EIGHTH EUROPEAN ROTORCRAFT FORUM

Paper No.3.3

AUTOMATIC GENERATION OF HELICOPTER ROTOR AEROELASTIC EQUATIONS OF MOTION

M.P. Gibbons †

G.T.S. Done*

† The Florida State University, Tallahassee, U.S.A.

* The City University, London, U.K.

August 31 through September 3, 1982

ASSOCIATION AERONAUTIQUE ET ASTRONAUTIQUE DE FRANCE

AUTOMATIC GENERATION OF HELICOPTER ROTOR AEROELASTIC EQUATIONS OF MOTION

M.P. Gibbons
The Florida State University, Tallahassee, U.S.A.
G.T.S. Done
The City University, London, U.K.

SUMMARY

A method is presented which allows much of the tedious algebraic manipulation required when formulating aeroelastic equations of motion to be effectively performed instead by the computer. The basic approach is outlined, and some results for a simple rotor model are presented which allow the numerical accuracy to be assessed.

1. INTRODUCTION

In order to generate reasonably comprehensive aeroelastic equations of motion for a helicopter rotor, several axes of reference are usually required in the analysis. Thus, a material point on a rotor blade can most conveniently have its position co-ordinates defined by means of successive axis transformations. For example, transformations between fixed and blade root rotating axes would be required, with further sets of axes with reference to the blade root to help account for the effects of twist and flapping and lagging bending flexibility along the blade. None of the transformations that change the co-ordinates of a point from one axis system to another may be particularly complicated, but when equations of motion to study blade dynamics are derived through use of, for example, Lagrange's equations the exercise can prove quite arduous for the aeroelastician.

It is common practice for the generalised co-ordinates in which the equations are expressed to be associated with specially chosen polynomial or other modes, or with previously calculated rotating or non-rotating normal modes. The derivation of the equations in these co-ordinates involves a certain amount of differentiation which when combined with the successive transformations leads to an enormous amount of work on paper for the analyst, if more than a few modes are involved. Furthermore, the possibility of an error creeping into the analysis is increased considerably for a multi-mode problem.

The approach adopted in the present paper is to generate the equations of motion automatically on the digital computer, as far as is possible. It relies on the fact that the above-mentioned series of transformations may be expressed as:

$$\underset{\sim}{R} = \underset{\sim\sim}{A}\underset{\sim}{r} + \underset{\sim}{B}$$

where $\underset{\sim}{R}$ is the position vector of a point on the blade in fixed co-ordinates, $\underset{\sim}{r}$ is the position vector of the point in blade co-ordinates and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are matrices that are functions of the modal co-ordinates, time and spanwise position. Thus, for a particular material point on a rotating blade, the co-ordinates relative to fixed axes can be computed for any given configuration and instant. Each of the matrices in the equations of motion has a contribution deriving from blade mass, and it can be shown, via Lagrange, that an element in one of these matrix contributions is the integral over the blade volume of a multiple involving blade density and certain differentials of the matrices $\underset{\sim}{A}$ and

$\underline{R}$. The differentials are with respect to q_i, $\dot{q}_i$ and/or t and the particular ones involved depend on the equation coefficient matrix being formed.

The above transformation may be used in order to generate air speeds and incidence angles relative to a local blade section, and through application of strip theory the aerodynamic generalised force contribution to the matrices can similarly be expressed as the integral over blade surface of multiples of certain differentials.

This knowledge permits the organisation of a systematic program to compute the coefficient matrices of the equations of motion automatically. The position vector transformation is programmed as a function which generates one position vector from the other, and the various differentiations of it are done numerically. The operations necessary are repeated for a set of points on the blade and the results integrated over all points concerned. The input data for mass and aerodynamics are arranged such that a simple spanwise integration is all that is needed. The contribution from blade stiffness is relatively straightforward and is dealt with separately.

The set of linearised equations in the desired generalised co-ordinates are therefore obtained at a given instant of time, and for the appropriate equilibrium values of the co-ordinates. The latter are found by solving the corresponding non-linear equations in which $\dot{\underline{q}}$ and $\ddot{\underline{q}}$ are set to zero.

The method is in the early stages of development and parallels that described by Lytwyn (1). In order to investigate the degree of accuracy required in the numerical differentiations and integrations, the approach has been tried out on a simple rotating rigid blade having flap, lag and pitch freedoms. Translation of the blade rotation axis is excluded in this model.

The paper outlines the basis of the method used and the investigations carried out regarding numerical accuracy. One set of results for the simple model is presented, and this includes and illustrates the variation of stability boundaries with important blade dynamic parameters. The results are seen to agree well with those obtained by more conventional means. In the light of continuing progress of the method, the future potential is outlined and discussed.

2. THE EQUATIONS OF MOTION IN DIFFERENTIAL FORM

2.1. General expressions

The system under consideration is taken to be a mathematical model of a rotor blade, complete rotor or the whole helicopter as desired. It is allowed to have n degrees of freedom, each associated with a generalised co-ordinate q_i and a corresponding mode shape. The mode shapes may be arbitrary, previously calculated or obtained by experiment, but they must all be definable mathematically or numerically. The equations of motion are obtained using Lagrange's equations, but instead of evolving them by writing down the energy expressions in full and carrying through the various differentiations involved within the algebra, we first of all express the equations in terms of the necessary differentiations. To do this, we must note that a modal approach implies that modal displacements are necessarily associated with axes which are non-inertial. As in the usual fully algebraic development, we need to express displacements at some stage in the proceedings with reference to inertial axes.

