good, especially at $\theta = 0$. The presence of the single-underlined terms (k1 = 1) has no effect on the results for $\theta = 0$ and very little effect for $\theta = 0.3$. The double-underlined terms (k2 = 1) raise the torsion frequency at $\theta = 0.3$ from its value at $\theta = 0$ instead of lowering it as indicated in [1]. The effects of k2 on lead-lag damping are minor. The difference in the torsion frequency due to the k2 terms is about 2%. In a displacement formulation, these terms of second degree in bending curvatures are neglected in the torsion stiffness (i.e., third-degree terms, such as $\phi w''^2$, $\phi v''w''$, etc., are neglected in the second-degree torsion equation). This is consistent in displacement formulations such as [1], [15], [18], and [19]. It is not

TABLE 3. VALUES OF STATIC EQUILIBRIUM DISPLACEMENTS AT BLADE TIP. (k1 = 0, k2 = 1 for COLYS results)

	$\bar{v}/R$	$\bar{w}/R$	$\bar{\phi}$
(a) $\omega_v = 0.7\Omega$, $\theta = 0.0$			
COLSYS	-0.002321	0.0	0.0
Ref. [1]	-0.002326	0.0	0.0
(b) $\omega_v = 0.7\Omega$, $\theta = 0.3$			
COLSYS	-0.03943	0.09315	-0.01332
Ref. [1]	-0.03940	0.09314	-0.01336
(c) $\omega_v = 1.5\Omega$, $\theta = 0.0$			
COLYSY	-0.000522	0.0	0.0
Ref. [1]	-0.000522	0.0	0.0
(d) $\omega_v = 1.5\Omega$, $\theta = 0.3$			
COLSYS	-0.03139	0.09432	-0.01164
Ref. [1]	-0.03129	0.09412	-0.01214

TABLE 4. COMPARISON OF STABILITY EIGENVALUE RESULTS OBTAINED BY USING VARIOUS VALUES OF k1 and k2 (16 segments). Eigenvalues are given per unit Ω.

k1	k2	Lead-lag	Flapping	Torsion
(a) $\theta = 0$				
$\omega_v = 0.7\Omega$				
0	0	-0.0010 ±0.6929i	-0.3235 ±1.0788i	-0.3615 ±4.9809i
0	1	-0.0010 ±0.6929i	-0.3235 ±1.0788i	-0.3615 ±4.9809i
1	1	-0.0010 ±0.6929i	-0.3235 ±1.0788i	-0.3615 ±4.9809i
Ref.	[1]	-0.0011 ±0.7014i	-0.3245 ±1.0751i	-0.3622 ±4.9875i
$\omega_v = 1.5\Omega$				
0	0	-0.0011 ±1.4938i	-0.3235 ±1.0767i	-0.3617 ±4.9822i
0	1	-0.0011 ±1.4938i	-0.3235 ±1.0767i	-0.3617 ±4.9822i
1	1	-0.0011 ±1.4938i	-0.3235 ±1.0767i	-0.3617 ±4.9822i
Ref.	[1]	-0.0011 ±1.5002i	-0.3246 ±1.0741i	-0.3625 ±4.9888i
(b) $\theta = 0.3$				
$\omega_v = 0.7\Omega$				
0	0	-0.0230 ±0.6823i	-0.3145 ±1.0749i	-0.3552 ±4.9592i
0	1	-0.0228 ±0.6819i	-0.3146 ±1.0746i	-0.3564 ±4.9720i
1	1	-0.0228 ±0.6815i	-0.3147 ±1.0743i	-0.3574 ±4.9836i
Ref.	[1]	-0.0249 ±0.6931i	-0.3117 ±1.0676i	-0.3628 ±4.9637i
$\omega_v = 1.5\Omega$				
0	0	-0.0641 ±1.6143i	-0.2821 ±0.9547i	-0.3520 ±4.9550i
0	1	-0.0665 ±1.5698i	-0.2792 ±0.9545i	-0.3494 ±5.0471i
1	1	-0.0667 ±1.5663i	-0.2789 ±0.9545i	-0.3491 ±5.0524i
Ref.	[1]	-0.0632 ±1.5823i	-0.2821 ±0.9507i	-0.3510 ±4.9150i

obvious, however, that these third-degree terms are the only ones that have an observable effect on the trends. The only way to resolve that question is to examine results from a consistent analysis in which $O(\varepsilon^3)$ terms are neglected with respect to unity, which has not yet been developed.

