An approach to calculate inviscid compressible flows by a low order panel method using the dual reciprocity method

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Abstract

In the present paper an approach is presented which allows the calculation of compressible potential flows with a low-order panel method. The integral representation of the governing full potential equation leads to an additional domain integral which is converted into a boundary integral by employing the dual reciprocity method (DRM). Using a low-order panel method creates difficulties to compute the field source strength in the vicinity of the airfoil since the latter is dominated by double space derivatives of the flow potential. This is due to the fact that in a low-order panel method the surface is actually represented by a number of discrete point vortices leading to an unrealistic wavy behaviour of the velocity and its gradients close to the surface. This problem can be overcome by representing both the surface and the dipole and source strength by cubic splines. Numerical results for the two-dimensional flow around a circle and a NACA0012 airfoil are in very good agreement with field panel and Euler methods and confirm the accuracy of the present approach.

Nomenclature

ϕ	total velocity potential
ρ	local fluid density
κ	ratio of specific heat capacities
∇	nabla operator
v	Velocity vector
M	Mach number
σ	right hand side of the Poisson equation, field source strength
H	influence coefficient matrix for doublets
G	influence coefficient matrix for sources
α	coefficients used for the interpolation

subscripts

i	references a source point
j	references a collocation point
k	references a boundary point
h	homogen
p	particular
∞	free stream conditions

Introduction

In aerodynamics panel methods are used because of their simple implementation and computational efficiency. The advantage lies in the fact that for solving the potential equation only the surface must be discretized and in most cases the later one is given in the specification and therefore no domain discretization is necessary. This method however is restricted to incompressible flows.

When analyzing the aerodynamics of wings and especially rotors the compressibility however must be taken into account and consequently the classical panel methods lose their function as an aerodynamic design tool.

In order to make use of the methods in aerodynamics with compressible effects as well the full potential equation has to be solved. By doing so, the integral representation shows a domain integral which must now be considered when applying the potential method.

Röttgermann [1] and Sinclair [2] have, among others, developed field panel methods which explicitly calculate the domain integral.

Theurer [3] follows the dual reciprocity method by Nardini and Brebbia [4] which can be looked up in detail in Partridge et. al. [5]. This method makes it possible to transform the volume integral back to boundary integrals. Referring to this approach it is

the objective of our studies to integrate the dual reciprocity method as a module in existing three dimensional panel codes.
Considerations on this led to the conclusion that because most widely used, it should be applied to simple most discretisation of panel methods (constant singularity distribution on straight panels). This requires fundamental examination of the dual reciprocity method with regard to low order panel methods which will be outlined subsequently.

Potential Flow Model

Assuming an inviscid, irrotational flow a velocity potential ϕ can be defined reducing the number of unknowns from three velocity components to a single potential. The equation to determine the potential is obtained directly from the continuity equation. The latter reads for steady flows in conservative form

$$\nabla(\rho\nabla\phi) = 0 \tag{1}$$

An additional equation for the density ρ is obtained from the energy equation for a perfect gas by assuming an isentropic flow

$$\frac{\rho}{\rho_\infty} = \left(1 + \tfrac{\kappa-1}{2} M_\infty^2 \left(1 - \nabla\phi^2\right)\right)^{\frac{1}{\kappa-1}} \tag{2}$$

Equations (1) and (2) constitute a system of nonlinear partial differential equations which can be solved by the dual-reciprocity method. Equation (1) is called the full potential equation.