Thus, let the position vector $\underset{\sim}{R} \equiv \underset{\sim}{R}(\underset{\sim}{r},\underset{\sim}{q},t)$ express the position of a point in the system with reference to fixed axes. The position vector of the point with reference to axes suitable for describing the arbitrary or other given modes is $\underset{\sim}{r}$, and the generalised co-ordinates corresponding to the modes are denoted by the vector $\underset{\sim}{q}$. Time is denoted by t. The external instantaneous force per unit area (i.e. aerodynamic pressure in the present case) is given by $\underset{\sim}{F} \equiv \underset{\sim}{F}(\underset{\sim}{r},\underset{\sim}{q},\dot{\underset{\sim}{q}},t)$ and is again with reference to fixed axes. Lagrange's equations provide:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \qquad i = 1,2..n \tag{2.1}$$

where

$$T = \tfrac{1}{2}\int \dot{\underset{\sim}{R}}.\dot{\underset{\sim}{R}}dm \tag{2.2}$$

is the kinetic energy and the integral is over all elemental masses dm comprising the system,

$$Q_i = \int \underset{\sim}{F}.\frac{\partial \underset{\sim}{R}}{\partial q_i}\, dS \qquad i = 1,2..n \tag{2.3}$$

is the aerodynamic generalised force and is given by the integral over the surface S of the system, and U is the potential energy. Structural damping is neglected. We are primarily interested in aeroelastic stability in the small, so the solution of eqn.(1.1) may be conveniently separated into a steady-state solution and small perturbations about the steady-state, i.e. we let $\underset{\sim}{q} = \underset{\sim}{q}_o + \varepsilon \underset{\sim}{q}_1$ where ε is small.

The steady-state solution is obtained from a set of non-linear equations in $\underset{\sim}{q}_o$ as indicated below. The equations of motion in the perturbed co-ordinates $\underset{\sim}{q}_1$ will also be developed below, and is seen to be:

$$\underset{\sim}{P}\ddot{\underset{\sim}{q}}_1 + \underset{\sim}{Q}\dot{\underset{\sim}{q}}_1 + \underset{\sim}{R}\underset{\sim}{q}_1 = \underset{\sim}{0} \tag{2.4}$$

in which

$$\begin{aligned} \underset{\sim}{P} &\equiv \underset{\sim}{P}_m \\ \underset{\sim}{Q} &\equiv \underset{\sim}{Q}_m + \underset{\sim}{Q}_a \\ \underset{\sim}{R} &\equiv \underset{\sim}{R}_m + \underset{\sim}{R}_a + \underset{\sim}{R}_e \end{aligned} \tag{2.5}$$

where the suffices 'm', 'a' and 'e' indicate contributions deriving from the system mass, the aerodynamic forces and the elastic forces respectively. Gravitational and other potential terms are assumed absent.

2.2. Respective contributions

2.2.1. Mass Terms

The kinetic energy contribution can be shown to be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \int\left\{ \frac{d}{dt}\left(\dot{\underset{\sim}{R}}.\frac{\partial \dot{\underset{\sim}{R}}}{\partial \dot{q}_i}\right) - \dot{\underset{\sim}{R}}.\frac{\partial \dot{\underset{\sim}{R}}}{\partial q_i} \right\}dm \tag{2.6}$$

The expression may be fully expanded on substituting:

$$\dot{R} \equiv \frac{dR}{dt}$$

$$= \frac{\partial \underset{\sim}{R}}{\partial t} + \sum_i \frac{\partial \underset{\sim}{R}}{\partial q_i} \dot{q}_i \qquad (2.7)$$

Then, noting that $\underset{\sim}{R}$ is not a function of $\dot{q}_i$ and setting $q_i = q_{io} + \varepsilon q_{i1}$, $\dot{q}_i = \varepsilon \dot{q}_{i1}$, etc., the various terms may be separated out, and terms involving ε^2 and higher orders ignored. The contributions to the matrices premultiplying $\ddot{\underset{\sim}{q}}_1$, $\dot{\underset{\sim}{q}}_1$ and $\underset{\sim}{q}_1$ in eqn.(2.4) are found to be:

$$\underset{\sim}{P}_m = \int \left(\frac{\partial \underset{\sim}{R}}{\partial q_i}\right)_o . \left(\frac{\partial \underset{\sim}{R}}{\partial q_j}\right)_o dm \qquad (2.8)$$

$$\underset{\sim}{Q}_m = 2\int \left(\frac{\partial \underset{\sim}{R}}{\partial q_i}\right)_o . \left(\frac{\partial^2 \underset{\sim}{R}}{\partial q_i \partial t}\right)_o dm \qquad (2.9)$$

$$\underset{\sim}{R}_m = \int \left\{ \left(\frac{\partial \underset{\sim}{R}}{\partial q_i}\right)_o . \left(\frac{\partial^3 \underset{\sim}{R}}{\partial q_j \partial t^2}\right)_o + \left(\frac{\partial^2 \underset{\sim}{R}}{\partial q_i \partial q_j}\right)_o . \left(\frac{\partial^2 \underset{\sim}{R}}{\partial t^2}\right)_o \right\} dm \qquad (2.10)$$

where the suffix 'o' indicates that the differentials take their values at the steady-state condition. The contribution to the steady-state equations is found at the same time to be

$$\int \frac{\partial \underset{\sim}{R}}{\partial q_i} \cdot \frac{\partial^2 \underset{\sim}{R}}{\partial t^2} dm$$

