SIXTH EUROPEAN ROTORCRAFT AND POWERED LIFT AIRCRAFT FORUM

Paper No. 71

OPTIMAL HIGHER HARMONIC BLADE PITCH CONTROL FOR MINIMUM VIBRATION OF A HINGED ROTOR

L. Beiner

Ben Gurion University
Beer Sheva, Israel

September 16-19, 1980
Bristol, England

THE UNIVERSITY, BRISTOL, BS8 1HR, ENGLAND

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Abstract

The paper presents a general analytical solution to the problem of of finding the optimal b/rev blade pitch control which minimizes the b-th blade lift harmonic of a given b-bladed hinged rotor under a given flight condition. It is proven that the resulting minimum is zero, thus suppressing the b/rev hub vertical force. The analysis assumes constant inflow ratio and blade lift-curve slope, linear twist, and second harmonic flapping. Explicit control laws are obtained for b = 2; 3; 4, showing as the major influence that the optimal blade pitch amplitude increases with airspeed as μ^b. Numerical examples carried out with b = 2-4, μ = 0.1-0.3, C_T/σ = 0.06-0.10, γ = 5-15, θ_1 = 6-10 deg, and $X/qd^2\sigma$ = 0.08-0.12, indicate that the required amount of optimal b/rev blade pitch amplitude increases also with C_T/σ, γ, and $X/qd^2\sigma$, and decreases with θ_1 and b, ranging from less than 1.5 deg for b = 2 to less than 0.04 deg for b = 4. These results are confirmed by wind-tunnel tests of 2- and 4-bladed hingeless rotors.

Notation

Standard NASA notation is used throughout the text. A few other symbols are defined below.

d	rotor diameter
J	performance index
q	dynamic pressure
t	total alternating blade lift coefficient
T_ψ	blade thrust at azimuth ψ
X	rotor propulsive force
T_{cn}	coefficient of cos $n\psi$ in expression for T_ψ
T_{sn}	coefficient of sin $n\psi$ in expression for T_ψ
α_r	blade element angle of attack
θ_{bP}	amplitude of optimal b/rev blade pitch
$\Delta\phi_{bP}$	phase angle of optimal b/rev blade pitch

*) Senior Lecturer, Mechanical Engineering Department. This work has been initiated during the author's stay as a MINERVA Visiting Scientist at the Lehrstuhl fuer Flugmechanik und Flugregelung, TU Munich (Prof. Dr.-Ing. G. Bruening)

1. Introduction

Higher harmonic control (HHC) superimposed on the conventional 0/rev and 1/rev blade pitch control has long been viewed as a promising approach to the reduction of vibratory hub loads and the resulting airframe vibration. The concept has been proven by both analysis and model testing (Refs. 1-5) which have shown that HHC is a powerful means of reducing vibratory loads. However, the amplitudes and phases of the blade pitch harmonics which give best results change significantly with flight conditions, so that research is now focused on the problem of inflight adjustment of the harmonics to provide optimum vibration reduction for all the flight conditions. In this connection, the goal of the present investigation is to extend the approach of Ref.6 to an arbitrary number of blades, so as to obtain explicit control laws yielding the optimal b/rev harmonic blade feathering required to suppress the b/rev hub vertical load of a b-bladed hinged rotor as function of the flight parameters. Such an analytical solution may be less accurate than numerical solutions based on more refined rotor models, but they are nevertheless instructive and indicative of the structure of the optimal control laws and the influence of various parameters. Simple expressions of similar structure for the optimal blade pitch control are obtained for b = 2, 3, and 4. Numerical applications performed with parameters in the range of usual applications resulted in satisfactory agreement with wind-tunnel test results.

2. Basic Assumptions

The present analysis is based on the following assumptions:

(1) constant inflow ratio λ
(2) constant blade lift-curve slope a
(3) linear blade twist θ_1
(4) untapered blades with zero flapping hinge offset
(5) neglection of reversed flow and tip losses
(6) second harmonic blade flapping
(7) neglection of powers of μ^4 and higher ($\mu \leqslant 0.3$)

3. Rotor Dynamic Analysis

By using tip-path plane axes in order to allow for deviations from the mean tip-path plane due to the higher harmonic flapping motion, the components of the forward velocity parallel and perpendicular to the mean tip-path plane are $\mu\Omega R$ and $\lambda\Omega R$, respectively. Thus, the velocities at a blade element are as follows:

- parallel to the mean tip-path plane and perpendicular to the blade

$$U_T = \Omega R\,(x + \mu \sin\psi) \tag{1}$$

- perpendicular to the mean tip-path plane and to the blade

$$U_p = \lambda\Omega R - xR\dot{\beta} - \mu\Omega R\cos\psi\,\beta \tag{2}$$

The blade pitch setting at any azimuth position can be expressed as

$$\theta = \theta_0 - \theta_1 x - A_1\cos\psi - B_1\sin\psi - A_b\cos b\psi - B_b\sin b\psi \tag{3}$$

where the b/rev harmonic blade pitch superimposed on the conventional 0/rev and 1/rev blade pitch control will serve as a tool for suppressing the b-th blade lift harmonic.

As it will be proven later by analysis, the amplitudes of the optimal HHC decrease very rapidly with order of harmonic, roughly by one order of magnitude per order of harmonic. Since the same holds true about the harmonics of the blade flapping motion (Ref. 7), it follows that for a consistent accuracy, a b/rev harmonic pitch control in Eq. (3) should be matched by a flapping angle expression including harmonics up to the same b-th order

$$\beta = a_0 - \sum_{k=2}^{b} (a_k \cos k\psi + b_k \sin k\psi) \tag{4}$$

where the first harmonic terms are zero by definition of axes. However, for b > 2 the algebra becomes quickly unmanageable and the results so complex that they tend to obscure the influence of various parameters which was set as primary goal of the investigation. For this reason, the flapping angle expansion was limited to the second harmonic

$$\beta = a_0 - a_2 \cos 2\psi - b_2 \sin 2\psi \tag{5}$$

which will imply a relative loss of accuracy for b > 2 and will limit the analysis to b = 4 due to the structure of the performance index (see Section 4). Differentiating (5) yields

$$\dot{\beta} = 2\Omega\,(a_2 \sin 2\psi - b_2 \cos 2\psi) \tag{6}$$

$$\ddot{\beta} = 4\Omega^2 (a_2 \cos 2\psi + b_2 \sin 2\psi) \tag{7}$$

Hence, the velocity through the disk is given by

$$U_P = \Omega R \big[\lambda - 2x\,(a_2 \sin 2\psi - b_2 \cos 2\psi) - \mu \cos\psi\,(a_0 - a_2 \cos 2\psi - b_2 \sin 2\psi)\big] \tag{8}$$

The differential thrust acting on a blade element is

$$dT_\psi = \frac{1}{2}\rho U_T^2 a \alpha_r c\,dr = \frac{1}{2}\rho a c R U_T^2 \left(\theta + \frac{U_P}{U_T}\right) dx \tag{9}$$

The corresponding differential thrust moment with respect to the flapping hinge is given by

$$dM_{T_\psi} = xR\,dT_\psi = \frac{1}{2}\rho a c R^2 U_T^2 \left(\theta + \frac{U_P}{U_T}\right) x\,dx \tag{10}$$

The blade thrust at any azimuth position is obtained by integrating Eq. (9) in conjunction with Eqs. (1), (3), and (8) along the blade

$$T_\psi = \frac{1}{2}\rho a c R \int_0^1 U_T^2 \left(\theta + \frac{U_P}{U_T}\right) dx =$$

$$= \frac{1}{2}\rho a c \Omega^2 R^3 \Big[\frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)\theta_0 + \frac{\lambda}{2} - \frac{1}{4}(1+\mu^2)\theta_1 - \frac{1}{2}\mu B_1 + \frac{1}{4}\mu^2 b_2$$

$$+ \cos\psi\Big[-\frac{1}{3}\left(1+\frac{3}{4}\mu^2\right) - \frac{1}{2}\mu a_0 - \frac{1}{4}\mu a_2\Big] + \sin\psi\Big[\mu\theta_0 + \lambda\mu - \frac{2}{3}\mu\theta_1$$

$$- \frac{1}{3}\left(1+\frac{9}{4}\mu^2\right) B_1 - \frac{1}{4}\mu b_2\Big]$$

$$+ \cos 2\psi \left(-\frac{1}{2}\mu^2\theta_0 + \frac{1}{4}\mu^2\theta_1 + \frac{1}{2}\mu B_1 + \frac{2}{3} b_2\right)$$

$$+ \sin 2\psi \left(-\frac{1}{2}\mu^2 a_0 - \frac{1}{2}\mu A_1 - \frac{2}{3} a_2\right)$$