Integral Representation

By writing the full–potential equation (1) as a Poisson equation,

$$\nabla^2\phi = \underbrace{-\frac{(\nabla\rho\nabla\phi)}{\rho}}_{\sigma} \tag{3}$$

the nonlinearity of the flow can be treated as non–homogeneity and the classical panel method can be extended by adding these term. In the remainder σ will be referred to as field source strength. Applying Green's theorem yields an integral equation for the variable ϕ

$$\begin{aligned} c\phi = &\int_\Omega \left(\nabla^2\phi\right) K \, d\Omega \\ &+ \int_\Gamma \left(\frac{\partial\phi}{\partial n} K - \phi\frac{\partial K}{\partial n}\right) d\Gamma + \phi_\infty \end{aligned} \tag{4}$$

in which c is a constant dependent on the Cauchy principle value integration of the singularity. Γ denotes the boundary which in aerodynamic applications is usually identical to the airfoil surface and the wake. Ω denotes the domain which surrounds the boundary. Apparently the existence of the non-homogeneity in Equation (3) gives raise to an additional domain integral which equals zero for an incompressible flow.

Dual Reciprocity Method

To circumvent the domain integration, Cheng et al. [6] suggest the concept of a particular solution by writing the solution ϕ as

$$\phi = \phi_h + \phi_p \tag{5}$$

where ϕ_h and ϕ_p are, respectively, the homogeneous solution and the particular solution that satisfy

$$\nabla^2\phi_h = 0 \tag{6}$$

$$\nabla^2\phi_p = \sigma \tag{7}$$

To solve for the homogeneous solution only the boundary integrals of Equation (4) are important since the domain integral vanishes

$$c\phi_h = \int_\Gamma \left(\frac{\partial\phi_h}{\partial n} K - \phi_h\frac{\partial K}{\partial n}\right) d\Gamma + \phi_\infty \tag{8}$$

Replacing ϕ_h with $(\phi - \phi_p)$ this results in

$$\begin{aligned} c(\phi - \phi_p) = &\int_\Gamma \left(\frac{\partial\phi}{\partial n} - \frac{\partial\phi_p}{\partial n}\right) K \, d\Gamma \\ &- \int_\Gamma \left(\phi - \phi_p\right)\frac{\partial K}{\partial n} \, d\Gamma + \phi_\infty \end{aligned} \tag{9}$$

It is necessary to determine the particular solution ϕ_p beforehand. For that purpose, we approximate the term σ in Equation (7) by a linear combination of simple interpolation functions

$$\nabla^2\phi_p = \sigma \approx \sum_{j=1}^{n+l} \alpha_j f_j \tag{10}$$

Introducing f_j in the following way

$$f_j = \nabla^2\hat{\phi}_j \tag{11}$$

results in an expression for the particular solution

$$\nabla^2\phi_p \approx \sum_{j=1}^{n+l} \alpha_j \nabla^2\hat{\phi}_j \tag{12}$$

$$\phi_p \approx \sum_{j=1}^{n+l} \alpha_j \hat{\phi}_j \tag{13}$$

where α_j are the undetermined coefficients. n and l are the numbers of the dual reciprocity nodes located on the boundary and in the domain, respectively. Finally one obtains an equation for the total potential at an arbitrary field point

$$\begin{aligned} c\phi = & \int_\Gamma \left(\frac{\partial\phi}{\partial n}K - \phi\frac{\partial K}{\partial n}\right) d\Gamma + \phi_\infty \\ & + \sum_{j=1}^{n+l} \alpha_j \left(\hat{\phi}_j + \int_\Gamma \left(\hat{\phi}_j \frac{\partial K}{\partial n} - \frac{\partial\hat{\phi}_j}{\partial n}K\right) d\Gamma\right) \end{aligned} \tag{14}$$

Hence we have achieved a boundary only formulation of the governing full potential equation. Equation (14) is in a general form, valid for the Laplace operator in two- or three dimensions. The basic solution K equals

$$K = \begin{cases} \frac{1}{2\pi}\ln(r) & \text{2–dimensional} \\ -\frac{1}{4\pi}\frac{1}{r} & \text{3–dimensional} \end{cases} \tag{15}$$

Further derivations are performed solely for the two–dimensional case.