4.2. Convergence of one-step methods

We now address the subject of convergence for the one-step methods including the C' terms. In Table 5, results obtained for the soft-in-plane case at $\theta = 0.3$ are tabulated as a function of the number of segments. The convergence is fairly rapid at first and tapers off as the converged value is approached. The smallest eigenvalues are generally within 1% for 16 segments.

TABLE 5. CONVERGENCE OF THE ONE-STEP METHOD BASED ON EQUATION (16) FOR VARIOUS NUMBERS OF SEGMENTS ($\omega_V = 0.7\Omega$, $\theta = 0.3$, k1 = 0, k2 = 1) Eigenvalues are given per unit Ω.

Segments	Lead-lag	Flapping	Torsion
4	-0.0159 ±0.7394i	-0.3099 ±1.2025i	-0.3365 ±4.8960i
8	-0.0234 ±0.6902i	-0.3108 ±1.0929i	-0.3523 ±4.9548i
12	-0.0230 ±0.6840i	-0.3136 ±1.0791i	-0.3553 ±4.9676i
16	-0.0228 ±0.6819i	-0.3146 ±1.0746i	-0.3564 ±4.9720i
20	-0.0227 ±0.6810i	-0.3151 ±1.0725i	-0.3569 ±4.9739i
24	-0.0227 ±0.6805i	-0.3153 ±1.0715i	-0.3572 ±4.9750i

In the one-step method, the C' terms may be neglected in some applications. This is certain to have an adverse effect on convergence for a rotating beam, however. Even for a beam with uniform properties, the perturbation equations will have variable coefficients due to tension and static equilibrium terms. Thus, it is important to study the convergence of the stability eigenvalues as a function of the number of segments for methods which do not use the C' terms. In Table 6, the lead-lag and flap eigenvalues are presented for three methods versus the number of segments. Method (1) is the complete second-order method with the C' terms included (Eq. (16)). Method (2) is the second order method without the C' terms. Method (3) is the first-order method (Eq. (15)). There is little difference in the rate of convergence for Methods (2) and (3). The inclusion of the C' terms, however, results in about an order of magnitude reduction in the number of segments required for convergence. Hundreds of segments may be needed for convergence when C' is neglected [23]. It should be noted that [6] uses only the tension terms in evaluation of C'. While that should result in a substantial improvement in convergence over Methods (2) and (3), it would still be inferior to Method (1) if the beam has portions of even modest nonuniformities in stiffness or inertial properties.

4.3. Results for a nonuniform blade

We have included results for a hypothetical nonuniform blade in Table 7. All properties are the same as those used in the configuration for comparison with [1], except that EI_y, EI_z, GJ, and m are equal to their values for the uniform blade times the factor (1 - 0.1x). Convergence for the nonuniform blade is a little slower, and all frequencies are increased *from* the values obtained the uniform blade. The lead-lag damping is only slightly decreased by the presence of taper for this case. This small effect is not surprising, since fundamental bending-torsion coupling

TABLE 6. EFFECT OF $C_i^! z_i$ TERMS ON CONVERGENCE OF STABILITY EIGENVALUES USING ONE-STEP METHODS. (θ = 0.3, ω_v = 0.7Ω, k1 = 0, k2 = 1). Eigenvalues are given per unit Ω.