2.2.2. Aerodynamic Terms

Using Taylor's series to expand $\underset{\sim}{F}$ and $\partial \underset{\sim}{R}/\partial q_i$ about the steady-state condition and substituting into the expression for generalised force, eqn.(2.3), the aerodynamic contributions to the matrices are readily determined as:

$$\underset{\sim}{Q}_a = -\int \left(\frac{\partial \underset{\sim}{R}}{\partial q_i}\right)_o . \left(\frac{\partial \underset{\sim}{F}}{\partial \dot{q}_j}\right)_o dS \qquad (2.11)$$

$$\underset{\sim}{R}_a = -\int \left\{ \left(\frac{\partial \underset{\sim}{R}}{\partial q_i}\right)_o . \left(\frac{\partial \underset{\sim}{F}}{\partial q_j}\right)_o + \left(\frac{\partial^2 \underset{\sim}{R}}{\partial q_i \partial q_j}\right)_o . \underset{\sim}{F}_o \right\} dS \qquad (2.12)$$

where, as before, the suffix 'o' implies values taken at the steady-state condition.

The contribution to the steady-state equation is found to be

$$-\int \frac{\partial \underset{\sim}{R}}{\partial q_i} . \underset{\sim}{F} \; dS.$$

2.2.3. Stiffness Terms

Usually in helicopter dynamic analysis, it is a relatively simple matter to form an expression for the potential energy, U, in terms of the generalised co-ordinates and the structural stiffness distributions. Transformations through several sets of axes are not involved and thus, it is not necessary to evolve the same type of procedure as in the case of the kinetic energy. The contributions to the equations of motion are therefore given directly in terms of U. Further explanation is provided in Section 4.3.

Substituting, as before, $\underset{\sim}{q} = \underset{\sim}{q}_o + \epsilon \underset{\sim}{q}$, into $\partial U/\partial q_i$ in eqn.(2.1) it follows that:

$$\underset{\sim}{R}_s = \frac{\partial^2 U}{\partial q_i \partial q_j} \tag{2.13}$$

and the contribution to the steady-state equations is $\underset{\sim}{R}_s \underset{\sim}{q}$.

2.3. Steady state equations

From sections 2.2.1, 2.2.2 and 2.2.3, the set of equations describing the steady-state is seen to be:

$$\int \frac{\partial \underset{\sim}{R}}{\partial q_i} \cdot \frac{\partial^2 \underset{\sim}{R}}{\partial t^2} \, dm - \int \frac{\partial \underset{\sim}{R}}{\partial q_i} \cdot \underset{\sim}{F} \, ds + \sum_j \frac{\partial^2 U}{\partial q_i \partial q_j} \cdot q_j = 0 \tag{2.14}$$

$$i = 1,2 \ldots n$$

This is a set of non-linear equations in the variables q_i, the solution q_{io} of which are independent of time if the helicopter rotor axis has components in the plane of the rotordisc neither of acceleration nor motion relative to the surrounding air. Otherwise, the solutions are periodic.

2.4. Equations of motion in perturbation co-ordinates

These are the equations (2.4) with the various matrix contributions given by eqns.(2.5), (2.8 - 2.13). Again, if there is no component in the plane of the rotor disc of acceleration or motion relative to the local air mass the matrix coefficients are constant, otherwise some or all are periodic.

3. FORM OF TRANSFORMATION FROM $\underset{\sim}{r}$ to $\underset{\sim}{R}$

A point in the moving system has position co-ordinates $\underset{\sim}{r}$ with respect to convenient axes, e.g. axes fixed to the blade root for a point on a rotor blade, but in the equations of motion the position co-ordinates relative to inertial axes, $\underset{\sim}{R}$, are required. The transformation from one set to another is quite standard and straightforward; nevertheless, because it has some bearing on subsequent computer program organisation, it is useful to examine the form the transformation takes.

In helicopter aeroelasticity, it is the rotor which is of primary interest, and Fig.1 shows a "semi-rigid" rotor blade in its deflected state on which is indicated sets of axes which enable the transformation from $\underset{\sim}{r}$ to $\underset{\sim}{R}$ to be made. The blade is considered initially coned and pretwisted with an offset axis, and applied angle of pitch. The elastic deflections are flap and lag of the blade axis and pitch or twist about the local blade axis. Deformation of a cross-section is excluded. The total flap $f_\beta(s)$ is given by the deflection due to coning angle and that from the flap modes, i.e.

$$f_\beta(s) = \beta_o s + \sum_i f_{\beta i}(s) q_i \tag{3.1}$$

where s is distance along the (possibly deformed) blade axis, β_o is the coning angle, q_i are the blade generalised co-ordinates and $f_{\beta i}(s)$ are the elastic flap modes of deformation in those co-ordinates. Similarly, the lag deflection $f_\zeta(s)$ is:

$$f_\zeta(s) = \sum_i f_{\zeta i}(s) q_i \qquad (3.2)$$

and the local blade section pitch $\theta(s)$ is:

$$\theta(s) = \theta_o(s) + \sum_i f_{\theta i}(s) q_i \qquad (3.3)$$

where $\theta_o(s)$ is the applied pitch plus built-in twist.