Solution procedure

To numerically solve Equation (14) the geometry is discretized into a set of n straight panels with constant-strength singularity distribution (of sources and doublets) see Katz and Plotkin [7]. This gives for a panel control point i the following expression

$$\begin{aligned} \phi_i = & \sum_{k=1}^{n} G_{ik}\frac{\partial\phi_k}{\partial n} - \sum_{k=1}^{n} H_{ik}\phi_k + \phi_\infty \\ & + \sum_{j=1}^{n+l} \alpha_j \left(\hat{\phi}_{ij} + \sum_{k=1}^{n} H_{ik}\hat{\phi}_{kj} - \sum_{k=1}^{n} G_{ik}\frac{\partial\hat{\phi}_{kj}}{\partial n}\right) \end{aligned} \tag{16}$$

H_{ik} describes the potential influence of the doublet distribution of panel number k at the control point i. In the same way G_{ik} represents the influence of a constant source distribution from the boundary at that control point. These elements can be computed once the geometry is known and stay the same throughout the computation.

Shape function

The shape function f_j is chosen to be

$$f_j(x, x_j) = 1 + r \tag{17}$$

where $r = |x_j - x|$ is the distance between an arbitrary point x and a dual reciprocity node x_j. For this choice of interpolation functions $\hat{\phi}$ and $\frac{\partial\hat{\phi}}{\partial n}$ can be calculated from $f_j = \nabla^2\hat{\phi}_j$ and $K = \frac{1}{2\pi}\ln(r)$ to

$$\hat{\phi}_j = \frac{r^2}{4} + \frac{r^3}{9} \tag{18}$$

$$\frac{\partial\hat{\phi}_j}{\partial n} = n \cdot \left(\frac{x}{2} + \frac{xr}{3}\right) \tag{19}$$

It may be noted that, since $\hat{\phi}$ and $\frac{\partial\hat{\phi}}{\partial n}$ are known functions once f is defined, there is no need to approximate their variation within each boundary element by using constant values as done for ϕ and $\frac{\partial\phi}{\partial n}$. However to do so implies that the same matrices H and G defined may be used on both sides of the equation.

Determination of the field source strength σ

An important part of the dual reciprocity method is the determination of the field source strength σ. By evaluating the outer derivative in Equation (3) it can be seen that σ contains single and double space derivatives of the flow potential, i.e. the velocity as well as the gradients of both velocity components. In principle these quantities can be determined directly from Equation (14) by performing the necessary spatial derivatives on the kernel K and by evaluating the integral over each panel.

However, if a panel method with constant-strength singularities is used, the velocity field is build up of point vortices at the panel edges whose strength is given by the jump in dipole strength between the adjacent panels. While the velocity field is increasingly accurate away from the surface it is therefore inaccurate close to the surface. This error is even more pronounced if the derivatives of the velocity components are computed.

An alternative to the above approach consists in approximating the flow potential by the same shape function which are used for the dual reciprocity method itself. The spatial derivatives can then be obtained by performing the latter on the shape functions. However, test simulations showed that for this procedure the resulting distribution of σ depends strongly on the choice of the l dual reciprocity nodes in the field which is an undesirable feature.

Therefore, in the present paper a different approach is followed. The surface of the airfoil as well as all distributions of source and dipole strengths are approximated by cubic splines. This results in a smooth airfoil contour and in a smooth distribution of ϕ and $\frac{\partial \phi}{\partial n}$. Since the derivatives which are needed for the computation of σ are taken with regard to an arbitrary field point, the derivative can again be performed on the kernel K.

The resulting line integral along the airfoil contour and the wake is solved numerically by employing a Runge-Kutta algorithm with adaptive step-size control. An alternative would be the use of Gaussian quadrature formulas. A general problem is that the integrand exhibits a strong peak if the field point is very close to the surface. In future, this problem will be tackled by splitting the integrand in a part which contains the singularity and can be integrated directly and a part which is solved numerically.