Segments	Method (1)	Method (2)	Method (3)
		Lead-lag	
8	-0.0234 ±0.6902i	-0.0201 ±0.7699i	-0.0181 ±0.8398i
16	-0.0228 ±0.6819i	-0.0216 ±0.7308i	-0.0203 ±0.7500i
24	-0.0227 ±0.6805i	-0.0220 ±0.7152i	-0.0211 ±0.7244i
32		-0.0221 ±0.7068i	-0.0214 ±0.7124i
48		-0.0223 ±0.6980i	-0.0218 ±0.7009i
64		-0.0224 ±0.6935i	-0.0220 ±0.6953i
80		-0.0224 ±0.6908i	-0.0222 ±0.6920i
96		-0.0225 ±0.6889i	-0.0222 ±0.6899i
		Flap	
8	-0.3108 ±1.0929i	-0.2849 ±1.1367i	-0.2922 ±1.1431i
16	-0.3146 ±1.0746i	-0.3008 ±1.1060i	-0.3034 ±1.1021i
24	-0.3153 ±1.0715i	-0.3059 ±1.0944i	-0.3075 ±1.0903i
32		-0.3085 ±1.0884i	-0.3095 ±1.0847i
48		-0.3110 ±1.0821i	-0.3116 ±1.0793i
64		-0.3122 ±1.0789i	-0.3127 ±1.0767i
80		-0.3129 ±1.0770i	-0.3133 ±1.0751i
96		-0.3134 ±1.0757i	-0.3137 ±1.0747i

TABLE 7. STABILITY EIGENVALUES FOR A NONUNIFORM BLADE (k1 = k2 = 1, θ = 0.3). Eigenvalues are given per unit Ω.

Segments	Lead-lag	Flap	Torsion
16	-0.0230 ±0.7001i	-0.3138 ±1.0819i	-0.3564 ±5.0798i
24	-0.0229 ±0.6987i	-0.3144 ±1.0790i	-0.3567 ±5.0837i
32	-0.0228 ±0.6983i	-0.3146 ±1.0781i	-0.3564 ±5.0851i

parameters [1] are not changed significantly. No significant computational penalty in the calculation of the nonlinear static equilibrium and the stability eigenvalues is incurred because of nonuniformity.

5. Concluding Remarks

Nonlinear equations for static equilibrium deformation and linearized perturbation equations for small motions about equilibrium, from which stability eigenvalues can be obtained, are written in a first-order, spatial derivative form. These equations are solved by COLSYS [12] and one-step integration techniques, respectively. Numerical results for uniform blades obtained from the equations are compared with published results [1] and [9]. and used to study the convergence of the one-step methods. Results are also presented for a hypothetical nonuniform blade. In the course of this study several conclusions have emerged.

1. The equations of the present mixed formulation are general enough to treat nonuniform pretwisted rotor blades with chordwise offsets between elastic center, mass center, and shear center. Certain higher order cross-section integrals are neglected, and symmetry about the major cross-section axis is assumed [15]. Neither the equations nor the solution methods, in their present form, apply to the forward-flight problem.

2. COLSYS is used to solve the nonlinear static equilibrium equations with the present mixed formulation. The calculation of the static equilibrium solution is neither particularly difficult nor time consuming even though nonlinearities and nonuniformities are involved. Thus, there is no need to limit the static equilibrium solution to some linear or otherwise ad-hoc estimate.

3. There are slight differences in the numerical results obtained from mixed and displacement formulations of rotating blades with geometric nonlinearity. The differences stem from the equations that relate moments to derivatives of bending rotations. A consistent ordering scheme applied to a mixed formulation may not produce exactly equivalent equations when applied to a displacement formulation. The main difference in the results is in the magnitude of the torsion frequency. The difference is of the order of rotations squared with respect to unity (about 2%). The present analysis yields the result that torsion frequency increases with increased pitch angle in direct contrast to results in [1].

4. One-step, second-order integration methods appear to be a viable means of calculating the stability eigenvalues. In order to obtain good convergence it is necessary to include the entire second-order term, for rotor blades, for which tension force, elastic, and inertial properties all may vary along the length of the blade.

5. The terms involving the torsion moment M_x in Equation (7) were found to be negligibly small for the limited range of parameters investigated. The double-underlined terms do affect a basic trend, as described in conclusion (3). This may indicate the need for consistently incorporating terms of the next higher order in elastic rotations. Unfortunately, the only way to ascertain the correctness of this assertion is to compare results from such an analysis, which does not yet exist, to those of this paper or [1].