$$+ \cos 3\psi \left(\frac{1}{4}\mu^2 A_1 + \frac{3}{4}\mu a_2\right) + \sin 3\psi \left(\frac{1}{4}\mu^2 B_1 + \frac{3}{4}\mu b_2\right)$$

$$+ \cos 4\psi \left(-\frac{1}{4}\mu^2 b_2\right) + \sin 4\psi \left(\frac{1}{4}\mu^2 a_2\right)$$

$$+ \cos\left[(b-2)\psi\right]\left(\frac{1}{4}\mu^2 A_b\right) + \sin\left[(b-2)\psi\right]\left(\frac{1}{4}\mu^2 B_b\right)$$

$$+ \cos\left[(b-1)\psi\right]\left(-\frac{1}{2}\mu B_b\right) + \sin\left[(b-1)\psi\right]\left(\frac{1}{2}\mu A_b\right)$$

$$+ \cos b\psi\left[-\frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)A_b\right] + \sin b\psi\left[-\frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)B_b\right]$$

$$+ \cos\left[(b+1)\psi\right]\left(\frac{1}{2}\mu B_b\right) + \sin\left[(b+1)\psi\right]\left(-\frac{1}{2}\mu A_b\right)$$

$$+ \cos\left[(b+2)\psi\right]\left(\frac{1}{4}\mu^2 A_b\right) + \sin\left[(b+2)\psi\right]\left(\frac{1}{4}\mu^2 B_b\right) \qquad (11)$$

The thrust moment at any azimuth position is obtained by integrating Eq. (10) in conjunction with Eqs. (1), (8), and (3) along the blade

$$M_{T_\psi} = \frac{1}{2}\rho a c R^2 \int_0^1 U_T^2 \left(\theta + \frac{U_P}{U_T}\right) x\, dx =$$

$$= \frac{1}{2}\rho a c \Omega^2 R^4 \left[\frac{1}{4}(1+\mu^2)\theta_0 - \frac{1}{5}\left(1+\frac{5}{6}\mu^2\right)\theta_1 + \frac{1}{3}\lambda - \frac{1}{3}\mu B_1 + \frac{1}{8}\mu^2 b_2\right]$$

$$+ \cos\psi\left[-\frac{1}{4}\left(1+\frac{1}{2}\mu^2\right)A_1 - \frac{1}{3}\mu a_0 - \frac{1}{6}\mu a_2\right]$$

$$+ \sin\psi\left[\frac{2}{3}\mu\theta_0 - \frac{1}{2}\mu\theta_1 + \frac{1}{2}\lambda\mu - \frac{1}{4}\left(1+\frac{3}{2}\mu^2\right)B_1 - \frac{1}{6}\mu b_2\right]$$

$$+ \cos 2\psi \left(-\frac{1}{4}\mu^2\theta_0 + \frac{1}{6}\mu^2\theta_1 + \frac{1}{3}\mu B_1 + \frac{1}{2} b_2\right)$$

$$+ \sin 2\psi \left(-\frac{1}{4}\mu^2 a_0 - \frac{1}{3}\mu A_1 - \frac{1}{2} a_2\right)$$

$$+ \cos 3\psi \left(\frac{1}{8}\mu^2 A_1 + \frac{1}{2}\mu a_2\right) + \sin 3\psi \left(\frac{1}{8}\mu^2 B_1 + \frac{1}{2}\mu b_2\right)$$

$$+ \cos 4\psi \left(-\frac{1}{8}\mu^2 b_2\right) + \sin 4\psi \left(\frac{1}{8}\mu^2 a_2\right)$$

$$+ \cos\left[(b-2)\psi\right]\left(\frac{1}{8}\mu^2 A_b\right) + \sin\left[(b-2)\psi\right]\left(\frac{1}{8}\mu^2 B_b\right)$$

$$+ \cos[(b-1)\psi](-\tfrac{1}{3}\mu B_b) + \sin[(b-1)\psi](\tfrac{1}{3}\mu A_b)$$

$$+ \cos b\psi[-\tfrac{1}{4}(1+\mu^2)A_b] + \sin b\psi[-\tfrac{1}{4}(1+\mu^2)B_b]$$

$$+ \cos[(b+1)\psi](\tfrac{1}{3}\mu B_b) + \sin[(b+1)\psi](-\tfrac{1}{3}\mu A_b)$$

$$+ \cos[(b+2)\psi](\tfrac{1}{8}\mu^2 A_b) + \sin[(b+2)\psi](\tfrac{1}{8}\mu^2 B_b) \tag{12}$$

Assuming an articulated rotor, the equilibrium condition about the flapping hinge (neglecting the weight moment) is

$$M_{T_\psi} - M_{CF} - M_I = 0 \tag{13}$$

where the centrifugal and inertia force moments are given by $M_{CF} = I\Omega^2\beta$ and $M_I = I\ddot{\beta}$, respectively. In nondimensional form and substituting for β and $\ddot{\beta}$ from Eqs. (6) and (7), the equilibrium condition becomes

$$\frac{M_{T_\psi}}{\frac{1}{2}\rho a c \Omega^2 R^4} = \frac{2}{\gamma}(a_0 + 3a_2\cos 2\psi + 3b_2 \sin 2\psi) \tag{14}$$

At this point b must be assigned a specific value in order to be able to solve Eq. (14) and proceed with the analysis, and by doing so it was found that both the steps of the solution and the structure of the results remain the same for b = 2, 3, and 4. Hence, it is this general algorithm of the solution that will be outlined below, with the detailed expressions of the coefficients being given in the Appendix.

Replacing b = 2; 3; 4 in Eqs. (11) and (12) and collecting like-terms, the thrust and thrust moment equations at any azimuth position can be rewritten as

$$T_\psi = T_0 + \sum_{n=1}^{b+2}(T_{cn}\cos n\psi + T_{sn}\sin n\psi) \tag{15}$$

$$M_{T_\psi} = M_{T_0} + \sum_{n=1}^{b+2}(M_{cn}\cos n\psi + M_{sn}\sin n\psi) \tag{16}$$

Comparing Eqs. (14) and (16) and equating the corresponding coefficients yields the following relationships

$$M_{T_0} - \frac{2a_0}{\gamma} = 0 \tag{17}$$

$$M_{c1} = 0 \tag{18}$$

$$M_{s1} = 0 \tag{19}$$

$$M_{c2} - \frac{6a_2}{\gamma} = 0 \tag{20}$$

$$M_{s2} - \frac{6b_2}{\gamma} = 0 \tag{21}$$

From Eq. (17) the inflow ratio is obtained as

$$\lambda = 3\left[-\frac{\theta_0}{4}(1+\mu^2) + \frac{\theta_1}{5}\left(1+\frac{5}{6}\mu^2\right) + \frac{\mu}{3}B_1 - \frac{\mu^2}{8}b_2 + \frac{2a_0}{\gamma}\underbrace{\left(-\frac{\mu^2}{8}A_2\right)}_{\text{for } b=2 \text{ only}}\right] \tag{22}$$

and by substituting it into Eq. (19), Eqs. (18)-(21) become free of λ . Next, the first harmonic of control A_1 and B_1 is obtained from Eqs. (18) and (19) and replaced in Eqs. (21) and (20), respectively; then, by solving for a_2 and b_2 and dividing out fractions and neglecting powers of μ^4 and higher (this will be consistently done in all subsequent calculations), the second harmonic flapping coefficients are finally obtained as (see Appendix for proper signs)

$$a_2 = \pm c_1(\mu,\gamma)A_b \pm c_2(\mu,\gamma)B_b + c_3(\mu,\gamma)\theta_1 + c_4'(\mu,\gamma)a_0 + c_4''(\mu,\gamma)\theta_0 \tag{23}$$

$$b_2 = \pm c_2(\mu,\gamma)A_b \pm c_1(\mu,\gamma)B_b + c_5(\mu,\gamma)\theta_1 + c_6'(\mu,\gamma)a_0 + c_6''(\mu,\gamma)\theta_0 \tag{24}$$

Substituting Eqs. (23) and (24) into the expressions of A_1 and B_1, the first harmonic blade pitch control takes the final form

$$A_1 = c_7(\mu,\gamma)A_b + c_8(\mu,\gamma)B_b + c_9(\mu,\gamma)\theta_1 + c_{10}'(\mu,\gamma)a_0 + c_{10}''(\mu,\gamma)\theta_0 \tag{25}$$

$$B_1 = c_{11}(\mu,\gamma)A_b + c_{12}(\mu,\gamma)B_b + c_{13}(\mu,\gamma)\theta_1 + c_{14}'(\mu,\gamma)a_0 + c_{14}''(\mu,\gamma)\theta_0 \tag{26}$$

