ON THE USE OF HAMILTONIAN CONCEPTS IN ROTORDYNAMICS

Th. van Holten
Stork Product Engineering
Amsterdam, Netherlands

Abstract

The paper presents a method to numerically calculate the motions of a non-linear dynamic system, without the need to derive the equations of motion. The method is described for the particular case of a system consisting of a chain of segments interconnected by hinges and springs. The system is specified by the inertia matrices of the segments and the transformation matrices of the hinges, after which a dynamic simulation of the motion can start immediately. The method is based on a transformation of the coordinate base towards generalized coordinates and generalized momenta, such as used in Hamilton's classical theory.

1. Notations

α	generalized angular coordinate
a_{ij}	elements of energy matrix [A]
$\underline{e}$	positionvector of hinge (fig.1)
K	total number of degrees of freedom
p	generalized momentum
q	generalized coordinate
Q	generalized force
$\underline{r}$	positionvector w.r.t. local frame
$\underline{R}$	positionvector w.r.t. inertial frame
T	kinetic energy
t	time
V	potential energy

indices:

cg	centre of gravity
i	row number in matrix
j	column number in matrix
k	denotes segment number
0	denotes inertial system

matrices:
The most general I x J matrix is symbolized by [A], with elements a_{ij}, where i=1 ... I is the rownumber, and j=1 ... J the columnnumber.
A row matrix is indicated by (A) with elements a_j, and a column matrix by (A) with elements a_i.
The transpose of a matrix is denoted by the superscript T: e.g. (A) = [A]^T.
The inverse is denoted as [A]$^{-1}$.
($\underline{E}$): column with unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$.
[H]: transformation w.r.t. inertial frame
(X): row containing local Cartesian coordinates
[J]: "inertia matrix" defined in (24)

2. Introduction

Formulating the dynamic equations of complex systems can be a formidable and tedious task. European Helicopter Forums since 1982 have given attention to new approaches in this area, intending to reduce the analytical effort involved, and shift the work to computerprocessing. Several references about the subject are mentioned under refs. 1 through 6.

The present paper deals with a method which is aimed at problems where the difficulties are still more serious than usual, viz. dynamic problems where linearizations are not allowed. The method outlined below is based on concepts found in the classical Hamiltonian formulation of mechanics.

During the 13th Forum a paper was contributed containing an explanation of the first principles of the method (ref.6). Since that time the method has been extended somewhat further. The earlier method discussed in ref.6 needed some analytical preparations before the purely numerical procedure for solving the dynamic problem could start. Although these analytical preliminaries involved much less effort than the classical Lagrangian approach of setting up the equations, it nevertheless has proved a handicap in the case of large numbers of degrees of freedom.

The paper describes a new extension of the theory so that no preliminary analysis is needed at all. After a specification of the system's configuration in terms of inertia- and transformation matrices, an entirely non-linear numerical integration of the motion can be performed. The large saving of preparation time is however bought at the expense of increased computertime. The method may therefore be used best for analyzing special, non-recurring types of problems. The theory is shown below for a special kind of configuration, consisting of a chain of elements connected by hinges and springs. Further extensions to other types of configurations will be obvious after the explanation below, and an example of this will be shown during the oral presentation during the Forum.

3. Preliminaries: the Hamiltonian concept of generalized momenta

We assume a dynamical system whose generalized coordinates are q_i (i=1 ,2 ,....,K).
If the kinetic energy of the system T is a homogeneous quadratic function of the velocities, the expression for T may be written as:

$$T = \tfrac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} a_{ij} \dot{q}_i \dot{q}_j \qquad (1)$$

$$= \tfrac{1}{2} (\dot{q}) .[A] .(\dot{q}) \qquad (2)$$

in which the symmetric matrix [A] will be called the energy-matrix, whose elements in general are functions of all the generalized coordinates.

In Hamilton's dynamical theory use is made of generalized momenta p_i. These are defined as:

$$p_i = \partial T/\partial \dot{q}_i \qquad (i=1 \ldots K) \qquad (3)$$

or, written in matrix motation:

$$(p) = (\partial T/\partial \dot{q}) \qquad (4)$$

Eq. (1) shows that for a system whose kinetic energy is a homogeneous quadratic function of the generalized coordinates, the generalized momenta will take the form:

$$p_i = \sum_{j=1}^{K} a_{ij} \dot{q}_j \qquad (i=1 \ldots K) \qquad (5)$$

or, again in matrix notation:

$$(p) = [A] .(\dot{q}) \qquad (6)$$

The generalized momenta are linear combinations of the velocities $\dot{q}_i$.