Evaluation of the α coefficients

The coefficients α_j from equation (10) are determined by collocation.

$$\sigma = \sum_{j=1}^{n+l} f_j \alpha_j \tag{20}$$

By taking the value for σ at the $n+l$ dual reciprocity nodes, a set of equations like the one above is obtained; this may be expressed in matrix form as

$$\boldsymbol{\sigma} = \boldsymbol{F\alpha} \tag{21}$$

where each column of $\boldsymbol{F}$ consists of a vector $\boldsymbol{f_j}$ containing the values of the function f_j at the dual reciprocity nodes. Thus $\boldsymbol{\alpha}$ is obtained by

$$\boldsymbol{\alpha} = \boldsymbol{F}^{-1}\boldsymbol{\sigma} \tag{22}$$

Iterative procedure

Since the governing Equation (1) is non–linear the solution must be obtained by an iterative procedure. In the first step the linear equation with the field source strength set to zero is solved resulting in a solution for the incompressible flow. In the next step the right hand side of Equation (3) is calculated at all $n+l$ dual reciprocity nodes. Once the field source strength is known the α coefficients can be calculated and the right hand side can be modified. This is done until the convergence of σ is under a preset value. The procedure is schematically shown in Figure 1.

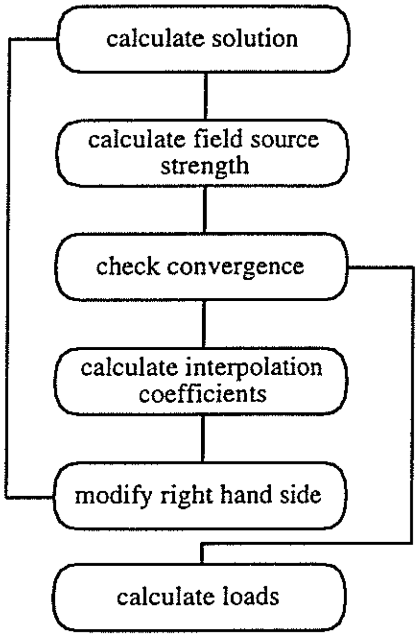

Figure 1: Iterative solution scheme.

Test cases

Flow around a circle

As a first test case we compute the flow around a circle at a free stream Mach number of 0.38. Since there is an analytical solution for the incompressible flow, this case is especially suitable for explaining the different methods to determine the field source strength σ.

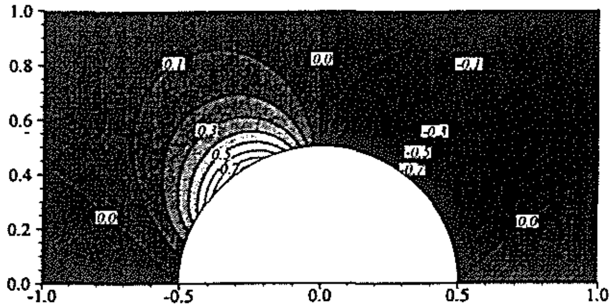

Figure 2: Analytical σ distribution.

Figure 2 shows the source strength σ which is computed from the incompressible flow field. This distribution represents the first iteration. Calculating the field source strength directly from the panel method with 100 straight panels of constant strength results in a correct solution away from the surface. However, close to the surface σ exhibits an alternating behaviour which is physically meaningless, see Figure 3.

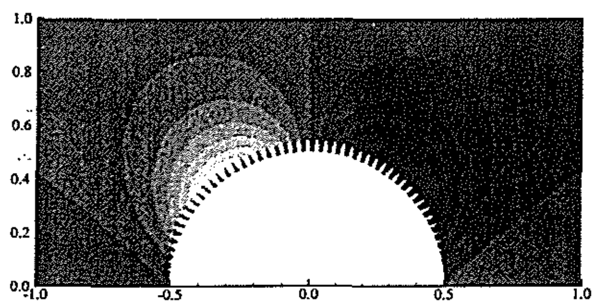

Figure 3: Field source strength with the low order panel method.