The average rotor thrust is found by integrating Eq. (11) around the azimuth

$$T = \frac{b}{2\pi}\int_0^{2\pi} T_\psi\, d\psi = \frac{1}{2}\rho a b c \Omega^2 R^3\left[\frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)\theta_0 - \frac{1}{4}(1+\mu^2)\theta_1 + \frac{\lambda}{2} - \frac{1}{2}\mu B_1 + \frac{1}{4}\mu^2 b_2 \underbrace{\left(+\frac{1}{4}\mu^2 A_2\right)}_{\text{for } b=2 \text{ only}}\right] \tag{27}$$

In nondimensional form and by substituting for λ , b_2, and B_1 from Eqs. (22), (24), and (26), respectively, the rotor thrust becomes

$$\frac{2C_T}{\sigma a} = \mu^2(\tau_1 A_b + \tau_2 B_b) + \left(\frac{\mu^2}{16}c_5 + \frac{1}{20}\right)\theta_1 + \left(\frac{\mu^2}{16}c_6' + \frac{3}{\gamma}\right)a_0 + \left[\frac{\mu^2}{16}c_6'' - \frac{1}{24}(1-3\mu^2)\right]\theta_0 \tag{28}$$

Equation (11) can be now rewritten in nondimensional form as

$$\frac{2bC_{T_\psi}}{\sigma a} = \frac{2C_T}{\sigma a} + t(\psi) \tag{29}$$

where

$$t(\psi) = \frac{1}{\frac{1}{2}\rho a c \Omega^2 R^3}\sum_{n=1}^{b+2}(T_{cn}\cos n\psi + T_{sn}\sin n\psi) \tag{30}$$

is the total alternating blade lift and $C_{T_\psi} = T_\psi / \rho\pi R^2(\Omega R)^2$.

4. The Optimization Problem

The parameter optimization problem to be solved can be formulated as follows: for a given hinged rotor under a given flight condition, find the b/rev blade pitch control which minimizes the squared amplitude of the b-th blade lift harmonic, while keeping the rotor thrust at a prescribed value. That is, for given μ , γ , θ_1 , and λ (i.e., $X/qd^2\sigma$, see Section 6), find A_b and B_b which minimize the performance index

$$J = T_{cb}^2 + T_{sb}^2 \tag{31}$$

and yield a prescribed C_T/σ value.

Upon substitution of a_2, b_2, A_1, and B_1 from Eqs. (23)-(26), respectively, the components of the b-th blade lift harmonic take the form

$$T_{cb} = c_{15}(\mu,\gamma)A_b + c_{16}(\mu,\gamma)B_b + c_{17}(\mu,\gamma)\theta_1 + c_{18}'(\mu,\gamma)a_o + c_{18}''(\mu,\gamma)\theta_o \tag{32}$$

$$T_{sb} = c_{19}(\mu,\gamma)A_b + c_{20}(\mu,\gamma)B_b + c_{21}(\mu,\gamma)\theta_1 + c_{22}'(\mu,\gamma)a_o + c_{22}''(\mu,\gamma)\theta_o \tag{33}$$

with the following relations between coefficients

$$c_{19} = -c_{16} \tag{34}$$

$$c_{20} = c_{15} \tag{35}$$

Thus, the performance index becomes

$$J = (c_{15}A_b + c_{16}B_b + c_{17}\theta_1 + c_{18}'a_o + c_{18}''\theta_o)^2 + (-c_{16}A_b + c_{15}B_b + c_{21}\theta_1 + c_{22}'a_o + c_{22}''\theta_o)^2 \tag{36}$$

The sufficient conditions for a minimum are

$$\frac{\partial J}{\partial A_b} = 0\,, \qquad \frac{\partial J}{\partial B_b} = 0 \tag{37}$$

leading to the following two equations for A_b and B_b

$$(c_{15}^2 + c_{16}^2)A_b + (c_{15}c_{17} - c_{16}c_{21})\theta_1 + (c_{15}c_{18}' - c_{16}c_{22}')a_o + (c_{15}c_{18}'' - c_{16}c_{22}'')\theta_o = 0 \tag{38}$$

$$(c_{15}^2 + c_{16}^2)B_b + (c_{16}c_{17} + c_{15}c_{21})\theta_1 + (c_{16}c_{18}' + c_{15}c_{22}')a_o + (c_{16}c_{18}'' + c_{15}c_{22}'')\theta_o = 0 \tag{39}$$

and also

$$\frac{\partial^2 J}{\partial A_b^2} = \frac{\partial^2 J}{\partial B_b^2} = c_{15}^2 + c_{16}^2 > 0\,, \tag{40}$$

all the conditions being thus satisfied.

The optimal values of A_b and B_b are obtained from Eqs. (38) and (39), respectively, as

$$A_b = \frac{1}{e_1(\mu,\gamma)}\left[-e_2(\mu,\gamma)\,\theta_1 - e_3'(\mu,\gamma)a_0 - e_3''(\mu,\gamma)\,\theta_0\right] \tag{41}$$

$$B_b = \frac{1}{e_1(\mu,\gamma)}\left[-e_4(\mu,\gamma)\,\theta_1 - e_5'(\mu,\gamma)a_0 - e_5''(\mu,\gamma)\,\theta_0\right] \tag{42}$$

where

$$e_1(\mu,\gamma) = c_{15}^2 + c_{16}^2 \tag{43}$$

$$e_2(\mu,\gamma) = c_{15}c_{17} - c_{16}c_{21} \tag{44}$$

$$e_3'(\mu,\gamma) = c_{15}c_{18}' - c_{16}c_{22}' \tag{45}$$

$$e_3''(\mu,\gamma) = c_{15}c_{18}'' - c_{16}c_{22}'' \tag{46}$$

$$e_4(\mu,\gamma) = c_{16}c_{17} + c_{15}c_{21} \tag{47}$$

$$e_5'(\mu,\gamma) = c_{16}c_{18}' + c_{15}c_{22}' \tag{48}$$

$$e_5''(\mu,\gamma) = c_{16}c_{18}'' + c_{15}c_{22}'' \tag{49}$$

and by replacing them back into the performance index (36), one gets

$$J_{min} = 0 \tag{50}$$

This first important result proves that the optimal b/rev blade pitch control (41)-(42) does actually reduce to zero the b-th harmonic of the alternating blade lift, thus cancelling the b/rev hub vertical force of a b-bladed hinged rotor.

Inspection of the results for b = 2; 3; 4 reveals the following interesting relationship between coefficients

$$\frac{c_{17}}{\overline{c}_{17}} = \frac{c_{18}'}{\overline{c}_{18}'} = \frac{c_{18}''}{\overline{c}_{18}''} = \frac{c_{21}}{\overline{c}_{21}} = \frac{c_{22}'}{\overline{c}_{22}'} = \frac{c_{22}''}{\overline{c}_{22}''} = \mu^b \tag{51}$$

which allows to rewrite coefficients $e_2 \div e_5''$ as

$$e_2 = \mu^b(c_{15}\overline{c}_{17} - c_{16}\overline{c}_{21}) = \mu^b\overline{e}_2 \tag{52}$$

$$e_3' = \mu^b(c_{15}\overline{c}_{18}' - c_{16}\overline{c}_{22}') = \mu^b\overline{e}_3' \tag{53}$$

$$e_3'' = \mu^b(c_{15}\overline{c}_{18}'' - c_{16}\overline{c}_{22}'') = \mu^b\overline{e}_3'' \tag{54}$$

$$e_4 = \mu^b(c_{16}\overline{c}_{17} + c_{15}\overline{c}_{21}) = \mu^b\overline{e}_4 \tag{55}$$

$$e_5' = \mu^b(c_{16}\overline{c}_{18}' + c_{15}\overline{c}_{22}') = \mu^b\overline{e}_5' \tag{56}$$

$$e_5'' = \mu^b(c_{16}\overline{c}_{18}'' + c_{15}\overline{c}_{22}'') = \mu^b\overline{e}_5'' \tag{57}$$

Thus, the optimal b/rev blade pitch components become

$$A_b = \mu^b \left[\frac{-\bar{e}_2(\mu,\gamma)\theta_1 - \bar{e}_3'(\mu,\gamma)a_0 - e_3''(\mu,\gamma)\theta_0}{e_1(\mu,\gamma)}\right] \tag{58}$$

$$B_b = \mu^b \left[\frac{-\bar{e}_4(\mu,\gamma)\theta_1 - \bar{e}_5'(\mu,\gamma)a_0 - \bar{e}_5''(\mu,\gamma)\theta_0}{e_1(\mu,\gamma)}\right] \tag{59}$$

yielding the corresponding amplitude and phase angle as

$$\theta_{bp} = \sqrt{A_b^2 + B_b^2} = \mu^b \cdot f(\mu,\gamma,\theta_1,a_0,\theta_0) \tag{60}$$

$$\Delta\phi_{bp} = \tan^{-1}\frac{B_b}{A_b} = g(\mu,\gamma,\theta_1,a_0,\theta_0) \tag{61}$$

This second important result expresses in a simple analytical form a predominant effect which was reported by all previous experimental investigations; namely, that the b/rev blade pitch amplitude required to null the b-th blade lift harmonic is proportional to μ^b, thus increasing as airspeed increases and decreasing as the number of blades increases, at nearly invariant phase angle with respect to μ.