4. Principles of the dynamic method

This chapter gives a summary of the method which was earlier reported in ref.6. The later derived extensions will be dealt with separately, in later chapters.
According to Lagrange's theory the equations of motion of a system with K generalized coordinates q_i (i=1 ... K) can be derived from:

$$d(\partial T/\partial \dot{q}_i)/dt = \partial T/\partial q_i - \partial V/\partial q_i + Q_i \qquad (i=1 \ldots K) \qquad (7)$$

Substituting the definition of the generalized momenta (3) into (7), and changing for convenience again to matrix notation:

$$(\dot{p}) = (\partial T/\partial q) - (\partial V/\partial q) + (Q) \qquad (8)$$

Combined with the inverse of eq.(6):

$$(\dot{q}) = [A]^{-1} .(p) \qquad (9)$$

the eqs. (8) and (9) form a system of 2K first order differential equations which constitute the complete set of equations of motion. The equations have in fact been transformed to a new coordinate-base consisting of the generalized coordinates q_i's and generalized momenta p_i's, instead of the usual coordinates q_i's and the associated velocities $\dot{q}_i$'s.
Note carefully however, that $(\partial T/\partial q)$ stands for a vector whose elements are the partial derivatives $(\partial T/\partial q_i)_{\dot{q}i=const}$, as in the original Lagrange equation (7). During a numerical integration of the system of equations (8) and (9), the calculation of the derivatives $(\partial T/\partial q_i)_{\dot{q}i=const}$ can be performed conveniently by a numerical differentiation of eq.(2).
The latter shows why the form of the equations of motion implied by (8) and (9) is of advantage when a purely numerical solution of the motion is aimed at: the sole preparation which is needed in order to perform a numerical integration, is to specify the functions $a_{ij}(q_r)$. A considerable number of analytical differentiations are thus avoided, which would usually be needed when the Lagrange-equations are being used in their original form (7).
During numerical work the matrix inversion
in (9) can be avoided when eq.(6) is treated as a system of linear equations, so that by more efficient methods the solutionvector $(\dot{q})$ may be determined.

5. Specification of the system

As shown above, the functions a_{ij} (q_r) must be specified before a numerical integration of motion can be performed. These functions implicitly describe the system's geometry and mass characteristics. For strongly non-linear problems or in cases with many degrees of freedom it soon becomes more convenient to calculate the values of a_{ij} just numerically at each time step of the integration. In the latter case, the geometry of the dynamical system has to be specified explicitly, in such a way that a subroutine can determine the numerical values of a_{ij} when the instantaneous values of the generalized coordinates q_r are given.

In this chapter it is shown how to accomplish this. Following this, an algorithm is derived which is able to calculate the instantaneous values of a_{ij}. For the time being the limitation of the approach shown below is, that the dynamical system must consist of a number of rigid segments, connected by hinges and springs. However, this limitation is not very fundamental.

It is assumed that the dynamical system is composed of a total number of K

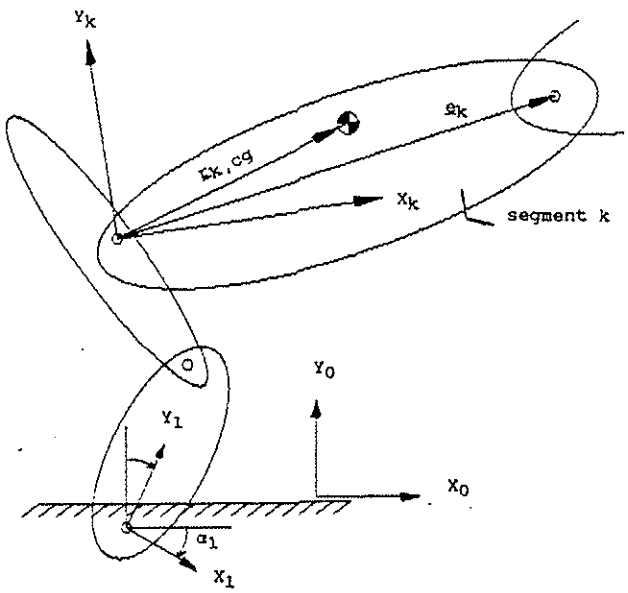

Fig.1: Schematic of configurations considered (drawn 2-dim for clarity)

segments, forming a simple chain in which the segments are interconnected by hinges (see fig.1). For the purpose of explaining the method it is assumed for the time being that no forking of the chain occurs, where segments would be connected to three instead of two other segments. The chain is at one end attached to an inertial frame of reference, through a hinge with the "fixed earth".