This result is obviously caused by the fact that the velocity distribution near the surface shows an alternating behaviour as shown in Figure 4. Numerical studies revealed that from a distance of 1.5 times the panel length on this wavy induction decays. However, this would imply that no nodes can be placed closer to the surface meaning that either the number of panels must be increased drastically or that the field source strength at the surface where it has its major impact on the solution is not well determined. It should be emphasized that this result is not only

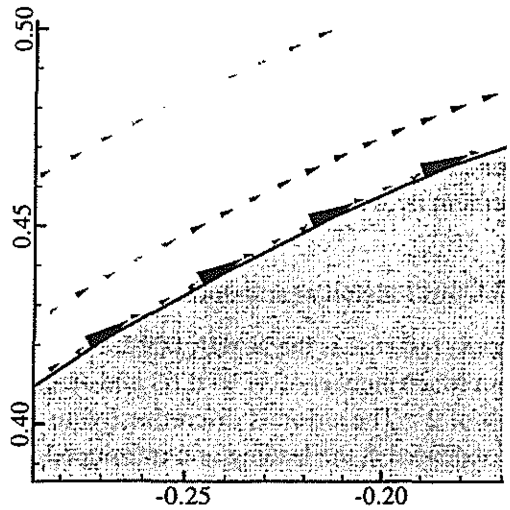

Figure 4: Induced velocity from constant singularity distributions.

due to the fact that a constant strength distribution is chosen but is also caused by the discretisation of the surface into *straight* panels.

This problem can be overcome by using cubic splines for approximating both the surface and the singularity strength on the surface and by solving the resulting line integral numerically. With this approach it is possible to accurately calculate the field source strength also in the vicinity of the boundary. This yields an excellent agreement with the analytical solution.

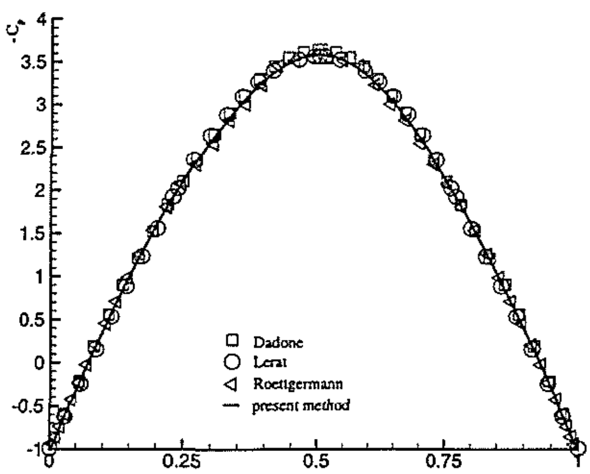

Figure 5: Comparison of surface pressure distribution.

Figure 5 shows the pressure distribution around the circle which was computed with the present method. The solution is compared with the ones obtained by Röttgermann with a field panel method and by Dadone and Lerat with Euler codes. There is a good agreement between all calculations.

NACA 0012 airfoil at sub-sonic flow conditions

The next test case is the NACA 0012 airfoil at a free stream Mach number of $M = 0.63$ and at an angle of attack of 2°. The airfoil is discretized with 100 panels condensed at the leading and trailing edge with a cosine distribution. For the evaluation of the interpolation coefficients a total number of 1500 nodes was distributed as seen in Figure 6. The location of the