5. Evaluation of the Coning Angle and Collective Pitch

Coning angle and collective pitch can be now determined from the inflow ratio and rotor thrust equations (22) and (28), respectively, where λ and C_T/σ have known values for a given flight condition. Replacing b_2, B_1, A_b, and B_b with their expressions (24), (26), (58), and (59), respectively, it is found that all terms in A_b and B_b are multiplied by powers of μ^4, being thus negligible. After some manipulations, Eqs. (22) and (28) take finally the following simple form

$$d_1 a_0 + d_2\theta_0 = d_3 \tag{62}$$

$$d_4 a_0 + d_5\theta_0 = d_6 \tag{63}$$

where

$$d_1 = \frac{2}{\gamma}(1+2\mu^2) \tag{64}$$

$$d_2 = -\frac{1}{4}\left(1-\frac{5}{9}\mu^2\right) \tag{65}$$

$$d_3 = \frac{\lambda}{3} - \frac{1}{5}\left(1-\frac{1}{2}\mu^2\right)\theta_1 \tag{66}$$

$$d_4 = \frac{3}{\gamma} \tag{67}$$

$$d_5 = -\frac{1}{24}(1-3\mu^2) \tag{68}$$

$$d_6 = \frac{2C_T}{\sigma a} - \frac{1}{20}\theta_1 \tag{69}$$

and by solving for a_0 and θ_0 one gets

$$a_0 = f_1(\mu,\gamma)\,\theta_1 + f_2(\mu,\gamma)\,\lambda + f_3(\mu,\gamma)\,\frac{C_T}{\sigma a} \qquad (70)$$

$$\theta_0 = f_4(\mu)\,\theta_1 + f_5(\mu)\,\lambda + f_6(\mu)\,\frac{C_T}{\sigma a} \qquad (71)$$

where

$$f_1(\mu,\gamma) = -\frac{\gamma}{160}\left(1+\frac{35}{6}\mu^2\right) \qquad (72)$$

$$f_2(\mu,\gamma) = -\frac{\gamma}{48}\left(1-\frac{5}{2}\mu^2\right) \qquad (73)$$

$$f_3(\mu,\gamma) = \frac{3\gamma}{4}\left(1-\frac{1}{18}\mu^2\right) \qquad (74)$$

$$f_4(\mu) = \frac{3}{4}\left(1-\frac{1}{2}\mu^2\right) \qquad (75)$$

$$f_5(\mu) = -\frac{3}{2}\left(1+\frac{1}{2}\mu^2\right) \qquad (76)$$

$$f_6(\mu) = 6\left(1+\frac{5}{2}\mu^2\right) \qquad (77)$$

Substituting now the above expressions of a_0 and θ_0 into Eqs. (58) and (59), the optimal b/rev blade pitch control takes its final form

$$A_b = \mu^b\left[m_1(\mu,\gamma)\,\theta_1 + m_2(\mu,\gamma)\,\lambda + m_3(\mu,\gamma)\,\frac{C_T}{\sigma a}\right] \qquad (78)$$

$$B_b = \mu^b\left[m_4(\mu,\gamma)\,\theta_1 + m_5(\mu,\gamma)\,\lambda + m_6(\mu,\gamma)\,\frac{C_T}{\sigma a}\right] \qquad (79)$$

where

$$m_1(\mu,\gamma) = -\frac{1}{e_1}\left(\bar{e}_2 + \bar{e}_3' f_1 + \bar{e}_3'' f_4\right) \qquad (80)$$

$$m_2(\mu,\gamma) = -\frac{1}{e_1}\left(\bar{e}_3' f_2 + \bar{e}_3'' f_5\right) \qquad (81)$$

$$m_3(\mu,\gamma) = -\frac{1}{e_1}\left(\bar{e}_3' f_3 + \bar{e}_3'' f_6\right) \qquad (82)$$

$$m_4(\mu,\gamma) = -\frac{1}{e_1}\left(\bar{e}_4 + \bar{e}_5' f_1 + \bar{e}_5'' f_4\right) \qquad (83)$$

$$m_5(\mu,\gamma) = -\frac{1}{e_1}\left(\bar{e}_5' f_2 + \bar{e}_5'' f_5\right) \qquad (84)$$

$$m_6(\mu,\gamma) = -\frac{1}{e_1}\left(\bar{e}_5' f_3 + \bar{e}_5'' f_6\right) \qquad (85)$$

Equations (70) and (71) put into evidence the third important result of the analysis, by showing that the optimal b/rev blade pitch control components A_b and B_b vary linearly with θ_1, λ, and $C_T/\sigma a$, and nonlinearly with b, μ, and γ.

6. Numerical Examples and Discussion of Results

In order to gain some insight into the relative influence of various parameters, several numerical applications have been carried out with data covering the following ranges of usual operating conditions and rotor designs

$$\mu = 0.1 - 0.3$$

$$b = 2 - 4$$

$$C_T/\sigma = 0.06 - 0.10$$

$$\theta_1 = 6 - 10 \text{ deg}$$

$$X/qd^2\sigma = 0.08 - 0.12$$

The connection between λ - which actually appears in the formulas and the propulsive force requirement $X/qd^2\sigma$ is established as follows: assuming small α_D, one has $\sin\alpha_D \cong \alpha_D$, $\cos\alpha_D \cong 1$, $\mu = V/\Omega R$, and $X = T\alpha_D$, then

$$\frac{C_T/\sigma}{X/qd^2\sigma} = \frac{[T/\rho\pi R^2(\Omega R)^2]\cdot(\rho V^2/2)\cdot(2R)^2}{T\alpha_D} = \frac{2\mu^2}{\pi\alpha_D} \tag{86}$$

wherefrom

$$\alpha_D = -\frac{2\mu^2}{\pi}\frac{(X/qd^2\sigma)}{(C_T/\sigma)} \tag{87}$$

and by using the rotor-disk angle of attack expression

$$\alpha_D = \frac{\lambda}{\mu} + \frac{C_T}{2\mu\sqrt{\mu^2+\lambda^2}} \tag{88}$$

one gets for $\lambda << \mu$

$$\lambda = \mu\alpha_D - \frac{C_T}{2\mu} = -\frac{2\mu^3(X/qd^2\sigma)}{\pi(C_T/\sigma)} - \frac{C_T}{2\mu} \tag{89}$$

The results of the calculations indicate that - as one would expect on physical grounds - the required amount of optimal b/rev blade pitch amplitude:

- increases with μ, C_T/σ, γ, and $X/qd^2\sigma$ (i.e., with aerodynamic loading)

- decreases with θ_1 (twist reduces air loads) and b (less b/rev blade pitch is required to eliminate blade lift harmonics of higher order and smaller magnitude).

The main parameters affecting the optimal blade pitch control are - for a given rotor - the advance ratio μ and thrust coefficient C_T/σ , and for a given flight condition - the blade number b . Their influence is presented in Figs. 1 - 5, where it can be seen that the required blade pitch amplitude θ_{bp} increases as μ^b with μ and almost linearly with C_T/σ, while not exceeding values of between 0.04 deg for b = 2 up to 1.5 deg for b = 2. On the other hand, the corresponding phase angles depicted in Fig. 3 appear to be influenced mainly by the blade inertia number γ.

In order to check the validity of the present analysis, a comparison with experimental data has been sought. Since no test data pertaining to hinged rotors was available, the comparison has been done using the wind-tunnel test results of McHugh and Shaw (Ref. 3) obtained on a 2- and 4-blade hingeless rotor. Accordingly, the comparative values listed in Table 1 should be regarded as an order of magnitude check ensuring that no gross errors have occured during the lengthy manipulations of the present analysis.