Each segment is assumed to be a rigid body without deformations. The generalized coordinates correspond with the rotations permitted by the hinges only, as there are no other deformations.

The segments are numbered 1, 2, ..., k, ..., K. The segment at the end of the chain which is connected to the "fixed earth" is numbered k=1. The last segment at the free end of the chain is indicated by k=K.
It is sometimes convenient to define "dummy" segments, such that there are as many segments as there are rotational degrees of freedom. We thus assume that every segment has just one degree of freedom with respect to the previous segment. The actual hinges of the physical configuration may of course permit more degrees of freedom than just one. In such a case a "dummy" segment is interposed so that each rotation (degree of freedom) permitted by the hinge is followed by a segment. A dummy segment however, will have no mass and no spatial extension, in contrast to the real segments.

By using the artifice of dummy segments, the numbering of the degrees of freedom is the same as that of the segments. The degree of freedom corresponding to the variable angle between the "fixed earth" and the first segment will be called α_1, the angle between segment number 1 and 2 will be denoted as α_2, etc.
The degrees of freedom are thus given by $\alpha_1, \ldots \alpha_k, \ldots, \alpha_K$.

In figure 1 the local Cartesian coordinate system of the k'th segment is drawn. Its origin coincides with the hinge point where the element is connected to the <u>previous</u> segment with number k-1. The unit vectors along the axes of the local coordinate system are $\underline{i}_k$, $\underline{j}_k$, and $\underline{k}_k$. In the following the notation $(\underline{E}_k)$ is used for the column $(\underline{E}_k) = (\underline{i}_k, \underline{j}_k, \underline{k}_k)$.

The angular orientation of the local coordinate system of the k'th segment with respect to the previous segment (the k-1'th) is defined by a transformation matrix:

$$(\underline{E}_k) = [\alpha_k].(\underline{E}_{k-1}) \qquad (10)$$

where the elements of the transformation matrix $[\alpha_k]$ are goniometric functions of the angle α_k (i.e. the generalized coordinate describing the rotation of segment k with respect to the previous segment k-1.

The angular orientation of the local coordinate system with respect to an <u>inertial</u> frame of reference is denoted as:

$$\begin{aligned} (\underline{E}_k) &= [H_k] (\underline{E}_0) = \\ &= [\alpha_k].[\alpha_{k-1}] \ldots \\ &\quad .. [\alpha_1].(\underline{E}_0) \end{aligned} \qquad (11)$$

where $(\underline{E}_0)$ stands for the column of unit vectors of the inertial coordinate system.

Using the above notations, a point P of the k'th segment with local coordinates $(X_k) = (x_k, y_k, z_k)$ has a positionvector $\underline{r}_k$ with respect to the local origin of the k'th segment given by:

$$\underline{r}_k = (X_k).(\underline{E}_k) \qquad (12)$$

In particular, the centre of gravity of the k'th segment has a position with respect to its local coordinate system denoted by

$$\underline{r}_{k,cg} = (X_k)_{cg} .(\underline{E}_k) \qquad (13)$$

and the hinge between the k'th segment and the next segment k+1 has a position $\underline{e}_k$ (see fig.1) defined by:

$$\underline{e}_k = (e_k). (\underline{E}_k) \qquad (14).$$

6. <u>Kinetic energy due to translation</u>

The kinetic energy due to translation of the centre of gravity of a segment is first considered in the following. Using the notations introduced above, it

follows that the position of the centre of gravity of the k'th segment with respect to the inertial frame of reference is given by:

$$\underline{R}_{k,cg} = (\ X_k\ )_{cg}.(\ \underline{E}_k\ ) + (\ e_{k-1}).(\ \underline{E}_{k-1}\ ) + \ldots + (\ e_1\ ).(\ \underline{E}_1\ ) =$$