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APPENDIX A

Steady State Equations

The static equilibrium equations are given in the following first-order form with the sign conventions and nomenclature of [15]. As in [15], section properties of higher order, B_1* and B_2*, warping rigidity and shear center offset terms, and C_1 and C_1* are neglected. In presenting these equations, care has been taken to neglect terms that are $O(\varepsilon^2)$ compared to unity. To avoid confusion with d/dx, the primes have been removed from the subscripts of EI_y, EI_z, and V_x.

Lead-lag equations:

$$\bar{V}_y' = -m\Omega^2[\bar{v} + e\cos(\theta + \bar{\phi})] - \bar{L}_v \tag{A1}$$

$$\bar{M}_z' = -\bar{V}_y + \bar{V}_x\bar{\zeta} + me\Omega^2 x\cos(\theta + \bar{\phi}) \tag{A2}$$

$$\bar{v}' = \bar{\zeta} \tag{A3}$$

$$\bar{\zeta}' = (a_1\bar{\zeta} - a_2\bar{\beta})\bar{M}_x + (-a_1 + a_4\bar{\phi})M_y + (a_2 - 2a_1\bar{\phi})\bar{M}_z + e_A V_x\cos(\theta + \bar{\phi})/Z_0 \tag{A4}$$

Flap equations:

$$\bar{V}_z' = -\bar{L}_w \tag{A5}$$

$$\bar{M}_y' = \bar{V}_z - \bar{V}_x\bar{\beta} - me\Omega^2 x\sin(\theta + \bar{\phi}) \tag{A6}$$

$$\bar{w}' = \bar{\beta} \tag{A7}$$

$$\bar{\beta}' = (a_3\bar{\zeta} - a_1\bar{\beta})\bar{M}_x + (-a_3 - 2a_1\bar{\phi})\bar{M}_y + (a_1 - a_4\bar{\phi})\bar{M}_z + e_A\bar{V}_x\sin(\theta + \bar{\phi})/Z_0 \tag{A8}$$

Torsion equations:

$$\bar{M}' = -\bar{V}_z\bar{\zeta} + \bar{V}_y\bar{\beta} + me\Omega^2\bar{v}\sin(\theta + \bar{\phi}) + m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\cos(\theta + \bar{\phi})\sin(\theta + \bar{\phi}) - \bar{M}_\phi \tag{A9}$$

$$\bar{\phi}' = 1/\bar{G}_0[\bar{M}_x + \bar{M}_y(\bar{\zeta} + a_6\theta'\sin\theta) + \bar{M}_z(\bar{\beta} - a_6\theta'\cos\theta) - a_7\theta'\bar{V}_x] \tag{A10}$$

where, for brevity,

$$\bar{G}_0 = GJ + a_5\bar{V}_x - a_6\bar{M}_y\sin\theta + a_6\bar{M}_z\cos\theta$$

$$Y_0 = EI_y$$

$$Z_0 = EI_z - EA\,e_A^{\,2} \tag{A11}$$

$$a_1 = -[(1/Y_0) - (1/Z_0)]\sin\theta\cos\theta$$

$$a_2 = (\sin^2\theta/Y_0) + (\cos^2\theta/Z_0)$$

$$a_3 = (\cos^2 \theta/Y_0) + (\sin^2 \theta/Z_0)$$

$$a_4 = a_3 - a_2$$

$$a_5 = k_A{}^2 EI_z/Z_0 \qquad \text{(A11 concluded)}$$

$$a_6 = k_A{}^2 EA\ e_A/Z_0$$

$$a_7 = (EI_z/Z_0)(k_A{}^2 - EJ/EA)$$

The term involving a_7 in Equation (A10) has been altered to agree with the recent results of [24].