The overall transformation from $\underset{\sim}{r} \equiv \{x_4, y_4, z_4\}$ to $\underset{\sim}{R} \equiv \{X, Y, Z\}$ is obtained by successive transformations between sets of axes, and it can be shown that:

$$\underset{\sim}{R} = \underset{\approx}{A}\underset{\sim}{r} + \underset{\sim}{B} \qquad (3.4)$$

where

$$\underset{\sim}{A} = \underset{\sim}{T}_o \underset{\sim}{T}_2 \underset{\sim}{T}_3 \underset{\sim}{T}_4 \qquad (3.5)$$

$$\underset{\sim}{B} = \underset{\sim}{T}_o \left[\begin{bmatrix} s - \frac{1}{2}\int_o^s \beta(s)^2 ds \\ Y_o \\ Z_o + f_\beta(s) \end{bmatrix} + \underset{\sim}{T}_2 \begin{bmatrix} -\frac{1}{2}\int_o^s \zeta(s)^2 ds \\ f_\zeta(s) \\ 0 \end{bmatrix} \right] \qquad (3.6)$$

$$\underset{\sim}{T}_o = \begin{bmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.7)$$

$$\underset{\sim}{T}_2 = \begin{bmatrix} \cos\beta(s) & 0 & -\sin\beta(s) \\ 0 & 1 & 0 \\ \sin\beta(s) & 0 & \cos\beta(s) \end{bmatrix} \qquad (3.8)$$

$$\underset{\sim}{T}_3 = \begin{bmatrix} \cos\zeta(s) & -\sin\zeta(s) & 0 \\ \sin\zeta(s) & \cos\zeta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.9)$$

$$\underset{\sim}{T}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{bmatrix} \qquad (3.10)$$

$$\beta(s) = \beta_o + \sum_i \frac{\partial f_{\beta i}(s)}{\partial s} q_i \qquad (3.11)$$

$$\zeta(s) = \sum_i \frac{\partial f_{\zeta i}(s)}{\partial s} q_i \qquad (3.12)$$

ℓ is the axis-length of the blade, and Y_o and Z_o define the amount of offset of the blade axis. Note that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are functions of s, and not of y and z.

4. INTEGRATION OF TERMS IN THE EQUATIONS OF MOTION

The integrations required of the terms comprising the equations of motion (2.8) to (2.12) and that describing the steady-state (2.14), are either over the volume of the system (mass terms) or the surface area (aerodynamic terms). These can normally be much simplified, and since our main interest continues to be concentrated on a rotor blade, the integrations are now formulated for this part of the system.

4.1. Mass Terms

Inspection of eqns.(2.8) to (2.10) and (2.14) shows that the integral is of the elemental mass times the scalar product of two (3 x 1) vectors, each being a particular differential of $\underset{\sim}{R}$, e.g. $\partial R/\partial q_i$, $\partial^2 R/\partial q_j \partial t$, $\partial^2 R/\partial t^2$, etc. If the blade pitch is fixed (no applied cyclic pitch) then $\underset{\sim}{T}_o$ is the only time varying component in $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and so in eqns. (2.8) to (2.10) $\partial R/\partial t$ introduces a scalar Ω and a skew-symmetric matrix $\tilde{S}_1$ where $\Omega S_1 = \partial T_o/\partial t$, and $\partial^2 R/\partial t^2$ introduces a scalar Ω^2 and a symmetric matrix $\tilde{S}_2$ where $\Omega^2 \tilde{S}_2 = \partial^2 T_o/\partial t^2$. Then the only differentials required are $\partial R/\partial \tilde{q}_i$ and $\partial^2 \tilde{R}/\partial q_i \partial q_j$. Let the two particular differentials under consideration be denoted $\underset{\sim}{R}^{(p)}$ and $\underset{\sim}{R}^{(q)}$. Then:

$$
\begin{aligned}
\underset{\sim}{R}^{(p)}.\underset{\sim}{R}^{(q)} &= (\underset{\sim}{A}^{(p)}\underset{\sim}{r} + \underset{\sim}{B}^{(p)}).(\underset{\sim}{A}^{(q)}\underset{\sim}{r} + \underset{\sim}{B}^{(q)}) \\
&= \underset{\sim}{r}^T(\underset{\sim}{A}^{(p)T}\underset{\sim}{A}^{(q)})\underset{\sim}{r} \\
&+ (\underset{\sim}{B}^{(q)T}\underset{\sim}{A}^{(p)} + \underset{\sim}{B}^{(p)T}\underset{\sim}{A}^{(q)})\underset{\sim}{r} \\
&+ \underset{\sim}{B}^{(p)T}\underset{\sim}{B}^{(q)}
\end{aligned}
\tag{4.1}
$$

in which suffix 'T' indicates a transpose, and normal matrix algebra applies.

From the previous section, the last term is seen to be a scalar function of s, the spanwise blade co-ordinate, so that:

$$\int \underset{\sim}{B}^{(p)T}\underset{\sim}{B}^{(q)} dm = \int C^{pq}(x)m(x)dx \tag{4.2}$$

where x is written in place of s, $m(x)$ is the spanwise mass distribution and $C^{pq}(x)$ is the scalar function. The dx integral is over the complete span and the fact that C^{pq} is a function of x only relies on $\underset{\sim}{B}$ being independent of y and z. Similarly, the middle term of eqn. (4.1) leads to:

$$
\begin{aligned}
&\int (\underset{\sim}{B}^{(q)T}\underset{\sim}{A}^{(p)} + \underset{\sim}{B}^{(p)T}\underset{\sim}{A}^{(q)})\underset{\sim}{r}\, dm \\
&= \int (D_y^{pq} m(x)\bar{y} + D_z^{pq} m(x)\bar{z})dx
\end{aligned}
\tag{4.3}
$$