Table 1. Comparison with windtunnel test results of a hingeless rotor

(γ = 12.4, θ_1 = 9 deg)

			b/rev pitch amplitude required to null b/rev shaft axial force, deg	
			Wind-tunnel tests (Ref.3, hingeless rotor)	Present method (hinged rotor)
		b = 2		
μ = 0.3	C_T/σ = 0.066	$X/qd^2\sigma$ = 0.13	0.60	0.77
μ = 0.3	C_T/σ = 0.122	$X/qd^2\sigma$ = 0.10	1.80 *)	1.63
		b = 4		
μ = 0.3	C_T/σ = 0.060	$X/qd^2\sigma$ = 0.10	0.030	0.022

*) test value obtained by extrapolation

7. Concluding Remarks

Using a simple rotor model, closed form expressions of the optimal b/rev blade pitch required to suppress the b/rev hub axial force of a b-bladed hinged rotor are derived for b = 2; 3; 4, allowing to discern the influence of various parameters. As a predominant influence, the required b/rev blade pitch amplitude is shown to vary as μ^b (increase with μ and decrease with b), while also increasing with C_T/σ , γ , and $X/qd^2\sigma$ and decreasing with θ_1. For flight conditions and rotor characteristics in the range of usual applications, the optimal pitch amplitude does not exceed values comprised between 0.04 deg for b = 4 up to 1.5 deg for b = 2. A limited comparison with wind-tunnel test results of a 2- and 4-bladed hingeless rotor is performed as an order of magnitude check, resulting in satisfactory agreement.

8. References

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3. F.J. McHugh, J. Shaw, Jr., Benefits of Higher-Harmonic Blade Pitch: Vibration Reduction, Blade-Load Reduction, and Performance Improvement. AHS Mideast Region Symposium on Rotor Technology, August 1976.

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Appendix

b = 2; 3; 4

$$c_3 = -\frac{\mu^2}{5\Delta}\left(1-\frac{14}{9}\mu^2\right) \tag{A-1}$$

$$c_4' = \frac{\mu^2}{\Delta}k_1 \tag{A-2}$$

$$c_4'' = \frac{\mu^2}{\Delta}k_2 \tag{A-3}$$

$$c_5 = \frac{\mu^2}{10\Delta}\left(1-2\mu^2\right) \tag{A-4}$$

$$c_6' = \frac{\mu^2}{\Delta}k_3 \tag{A-5}$$

$$c_6'' = \frac{\mu^2}{\Delta}k_4 \tag{A-6}$$

$$\Delta = \frac{12}{\gamma}\left(1+\frac{4}{9}\mu^2\right) + \frac{\gamma}{12}\left(1-\frac{4}{9}\mu^2\right) \tag{A-7}$$

$$k_1 = \frac{7\gamma}{216}\left(1-\frac{8}{7}\mu^2\right) + \frac{8}{\gamma}\left(1+\frac{17}{18}\mu^2\right) \tag{A-8}$$

$$k_2 = \frac{5}{18}\left(1-\frac{79}{45}\mu^2\right) \tag{A-9}$$

$$k_3 = \frac{7}{18}\left(1-\frac{44}{63}\mu^2\right) \tag{A-10}$$

$$k_4 = -\frac{5\gamma}{216}\left(1-\frac{11}{5}\mu^2\right) \tag{A-11}$$

$$c_9 = -\frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_3 \tag{A-12}$$

$$c_{10}' = -\frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)(c_4' + 2) \tag{A-13}$$

$$c_{10}'' = -\frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_4'' \tag{A-14}$$

$$c_{13} = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_5 - \frac{4\mu}{3}\left(1-\frac{3}{4}\mu^2\right) \tag{A-15}$$

$$c_{14}' = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_6' \tag{A-16}$$

$$c_{14}'' = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_6'' + \frac{7\mu}{6}\left(1-\frac{11}{14}\mu^2\right) \tag{A-17}$$

<u>b = 2</u>

$$c_1 = \frac{1}{2\Delta}\left(1-\frac{1}{3}\mu^2\right) \tag{A-18}$$

$$c_2 = \frac{\gamma}{24\Delta}\left(1-\frac{7}{9}\mu^2\right) \tag{A-19}$$

$$a_2 = -c_1 A_2 - c_2 B_2 + c_3\theta_1 + c_4' a_0 + c_4''\theta_0 \tag{A-20}$$

$$b_2 = c_2 A_2 - c_1 B_2 + c_5\theta_1 + c_6' a_0 + c_6''\theta_0 \tag{A-21}$$

$$c_7 = \frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_1 \tag{A-22}$$

$$c_8 = \frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)(c_2-2) \tag{A-23}$$

$$c_{11} = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_2 + \frac{4\mu}{3}\left(1-\frac{1}{16}\mu^2\right) \tag{A-24}$$

$$c_{12} = \frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_1 \tag{A-25}$$

$$T_{c2} = -\frac{\mu^2\theta_0}{2} + \frac{\mu^2\theta_1}{4} + \frac{\mu B_1}{2} - \frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)A_2 + \frac{2}{3}b_2 =$$

$$= c_{15}A_2 + c_{16}B_2 + c_{17}\theta_1 + c_{18}' a_0 + c_{18}''\theta_0 \tag{A-26}$$

$$T_{s2} = -\frac{\mu^2 a_0}{2} - \frac{\mu A_1}{2} - \frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)B_2 - \frac{2}{3}a_2 =$$

$$= c_{19}A_2 + c_{20}B_2 + c_{21}\theta_1 + c_{22}' a_0 + c_{22}''\theta_0 \tag{A-27}$$

$$c_{15} = \frac{1}{3}\left(1-\frac{1}{2}\mu^2\right)(2c_2-1) = -\frac{1}{3}\left[1-\frac{\gamma}{12\Delta} - \frac{1}{2}\mu^2\left(1-\frac{23\gamma}{108\Delta}\right)\right] \tag{A-28}$$

$$c_{16} = -\frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c_1 = -\frac{1}{3\Delta}\left(1-\frac{5}{6}\mu^2\right) \tag{A-29}$$

$$c_{17} = \frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c_5 - \frac{3\mu^2}{20}\left(1-2\mu^2\right) =$$

$$= \mu^2 \left\{ -\frac{3}{20}(1-2\mu^2)\left[1-\frac{4}{9\Delta}\left(1-\frac{1}{2}\mu^2\right)\right]\right\} = \mu^2 \bar{c}_{17} \qquad \text{(A-30)}$$

$$c'_{18} = \frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c'_6 + \frac{6\mu^2}{\gamma}\left(1+\frac{1}{2}\mu^2\right) =$$

$$= \mu^2\left[\frac{2}{3\Delta}\left(1-\frac{1}{2}\mu^2\right)k_3 + \frac{6}{\gamma}\left(1+\frac{1}{2}\mu^2\right)\right] = \mu^2 \bar{c}'_{18} \qquad \text{(A-31)}$$

$$c''_{18} = \frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c''_6 + \frac{\mu^2}{12}\left(1-\frac{11}{12}\mu^2\right) =$$

$$= \mu^2\left[\frac{2k_4}{3\Delta}\left(1-\frac{1}{2}\mu^2\right) + \frac{1}{12}\left(1-\frac{11}{12}\mu^2\right)\right] = \mu^2 \bar{c}''_{18} \qquad \text{(A-32)}$$

$$c_{19} = -c_{16} \qquad \text{(A-33)}$$

$$c_{20} = c_{15} \qquad \text{(A-34)}$$

$$c_{21} = -\frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c_3 = \mu^2\left[\frac{2}{15\Delta}\left(1-\frac{37}{18}\mu^2\right)\right] = \mu^2 \bar{c}_{21} \qquad \text{(A-35)}$$

$$c'_{22} = -\frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c'_4 + \frac{\mu^2}{6}\left(1-2\mu^2\right) =$$

$$= \mu^2\left[-\frac{2k_1}{3\Delta}\left(1-\frac{1}{2}\mu^2\right) + \frac{1}{6}\left(1-2\mu^2\right) = \mu^2 \bar{c}'_{22} \qquad \text{(A-36)}$$

$$c''_{22} = -\frac{2}{3}\left(1-\frac{1}{2}\mu^2\right)c''_4 = \mu^2\left[-\frac{2k_2}{3\Delta}\left(1-\frac{1}{2}\mu^2\right)\right] = \mu^2 \bar{c}''_{22} \qquad \text{(A-37)}$$