In the above equations, the tension in the inextensible blade is given by the uncoupled expression

$$\bar{V}_x = \Omega^2 \int_x^R mx\ dx \qquad \text{(A12)}$$

The aerodynamic loading is derived in [1]. The steady component is

$$\bar{L}_v = (\rho_\infty ac/2)[v_i{}^2 - \Omega^2 x^2 (c_{d_0}/a) - \Omega x v_i(\theta + \bar{\phi})] \qquad \text{(A13)}$$

$$\bar{L}_w = (\rho_\infty ac/2)[-\Omega x v_i + \Omega^2 x^2 (\theta + \bar{\phi} + \bar{\psi}) - \Omega^2 x \bar{v} \bar{\beta} + (\Omega^2 x c \bar{\beta}/2)] \qquad \text{(A14)}$$

$$\bar{M}_\phi = 0 \qquad \text{(A15)}$$

The term $\bar{\psi}$ in Equation (A15) is defined

$$\bar{\psi} = \int_0^x \bar{\zeta}\bar{\beta}'\ dx \qquad \text{(A16)}$$

and is represented in the analysis as the following additional ordinary differential equation

$$\bar{\psi}' = \bar{\zeta}\bar{\beta}' \qquad \text{(A17)}$$

with the initial boundary value specified by the values of ζ and β' at $x(0)$. The induced velocity is given by

$$v_i = \mathrm{sgn}(\theta + \phi_0)\Omega R(\pi\sigma/8)[\sqrt{1 + (12/\pi\sigma)|\theta + \phi_0|} - 1] \qquad \text{(A18)}$$

where

$$\phi_0 = \bar{\phi} \quad \text{at} \quad x/R = 0.75 \qquad \text{(A19)}$$

and

$$\sigma = bc/\pi R \qquad \text{(A20)}$$

APPENDIX B

Linearized Perturbation Equations

The perturbation equations given in [15] have been converted to the linear ordinary differential equations in first-order spatial derivative form with the eigenvalue λ. Although not necessary for the method of solution finally adopted, they are presented linear in λ as well.

Lead-lag equations:

$$V_y' = m\{\lambda\hat{v}^* - e\lambda\hat{\phi}^* \sin\theta - \Omega^2[\hat{v} - e\hat{\phi}\sin(\theta + \bar{\phi})] + 2\Omega[\lambda\hat{u} - e(\lambda\hat{\zeta}\cos\theta + \lambda\hat{\beta}\sin\theta)]\} - \hat{L}_v \tag{B1}$$

$$\hat{M}_z' = -\hat{V}_y + \bar{V}_x\hat{\zeta} + \bar{\zeta}\hat{V}_x - me\Omega^2 x\hat{\phi}\sin(\theta + \bar{\phi}) + 2me\Omega\lambda\hat{v}\cos\theta \tag{B2}$$

$$\hat{v}' = \hat{\zeta} \tag{B3}$$

$$\hat{\zeta}' = (a_1\bar{\zeta} - a_2\bar{\beta})\hat{M}_x + (-a_1 + a_4\bar{\phi})\hat{M}_y + (a_2 - 2a_1\bar{\phi})\hat{M}_z + \bar{M}_x(a_1\hat{\zeta} - a_2\hat{\beta}) + [a_4\bar{M}_y - 2a_1\bar{M}_z - (e_A/Z_0)\sin(\theta + \bar{\phi})\bar{V}_x]\hat{\phi} + e_A\hat{V}_x\cos(\theta + \bar{\phi})/Z_0 \tag{B4}$$

Flap equations:

$$\hat{V}_z' = m(\lambda\hat{w}^* + e\lambda\hat{\phi}^*\cos\theta) - \hat{L}_w \tag{B5}$$

$$\hat{M}_y' = \hat{V}_z - \bar{V}_x\hat{\beta} - \bar{\beta}\hat{V}_x - me\Omega^2 x\hat{\phi}\cos(\theta + \bar{\phi}) - 2me\Omega\lambda\hat{v}\sin\theta \tag{B6}$$

$$\hat{w}' = \hat{\beta} \tag{B7}$$

$$\hat{\beta}' = (a_3\bar{\zeta} - a_1\bar{\beta})\hat{M}_x - (a_3 + 2a_1\bar{\phi})\hat{M}_y + (a_1 - a_4\bar{\phi})\hat{M}_z + \bar{M}_x(a_3\hat{\zeta} - a_1\hat{\beta}) - [2a_1\bar{M}_y + a_4\bar{M}_z - (e_A/Z_0)\bar{V}_x\cos(\theta + \phi)]\hat{\phi} + e_A\hat{V}_x\sin(\theta + \phi)/Z_0 \tag{B8}$$