which is another scalar function of x, and in which $\bar{y}$ and $\bar{z}$ are the co-ordinates of the centre of mass of the blade section at x, and x_4 in $\underset{\sim}{r}$ is zero. Finally, in the first term of eqn.(4.1) the matrix multiple $A^{(p)T}A^{(q)}$ which is again independent of y and z can be written $\tilde{E}^{pq}$; this is comprised of 9 elements E_{xx}^{pq}, E_{xy}^{pq}, E_{xz}^{pq}, E_{yx}^{pq}......etc. Then it can be shown that:

$$\int \underset{\sim}{r}^T (\underset{\sim}{A}^{(p)T} \underset{\sim}{A}^{(q)}) \underset{\sim}{r} \, dm$$

$$= \int (E_{yy}^{pq} I_{zz} + E_{zz}^{pq} I_{yy} + 2E_{yz}^{pq} I_{yz}) dx \qquad (4.4)$$

in which I_{yy}, I_{xy} etc. are the usual blade section moments and products of inertia. These quantities are usually expressed as "flatwise" and "edgewise" inertias, so the axes of reference are synonymous with x_4, y_4 and z_4 in Fig.1.

If the applied blade pitch is time-invariant, then the differentiations of A and B involving time can be avoided by making use of the (3 x 3) matrices $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ mentioned below eqn.(4.1).

4.2. Aerodynamic Terms

4.2.1. General expressions

As for the mass terms in Section 4.1, inspection of eqns.(2.11) and (2.12) indicates that integrals over the lifting surface of the scalar product of two (3 x 1) vectors, one being a certain differential of R, and the other being either F_o or a differential of F. Again we denote the differentiations required by superfixes 'p' and 'q' so that:

$$\underset{\sim}{R}^{(p)} . \underset{\sim}{F}^{(q)} = (\underset{\sim}{A}^{(p)} \underset{\sim}{r} + \underset{\sim}{B}^{(p)}) . \underset{\sim}{F}^{(q)}$$

$$= (\underset{\sim}{A}^{(p)} \underset{\sim}{r} + \underset{\sim}{B}^{(p)}) . (\underset{\sim}{A}^{(q)} \underset{\sim}{P} + \underset{\sim}{AP}^{(q)}) \qquad (4.5)$$

in which $\underset{\sim}{P}$ are pressures with reference to the local blade axes and the transformation between F and P is made through $\underset{\sim}{A}$. See Fig.2. Thus integrations of terms like:

$$\underset{\sim}{P}^{(p)T} \underset{\sim}{A}^{(q)T} \underset{\sim}{A}^{(r)} \underset{\sim}{r} \quad \text{and} \quad \underset{\sim}{P}^{(p)T} \underset{\sim}{A}^{(q)T} \underset{\sim}{B}^{(r)}$$

need to be made, in which either 'p', 'q' or 'r' could symbolise zero differentiation and normal matrix representation is used. The first of these can be shown to be:

$$\int \underset{\sim}{P}^{(p)T} \underset{\sim}{A}^{(q)T} \underset{\sim}{A}^{(r)} \underset{\sim}{r} \, dS$$

$$= \int E_{yx}^{qr} x \int P_y^{p} dy dx + \int E_{zx}^{qr} x \int P_z^{p} dy dx$$

$$+ \int E_{zy}^{qr} \int P_z^{p} y dy dx \qquad (4.6)$$

and the second

$$\int \underset{\sim}{P}^{(p)T} \underset{\sim}{A}^{(q)T} \underset{\sim}{B}^{(r)} dS$$

$$= \int D_y^{qr} \int P_y^p dy dx + \int D_z^{qr} \int P_z^p dy dx \qquad (4.7)$$

where the integrals are over the lifting surface and E_{yx}, D_y, etc. are as in Section 4.1.

These expressions have certain terms absent due to simplifying assumptions. The spanwise pressure component P_x is assumed zero. A term involving $P_y{}^Py$ can be shown to be associated with the movement of points on the aerofoil surface towards and away from the axis (i.e. in the y-direction) when the aerofoil is rotated. As rotations are of small order, the movements occurring are second order "shortening" effects, and they and the associated term $P_y{}^Py$ may be neglected. The same applies for a term $P_z{}^Pz$ in which the shortening effect is in the z-direction. A term involving $P_y{}^Pz$ can be identified with the contribution to the aerodynamic moment of the edgewise forces on top and bottom surfaces of the aerofoil section. For thin aerofoils, these forces are drag-like and the overall contribution can reasonably be neglected. These simplifications are necessary because the aerodynamic forces would normally be provided as a lift, drag and moment on a (non-deformable) section, a detailed pressure distribution not being available.

The dy integrals (over the chord) are readily identified as:

$$\begin{aligned}
\int P_z dy &= L\cos\alpha + D\sin\alpha \\
\int P_y dy &= L\sin\alpha - D\cos\alpha \\
\int P_z y dy &= M
\end{aligned} \tag{4.8}$$

in which α is the section angle of incidence, and L, D and M are the section instantaneous aerodynamic lift, drag and moment about the axis of reference as in Fig.3.