<u>b = 3</u>

$$c_1 = \frac{\gamma\mu}{18\Delta}\left(1-\frac{1}{2}\mu^2\right) \qquad \text{(A-38)}$$

$$c_2 = \frac{2\mu}{3\Delta}\left(1-\frac{1}{18}\mu^2\right) \qquad \text{(A-39)}$$

$$a_2 = c_1 A_3 - c_2 B_3 + c_3\theta_1 + c'_4 a_0 + c''_4\theta_0 \qquad \text{(A-40)}$$

$$b_2 = c_2 A_3 + c_1 B_3 + c_5\theta_1 + c'_6 a_0 + c''_6\theta_0 \qquad \text{(A-41)}$$

$$c_7 = -\frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_1 + \frac{1}{2}\mu^2\left(1-\frac{1}{2}\mu^2\right) \qquad \text{(A-42)}$$

$$c_8 = \frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_2 \qquad \text{(A-43)}$$

$$c_{11} = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_2 \qquad \text{(A-44)}$$

$$c_{12} = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_1 + \frac{\mu^2}{2}\left(1+\frac{1}{2}\mu^2\right) \qquad \text{(A-45)}$$

$$T_{c3} = \frac{1}{4}\mu^2 A_1 - \frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)A_3 + \frac{3}{4}\mu a_2 =$$

$$= c_{15} A_3 + c_{16} B_3 + c_{17} \theta_1 + c_{18}' a_o + c_{18}'' \theta_o \qquad \text{(A-46)}$$

$$T_{53} = \frac{1}{4}\mu^2 B_1 - \frac{1}{3}\left(1 + \frac{3}{2}\mu^2\right) B_3 + \frac{3}{4}\mu b_2 =$$

$$= c_{19} A_3 + c_{20} B_3 + c_{21}\theta_1 + c_{22}' a_o + c_{22}'' \theta_o \qquad \text{(A-47)}$$

$$c_{15} = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_1 - \frac{1}{3}\left(1 + \frac{3}{2}\mu^2\right) = -\frac{1}{3}\left[1 + \frac{3}{2}\mu^2\left(1 - \frac{\gamma}{12\Delta}\right)\right] \qquad \text{(A-48)}$$

$$c_{16} = -\frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_2 = -\frac{\mu^2}{2\Delta}\left(1 - \frac{5}{18}\mu^2\right) \qquad \text{(A-49)}$$

$$c_{17} = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_3 = \mu^3\left[-\frac{3}{20\Delta}\left(1 - \frac{16}{9}\mu^2\right)\right] = \mu^3 \bar{c}_{17} \qquad \text{(A-50)}$$

$$c_{18}' = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_4' - \frac{\mu^3}{3}\left(1 - \frac{1}{2}\mu^2\right) =$$

$$= \mu^3\left[\frac{3k_1}{4\Delta}\left(1 - \frac{2}{9}\mu^2\right) - \frac{1}{3}\left(1 - \frac{1}{2}\mu^2\right)\right] = \mu^3 \bar{c}_{18}' \qquad \text{(A-51)}$$

$$c_{18}'' = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_4'' = \mu^3\left[\frac{3k_2}{4\Delta}\left(1 - \frac{2}{9}\mu^2\right)\right] = \mu^3 \bar{c}_{18}'' \qquad \text{(A-52)}$$

$$c_{19} = -c_{16} \qquad \text{(A-53)}$$

$$c_{20} = c_{15} \qquad \text{(A-54)}$$

$$c_{21} = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_5 - \frac{\mu^3}{5}\left(1 - \frac{3}{4}\mu^2\right) =$$

$$= \mu^3\left[\frac{3}{40\Delta}\left(1 - \frac{20}{9}\mu^2\right) - \frac{1}{5}\left(1 - \frac{3}{4}\mu^2\right)\right] = \mu^3 \bar{c}_{21} \qquad \text{(A-55)}$$

$$c_{22}' = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_6' + \frac{3\mu^3}{\gamma}\left(1 + \frac{1}{2}\mu^2\right) =$$

$$= \mu^3\left[\frac{3k_3}{4\Delta}\left(1 - \frac{2}{9}\mu^2\right) + \frac{3}{\gamma}\left(1 + \frac{1}{2}\mu^2\right)\right] = \mu^3 \bar{c}_{22}' \qquad \text{(A-56)}$$

$$c_{22}'' = \frac{3\mu}{4}\left(1 - \frac{2}{9}\mu^2\right) c_6'' + \frac{7\mu^3}{24}\left(1 - \frac{11}{14}\mu^2\right) =$$

$$= \mu^3\left[\frac{3k_4}{4\Delta}\left(1 - \frac{2}{9}\mu^2\right) + \frac{7}{24}\left(1 - \frac{11}{14}\mu^2\right)\right] = \mu^3 \bar{c}_{22}'' \qquad \text{(A-57)}$$

<u>b = 4</u>

$$c_1 = \frac{\mu^2}{4\Delta}\left(1 + \frac{4}{9}\mu^2\right) \qquad \text{(A-58)}$$

$$c_2 = \frac{\gamma\mu^2}{48\Delta} \qquad \text{(A-59)}$$

$$a_2 = c_1 A_4 + c_2 B_4 + c_3 \theta_1 + c_4' a_o + c_4'' \theta_o \qquad \text{(A-60)}$$

$$b_2 = -c_2 A_4 + c_1 B_4 + c_5 \theta_1 + c_6' a_0 + c_6'' \theta_0 \tag{A-61}$$

$$c_7 = -\frac{2\mu}{3}\left(1 - \frac{1}{2}\mu^2\right) c_1 \tag{A-62}$$

$$c_8 = -\frac{2\mu}{3}\left(1 - \frac{1}{2}\mu^2\right) c_2 \tag{A-63}$$

$$c_{11} = \frac{2\mu}{3}\left(1 + \frac{13}{8}\mu^2\right) c_2 \tag{A-64}$$

$$c_{12} = -\frac{2\mu}{3}\left(1 + \frac{13}{8}\mu^2\right) c_1 \tag{A-65}$$

$$T_{c4} = -\frac{1}{3}\left(1 + \frac{3}{2}\mu^2\right) A_4 - \frac{1}{4}\mu^2 b_2 =$$

$$= c_{15} A_4 + c_{16} B_4 + c_{17} \theta_1 + c_{18}' a_0 + c_{18}'' \theta_0 \tag{A-66}$$

$$T_{s4} = -\frac{1}{3}\left(1 + \frac{3}{2}\mu^2\right) B_4 + \frac{1}{4}\mu^2 a_2 =$$

$$= c_{19} A_4 + c_{20} B_4 + c_{21} \theta_1 + c_{22}' a_0 + c_{22}'' \theta_0 \tag{A-67}$$

$$c_{15} = \frac{1}{4}\mu^2 (c_2 - 2) - \frac{1}{3} = -\frac{1}{3}\left[1 - \frac{3\mu^2}{4}\left(\frac{\gamma \mu^2}{48\Delta} - 2\right)\right] \tag{A-68}$$

$$c_{16} = -\frac{1}{4}\mu^2 c_1 = -\frac{\mu^4}{16\Delta}\left(1 + \frac{4}{9}\mu^2\right) \tag{A-69}$$

$$c_{17} = -\frac{1}{4}\mu^2 c_5 = \mu^4\left[-\frac{1}{40\Delta}\left(1 - 2\mu^2\right)\right] = \mu^4 \bar{c}_{17} \tag{A-70}$$

$$c_{18}' = -\frac{1}{4}\mu^2 c_6' = \mu^4\left(-\frac{k_3}{4\Delta}\right) = \mu^4 \bar{c}_{18}' \tag{A-71}$$

$$c_{18}'' = -\frac{1}{4}\mu^2 c_6'' = \mu^4\left(-\frac{k_4}{4\Delta}\right) = \mu^4 \bar{c}_{18}'' \tag{A-72}$$

$$c_{21} = \frac{1}{4}\mu^2 c_3 = \mu^4\left[-\frac{1}{20\Delta}\left(1 - \frac{14}{9}\mu^2\right)\right] = \mu^4 \bar{c}_{21} \tag{A-73}$$

$$c_{22}' = \frac{1}{4}\mu^2 c_4' = \mu^4\left(\frac{k_1}{4\Delta}\right) = \mu^4 \bar{c}_{22}' \tag{A-74}$$

$$c_{22}'' = \frac{1}{4}\mu^2 c_4'' = \mu^4\left(\frac{k_2}{4\Delta}\right) = \mu^4 \bar{c}_{22}'' \tag{A-75}$$