Torsion equations:

$$\hat{M}_x' = -\bar{V}_z\hat{\zeta} - \bar{\zeta}\hat{V}_z + \bar{V}_y\hat{\beta} + \bar{\beta}\hat{V}_y - me[\lambda\hat{v}^*\sin(\theta + \bar{\phi}) - \Omega^2\hat{v}\sin(\theta + \bar{\phi}) - \Omega^2\bar{v}\cos(\theta + \phi)\hat{\phi} - \lambda\hat{w}^*\cos(\theta + \bar{\phi}) + 2\Omega\lambda\hat{u}\sin\theta] + mk_m^2\lambda\hat{\phi}^* + m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\hat{\phi}\cos 2\theta - M_\phi \tag{B9}$$

(B9)

$$\hat{\phi}' = (1/\bar{G}_0)\{\hat{M}_x + [\bar{\zeta} + a_6(\theta' + \bar{\phi}')\sin\theta]\hat{M}_y + \bar{M}_y\hat{\zeta} + [\bar{\beta} - a_6(\theta' + \bar{\phi}')\cos\theta]\hat{M}_z + \bar{M}_z\hat{\beta}\} \tag{B10}$$

where the constants a_i and $\bar{G}_0$ are defined in appendix A.

Equations (B1) through (B10) have been linearized in terms of λ by using the relations

$$v^* = \lambda v \tag{B11}$$

$$w^* = \lambda w \tag{B12}$$

$$\phi^* = \lambda\phi \tag{B13}$$

Additional differential equations needed to supplement Equations (B1) through (B10) are

$$\hat{\psi}' = \bar{\zeta}\hat{\beta}' + \hat{\zeta}\bar{\beta}' \tag{B14}$$

$$\hat{v}^{*\prime} = \lambda\hat{v}' \tag{B15}$$

$$\hat{w}^{*\prime} = \lambda\hat{w}' \tag{B16}$$

$$\phi^{*\prime} = \lambda\hat{\phi}' \tag{B17}$$

The perturbed tension and displacement in the axial direction are

$$\hat{V}_x' = -2m\Omega\lambda v \tag{B18}$$

$$\begin{aligned}\hat{u}' = -\bar{\zeta}\hat{\zeta} - \bar{\beta}\hat{\beta} - (e_A/Z_0)(\hat{M}_y \sin\theta \\ - \hat{M}_z \cos\theta)\end{aligned} \tag{B19}$$

The aerodynamic load contributions [1] are

$$\begin{aligned}\hat{L}_v = (\rho_\infty ac/2)\{-\Omega x v_i\hat{\phi} - [2\Omega x(c_{d_0}/a) + (\theta + \bar{\phi})v_i]\lambda\hat{v} \\ + [2v_i - \Omega x(\theta + \bar{\phi})]\lambda\hat{w}\}\end{aligned} \tag{B20}$$

$$\begin{aligned}\hat{L}_w = (\rho_\infty ac/2)\{\Omega^2 x^2(\hat{\phi} + \hat{\psi}) - \Omega^2 x(\bar{v}\hat{\beta} + \bar{\beta}\hat{v}) + (\Omega^2 xc\hat{\beta}/2) \\ + [2\Omega x(\theta + \bar{\phi}) - v_i]\lambda\hat{v} - \Omega x\lambda\hat{w} + (3/4)c\Omega x\lambda\hat{\phi} - (c/4)\lambda w^*\}\end{aligned} \tag{B21}$$

$$\hat{M}_\phi = -(\rho_\infty ac/2)[(c^2/8)\Omega x\lambda\hat{\phi}] \tag{B22}$$

Thus, Equations (B1) through (B10), (B14) through (B19) form the complete set of 16 coupled ordinary first-order differential equations required to calculate the stability eigenvalues of a nonuniform, pretwisted blade.