4.2.2. Use of Strip Theory

Although more complicated formulations can be used, it is convenient to apply strip theory in obtaining L, D and M at a blade section. The evaluation of these forces requires, amongst other things, knowledge of the section aerodynamic coefficients C_L, C_D and C_M, the instantaneous incidence angle α and the instantaneous local (relative) airspeed V. The latter are given in terms of components of air velocity with respect to local blade section axes, i.e.:

$$\begin{aligned}
\alpha &= \tan^{-1}(-V_z/V_y) \\
V &= (V_y{}^2 + V_z{}^2)^{\frac{1}{2}}
\end{aligned} \tag{4.9}$$

as indicated in Fig.3. These components are elements of the relative air velocity vector $\underset{\sim}{V}$, where:

$$\underset{\sim}{V} = \underset{\sim}{A}^T(\dot{\underset{\sim}{A}}\underset{\sim}{r} + \dot{\underset{\sim}{B}} + \underset{\sim}{u}) \tag{4.10}$$

in which u is a vector providing velocity components due to motion of the helicopter as a whole through the surrounding air mass plus induced velocity through the rotor disc. It has to be remembered in eqn.(4.10) that the differentiations of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are "full" so that:

$$\dot{\underset{\sim}{A}} \equiv \frac{dA}{dt} = \left[\frac{\partial}{\partial t} + \sum_i \dot{q}_i \frac{\partial}{\partial q_i}\right] \underset{\sim}{A} \tag{4.11}$$

and similarly for $\dot{\underset{\sim}{B}}$. Thus, various differentials $A^{(p)}$ and $B^{(p)}$ are required in order to obtain $\underset{\sim}{V}$ and the instantaneous section $\tilde{L}$, D and M.

4.3. Stiffness Terms

It is assumed that the blade stiffness properties are provided as spanwise distributions of bending and torsional rigidities (i.e. $EI_{yy}(x)$, $EI_{zz}(x)$, $GJ(x)$) and orientation of section principal axes; then the bending and twisting strain energy can be easily formulated in terms of the arbitrary modes of eqns.(3.1) to (3.3). The contribution from track rod stiffness, if any, must also be included here. For a symmetric blade section the principal axes are aligned with the flat and edgewise blades axes (i.e. the local blade section axes), but the flap and lagwise directions are not normally coincident, so it is necessary to resolve the modal contributions through the local blade pitch angle in order to build up the bending energy terms.

The differentiations necessary (eqn.(2.13)) are of the mode shapes themselves and the integrations involved are over the span. A detailed description of the procedure is not given here, as it follows common practice.

5. COMPUTER PROGRAM

All the work done so far has been concerned with single rotor blade aeroelasticity and situations in which the variables in the steady-state case are independent of time, the coefficients in the aeroelastic stability equations also being constant. These equations are built up by summing the individual contributions, and a good guide to the overall program organisation is given by listing the sequential operations necessary to provide a mass term contribution, eqns. (2.8) to (2.10).

(i) Select matrix
(ii) Select coefficient
(iii) Select differentiation required
(iv) Select base values of t and q_i
(v) Select blade station, and read in mass and modal data
(vi) Form A and B
(vii) Compute required differentiation
(viii) Form the C, D and E coefficients
(ix) Perform required multiplications (eqns.4.2 to 4.4)
(x) Integrate over all blade stations
(xi) Repeat for all coefficients and matrices.

In order to form A and B in operation (vi) a routine is written involving the transformation matrices (3.6) to (3.10). The differentiation is done numerically using central difference formulae found in standard text books on numerical analysis, e.g. (2). In the program a choice can be made between 2-point (i.e. using one point on each side of the point of interest, and equivalent to obtaining the slope from a fitted parabola), 4-, 6- and 8-point formulae. The integration is performed using Simpson's rule although higher order Newton-Cotes procedures have been used also. The input data, i.e. the mass and inertia distribution and the mode shape data is calculated or interpolated at the chosen spanwise (integration point) stations.

The aerodynamic terms in eqns.(2.11) and (2.12) are obtained in a similar manner, with the additional complication of the need to form $\underset{\sim}{F}$, the vector of aerodynamic forces. From sections (4.2.1) and (4.2.2) it is seen that data on C_L, C_D and C_M at blade stations must be provided, and the full differentiations with respect to time of $\underset{\sim}{A}$ and $\underset{\sim}{B}$, namely $\dot{\underset{\sim}{A}}$ and $\dot{\underset{\sim}{B}}$ have to be formed in order to provide the incidence angle α and local relative air speed V.

The stiffness terms are relatively easily obtained and follow from integration of stiffness and modal input data.

The steady-state equations (2.14) are formed from the separate mass, aerodynamic and stiffness contributions for starting values of q_i (normally zero). The left hand sides will generally be non-zero initially and an iteration process obtains q_i for satisfaction of the equations. A standard NAG routine based on the Gauss-Newton method is used for the iteration which minimises the sum of the squares of the residuals, (3). The solutions or steady-state values q_{io} are then used as base values in estimating the matrix coefficients for the aeroelastic equations of motion.

6. NUMERICAL TESTS

Numerical tests have been carried out on a simple rotor model in order to establish confidence in the method and to assess the accuracy that can be expected. These tests are as follows:

(a) Comparison of matrix coefficients determined by the present method with those found analytically.

(b) Time independence check.

(c) Variation of accuracy with the number of points used in the central difference formulae for differentiation.

(d) Comparison of stability bounds with those obtained elsewhere.

These tests are described in some detail by Gibbons (4), but it is not necessary to do the same here; the main conclusions should be sufficient, and these are outlined in Section 6.2.