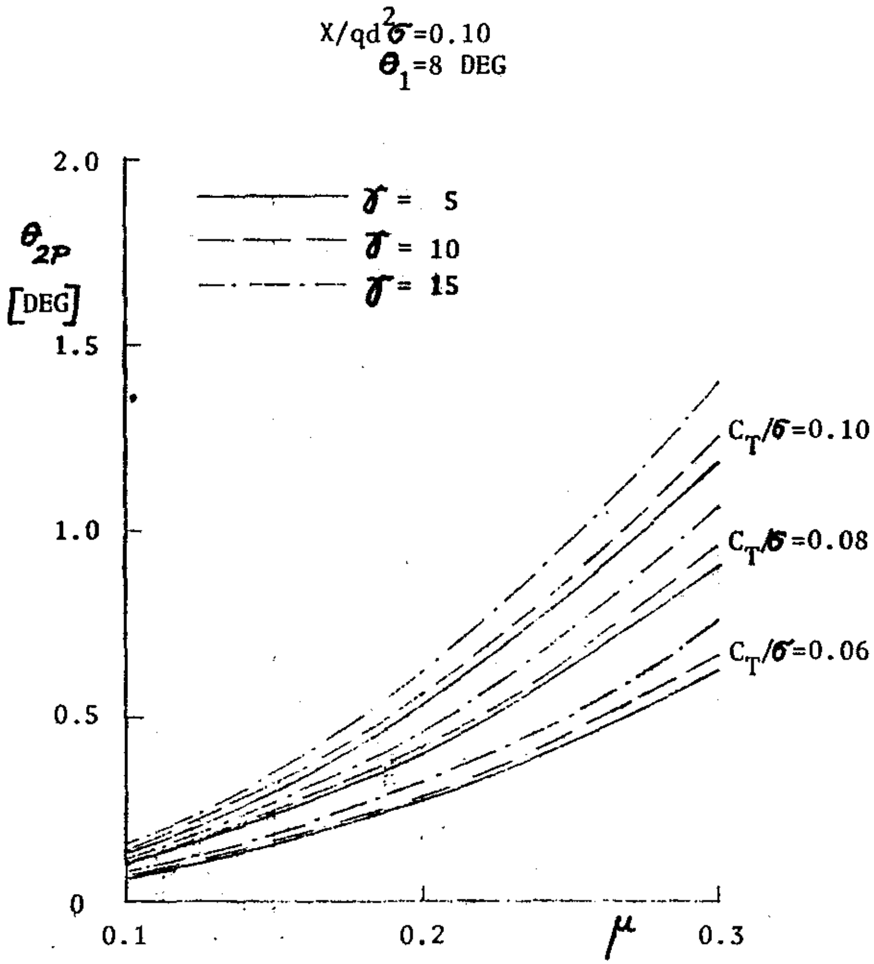

FIGURE 1. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VS. ADVANCE RATIO

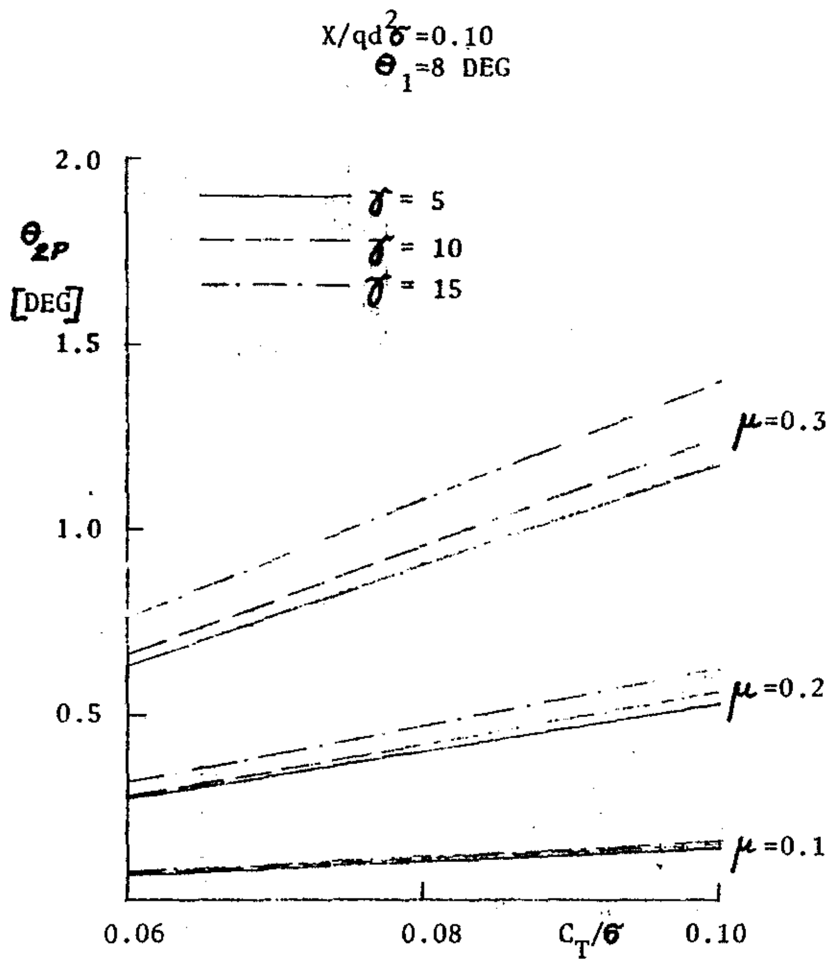

FIGURE 2. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VS. THRUST COEFFICIENT