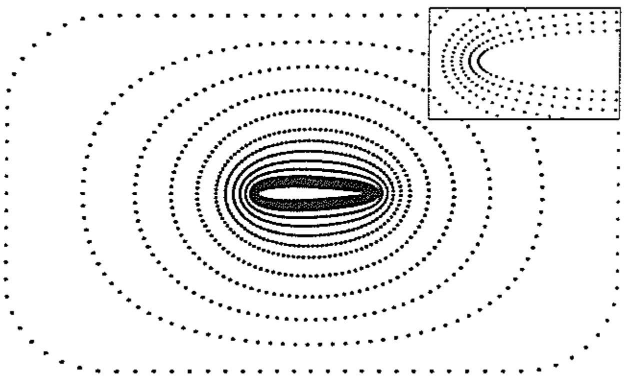

Figure 6: Distributed dual reciprocity nodes

dual reciprocity nodes is precomputed from a simple two–dimensional grid generator. In Figure 7 the computed pressure distribution of the airfoil is compared with the one obtained by Röttgermann, Sinclair and Theurer. The first two used a field panel method and the last–named used the dual reciprocity method for the solution of the full potential equation. Our result is in good agreement with the references. (The total lift coefficient with $C_l = 0.3319$ is in line with reference calculation cited in [8] giving values in the range of 0.3291 up to 0.336). Finally in Figures, 8 and 9 the calculated Mach number distribution and the field source strength distribution is presented These are in agreement with solutions given in [8]. It should be emphasized that these results are not sensitive to the exact distribution of the dual reciprocity nodes.

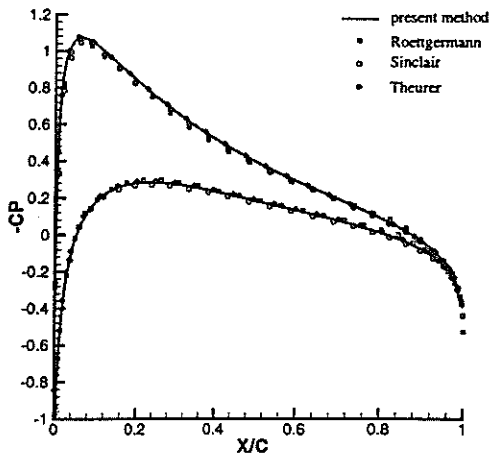

Figure 7: NACA 0012 surface pressure distribution.

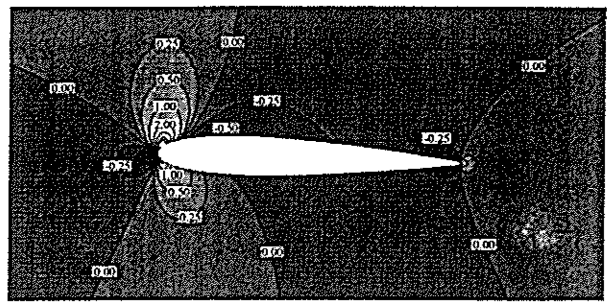

Figure 8: Field source strength distribution.

Conclusion

We have presented in this paper a possibility to use the dual reciprocity method for the solution of the full potential equation with a low order panel method. The DRM needs the evaluation of the velocity in the vicinity of the boundary which can not be calculated with a low order panel method accurately. This stems from the single vortices at the corner of each panel due to the discrete jump in doublet strength. Using cubic spline functions replaces the once discrete distributions with a smooth curve and simulates a higher order singularity distribution. This enables the evaluation of the velocity gradients in the vicinity of the boundary and calculations have shown that for the sub-sonic case one obtains very good agreement with field panel and Euler methods. As a remark it should be noted, that the result can be not be better than the discrete solution.

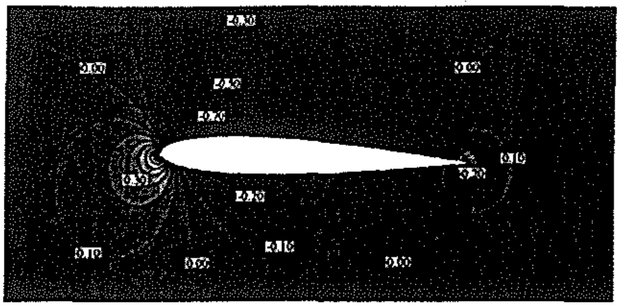

Figure 9: Field pressure coefficient distribution.

Further studies are going in two directions: Extension to the transonic flow regime and implementation of the approach in the existing 3–dimensional panel code by using bicubic spline functions.

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