6.1. The Model

The mathematical model is that which applies to a single rotor blade rotating at constant speed about a fixed axis. The blade is itself rigid, and has a uniform mass distribution concentrated in its own plane. It can flap, lag and pitch about axes which coincide with or pass through the rotation axis against linear stiffnesses. The aerodynamic forces are governed by a constant drag coefficient C_D and constant lift-curve slope (providing C_L), and axi-symmetry prevails (i.e. no periodic terms). See Fig.4.

6.2. The Tests

To check the numerical approach against analysis as in (a) above, the degrees of freedom were restricted to flap and lag, for a given fixed amount of pitch. The matrix coefficients were derived by straightforward algebraic analysis and compared with those obtained numerically. There was exact agreement.

Because the test problem does not involve periodicity, the base value of time t necessary to start the computer process should make no difference to the final results. This formed the basis of the time independence check (b) and was tested on both the flap-lag model and the full three degree of freedom model, by obtaining coefficient matrices for various values of t. The results were unaffected by the base value of t chosen.

The accuracy dependence on the differentiation formula used (test c) was investigated using the flap-lag model, and it was found that at least five significant figures were correctly obtained in the matrix coefficients of the equations of motion for the 2-point formula, and much better than that for the 4- and 8-point formulae.

In all of these cases, the integration was carried out spanwise across only 5 spanwise stations only using the 5-point Newton-Cotes quadrature formula. More stations would be needed in more realistic problems. Also, because at the time of the numerical tests the chordwise integration was not effectively expressed through the overall mass and aerodynamic properties (e.g. section moments of inertia, section drag and moment, etc.), it too required treatment; in this case, three points chordwise were used.

Finally, a check was made on stability bounds with those which appear in Bramwell (5). The stability of a particular configuration in which the flag, lag and pitch stiffnesses vary was found by computing the eigenvalues provided by the equations of motion, and the boundaries were obtained by interpolation. Although the case of three degrees of freedom was investigated, the stability diagram for flap and lag shown in Fig.5 compares directly with Fig.11.3 in Bramwell's book. In this case the induced velocity parameter is $\lambda_i = 0.04$. The point at which the amount of fixed pitch just eliminates instability can also be found ($\theta_{oC} = 0.318$) which agrees with that given by eqn.(11.10) in Bramwell.

7. DISCUSSION

The automatic generation of helicopter aeroelastic equations of motion relies on the existence of a computer program in which confidence has been established. This confidence can only come from continued use of the program in practical circumstances, and so far, experience is limited. The numerical exercises described in Section 6 at least demonstrate that the method works, and provide an indication of the accuracy to be expected. The results have been sufficiently encouraging for the technique to be applied to much more realistic problems in helicopter dynamics, and it is hoped that the results will be available in the not too distant future.

In a sense, the method can quite easily be applied to complicated models, because all that is needed basically is the ability to formulate the successive transformation matrices in eqns.(3.5) to (3.10). This follows from the modal definition of the deformed state. The complication arises in the input of long strings of data, which comes hand-in-hand with a practical system description.

At present, the evaluation of each matrix coefficient on a practical rotor blade example of this type requires about 10 seconds of CPU time on a Honeywell Twin 66/60 computer. This is using 25 integration points spanwise and a 4-point differentiation formula. There is much scope for reducing the computer usage, but the accuracy desired in the final results needs first to be defined. So far, no real attempt has been made to improve the efficiency of the program.

Computing time increases when translational flight is considered, because periodic coefficients appear in the equations of motion. The present method produces coefficient matrices at the instant of time taken and thus for stability to be assessed, using "Floquet" for example, the matrices would need to be computed at several time instants.

Provided that unforeseen problems do not prove insuperable, the technique described in this paper represents a means of avoiding some of the more laborious algebraic manipulations necessary in forming aeroelastic equations of motion, thereby allowing the saving of a considerable amount of time and mental energy.

8. ACKNOWLEDGEMENTS

The authors would like to express their appreciation to Dr.S.P. King of Westland Helicopters Ltd., and Mr.A.J.Sobey of the R.A.E., Farnborough for their continued support and encouragement. This work is funded by the Ministry of Defence.

9. REFERENCES

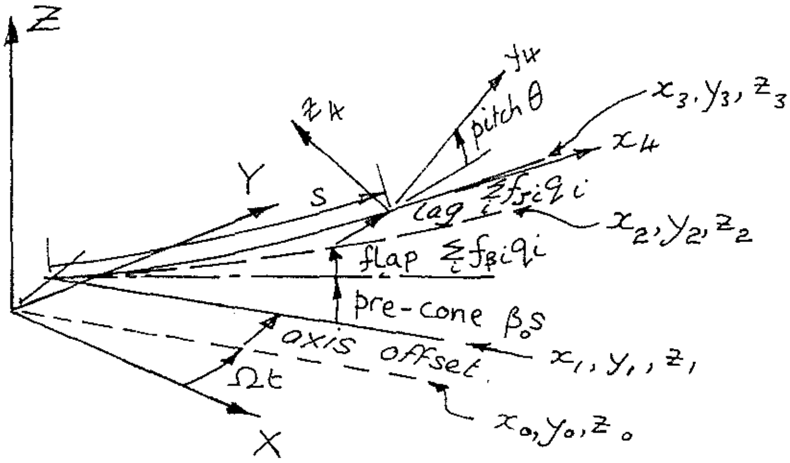
1. R.T. Lytwyn, Aeroelastic stability analysis of hingeless rotor helicopters in forward flight using blade and airframe normal modes. Proceedings of 36th Annual National Forum of the American Helicopter Society, Paper No.24, May, 1980.