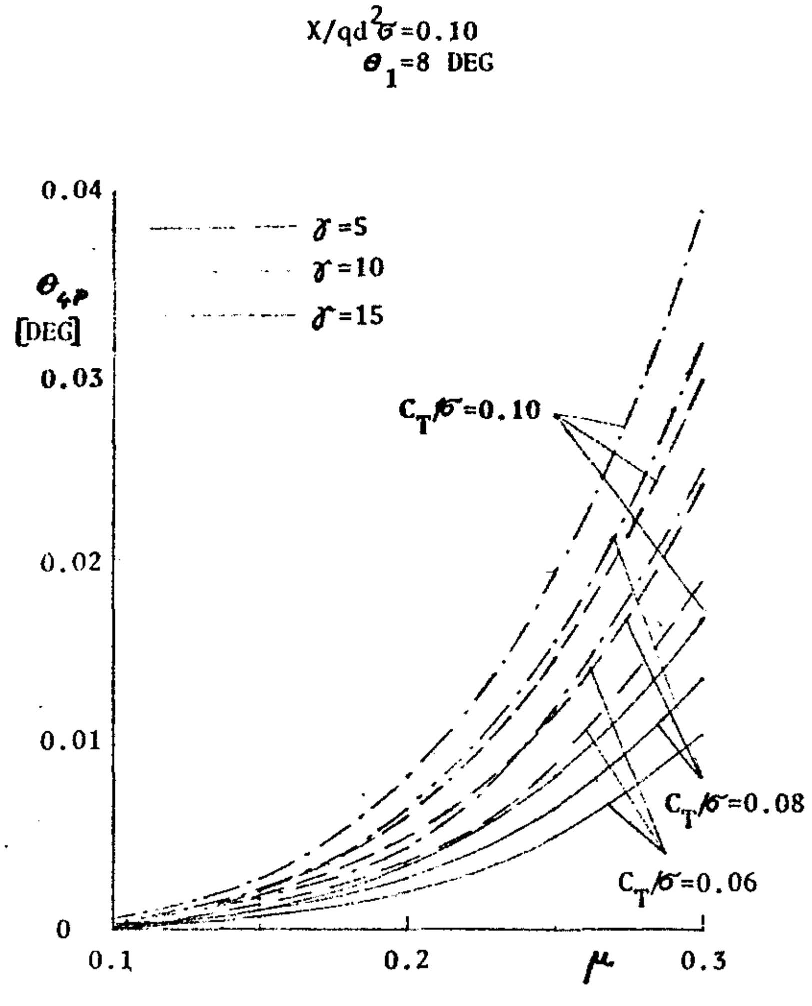

FIGURE 3. AMPLITUDE OF OPTIMAL 4/REV BLADE PITCH VS. ADVANCE RATIO

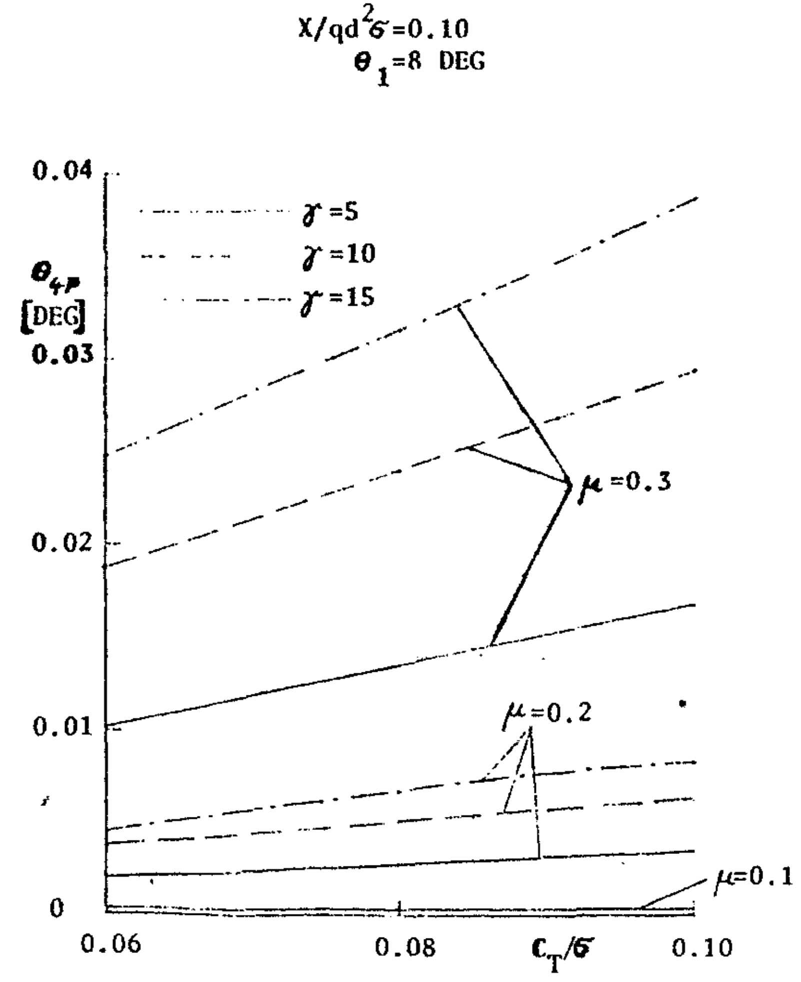

FIGURE 4. AMPLITUDE OF OPTIMAL 4/REV BLADE PITCH VS. THRUST COEFFICIENT

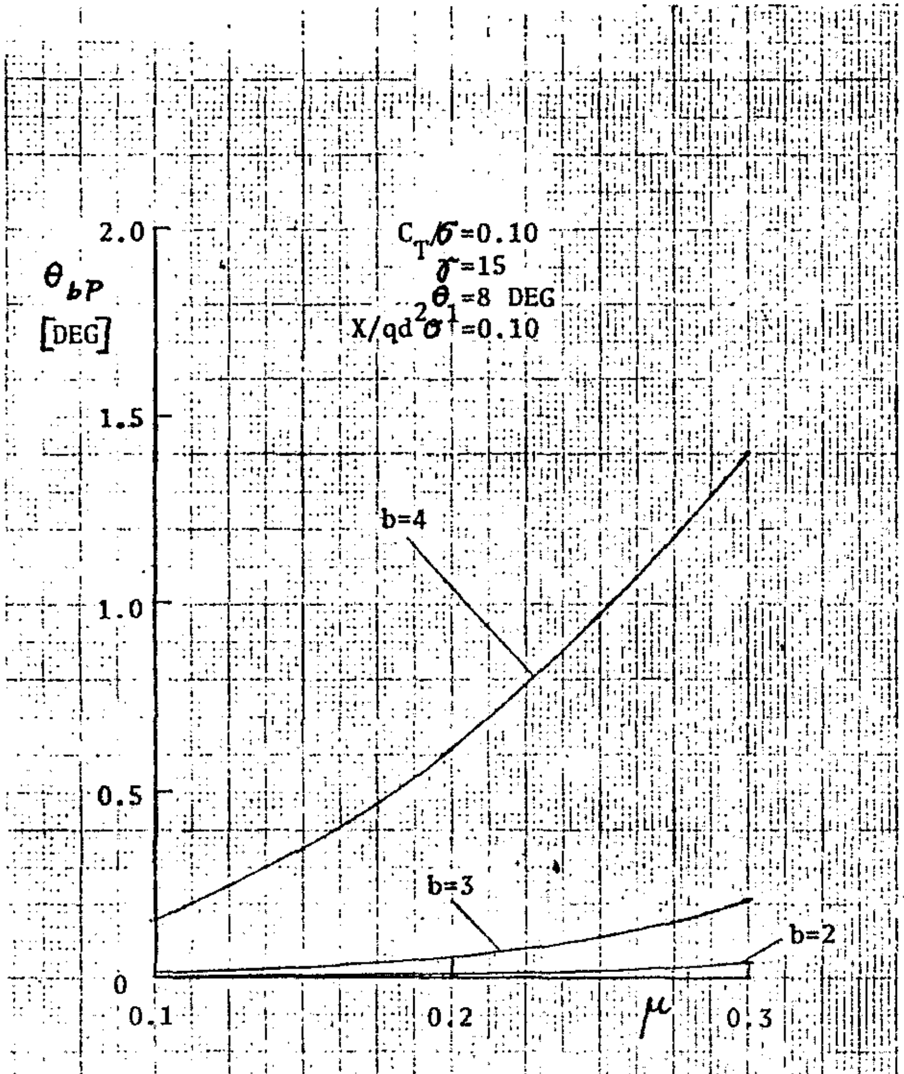

FIGURE 5. AMPLITUDE OF OPTIMAL b/REV BLADE PITCH VS. ADVANCE RATIO

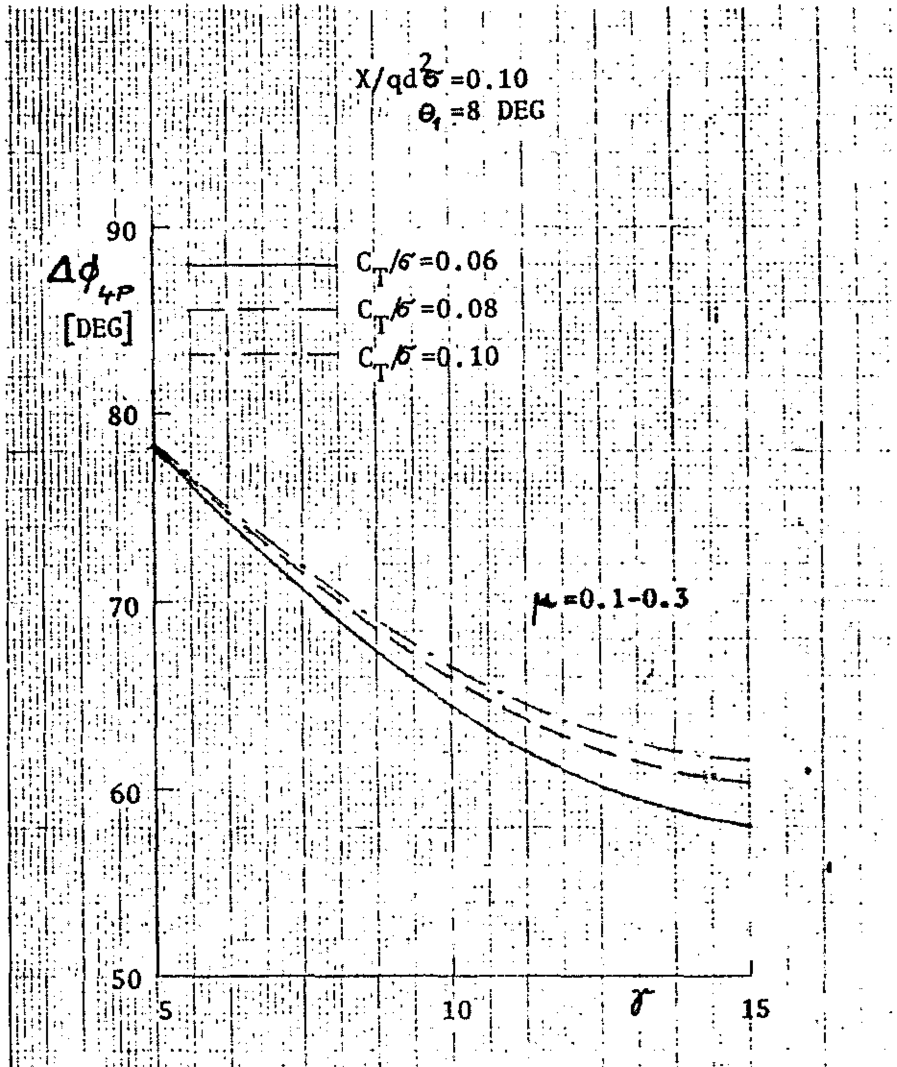

FIGURE 6. PHASE ANGLE OF OPTIMAL 4/REV BLADE PITCH VS. BLADE INERTIA NUMBER