2. Modern Computing Methods, 2nd ed. H.M.S.O. London, 1961.

3. P.E. Gill and W. Murray, Algorithms for the solution of the non-linear least-squares problem. SIAM J.Numer.Anal, 15, 5, Oct.1978.

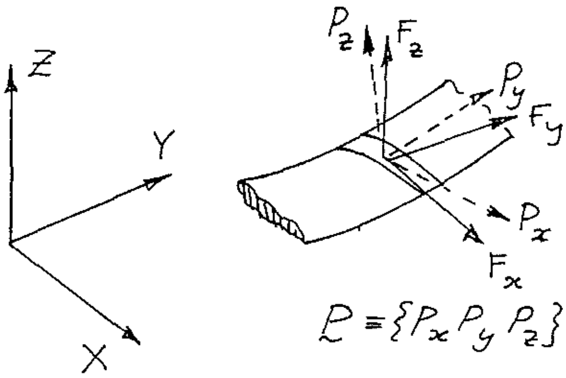
4. M.P. Gibbons, Automatic generation of helicopter rotor equations of motion, Unpublished report of The City University, Department of Mechanical Engineering, Aug.1981.

5. A.R.S. Bramwell, Helicopter dynamics, Edward Arnold, London, 1976.



$\underset{\sim}{R} \equiv \{X, Y, Z\}$ - fixed coords $\underset{\sim}{r} \equiv \{x_4, y_4, z_4\}$ - blade coords

Fig 1 - Sets of axes



$\underset{\sim}{P} \equiv \{P_x \, P_y \, P_z\}$ w.r.t. local blade axes.

$\underset{\sim}{F} \equiv \{F_x \, F_y \, F_z\}$ w.r.t. fixed axes

Fig 2 - Aerodynamic force components

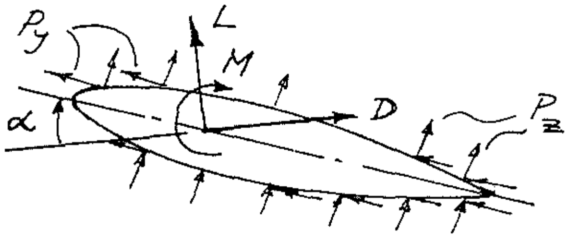
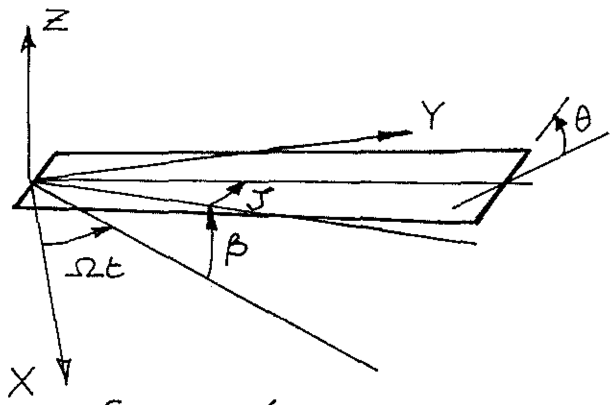


Fig 3. - Aerodynamic resultants.



Span = 6 m
Chord = 0·5 m
a_1 = 5·7
C_D = 0·05
Mass = 90 kg

Fig 4. – Model for numerical tests

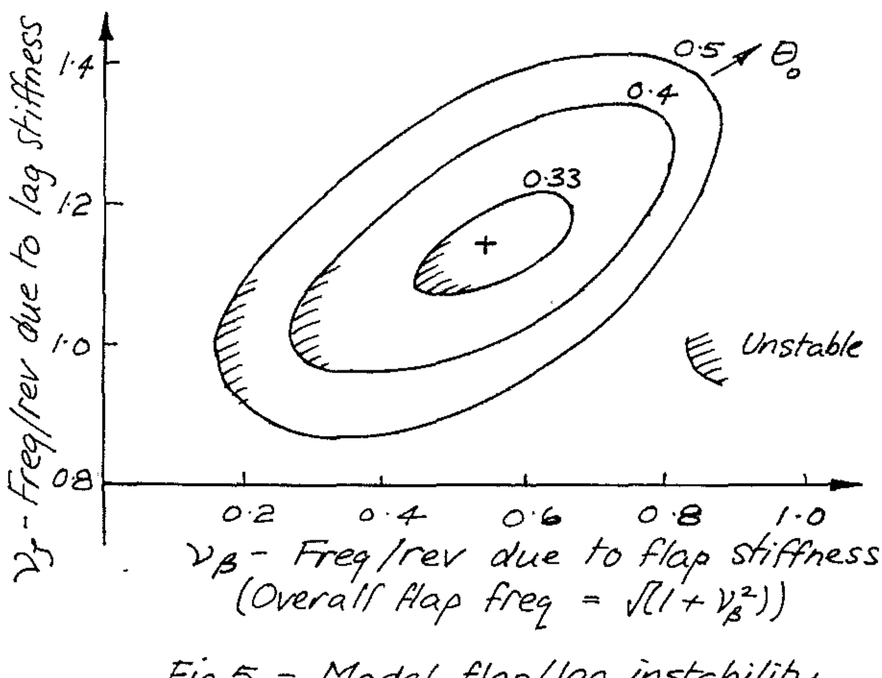


Fig 5 – Model flap/lag instability