$$= \left[(\ X_k\ )_{cg}.[\ H_k\ ] + \sum_{n=1}^{k-1} (\ e_n\ ).[\ H_n\ ]\right] . (\ \underline{E}_0\ ) \qquad (15)$$

or in short notation:

$$\underline{R}_{k,cg} = (\ R_k\ )_{cg}.(\ \underline{E}_0\ ) \qquad (16)$$

where the notation $\underline{R}$ in general will be used for the positionvector of a point with respect to the inertial frame of reference. The time derivative of the positionvector is:

$$\dot{\underline{R}}_{k,cg} = \sum_{i=1}^{k} (\partial R_k/\partial\alpha_i\ )_{cg}\ .\dot{\alpha}_i\ .(\ \underline{E}_0\ ) \qquad (17)$$

where $(\ \partial R_{cg}/\ \partial\alpha_i\ )$ symbolizes a row matrix whose elements consist of partial derivatives obtained from the elements of the $(\ R_k\ )_{cg}$ row matrix. The steps taken to derive the corresponding kinetic energy are: premultiply the transpose of (17) by (17), and multiply by $\frac{1}{2}\ M_k$ where M_k is the mass of the segment. Comparing with eq.(1) yields:

$$(a_{ij})_{k,transl} = M_k\ (\partial R_k/\ \partial\alpha_i\ )_{cg}.\ .(\ \partial R_k/\ \partial\alpha_j\ )_{cg} \qquad (18)$$

The row matrix $(\partial R_k/\ \partial\alpha_i\ )_{cg}$ in the r.h.s. of (18) is obtained from eq.(15):

$$(\partial R_k/\partial\alpha_i\ )_{cg} = (\ X_k\ )_{cg}.[\partial H_k/\partial\alpha_i] + \sum_{n=i}^{k-1} (\ e_n\ ).[\ \partial H_n/\partial\alpha_i\ ] \qquad (19)$$

where, for i<=k:

$$[\partial H_k/\partial\alpha_i\ ] = [\ \alpha_k\ ]\ldots[\ \alpha_i'\ ]\ldots[\ \alpha_1\ ] \qquad (20)$$

The matrix $[\ \alpha_i'\ ]$ is obtained from the transformation matrix $[\ \alpha_i\ ]$ by differentiating its elements with respect to α_i.

7. Kinetic energy due to rotation

The kinetic energy due to rotation around the centre of gravity is derived by hypothetically shifting the centre of gravity of the k'th segment towards the origin of the inertial frame of reference. The segment shall nevertheless describe the same rotations as the corresponding real segment. In that case the absolute positionvector of an arbitrary point of the segment is given by:

$$\underline{R}_k = ((\ X_k\ )-(\ X_k\ )_{cg}).\ [\ H_k\ ].(\ \underline{E}_0\ ) \qquad (21)$$

and its time derivative:

$$\dot{\underline{R}}_k = ((\ X_k\ )-(\ X_k\ )_{cg}).\sum_{i=1}^{k} [\partial H_k/\partial\alpha_i\ ]\ \dot{\alpha}_i.\ .(\ \underline{E}_0\ ) \qquad (22)$$

The procedure to obtain the corresponding kinetic energy is: premultiply (22) by its transposed. Observe that a product of the form $(\ \underline{E}\ ).[\ ].(\ \underline{E}\ ) = \mathrm{Tr}[\ ]$, which means the "trace" of the matrix. The trace of a square matrix is defined as the sum of the diagonal elements, and is denoted by Tr[B]. After this operation the kinetic energy is determined by multiplying with the local mass and integrating over the whole segment. The final result is compared with eq.(1), and yields:

$$(a_{ij})_{k,rot} = \mathrm{Tr}\left[\ [\partial H_k/\partial\alpha_i\ ]^T.[\ J_k\ ].\ .[\partial H_k/\partial\alpha_j\ ]\right] \qquad (23)$$

where the elements of [J] (not to be confused with the usual inertia matrix !) are given by:

$$[J_k] = \int ((\ X_k\ )-(\ X_k\ )_{cg}).\ ((\ X_k\ )-(\ X_k\ )_{cg})dm \qquad (24)$$

8. Computational procedure

When the above derived theory is used in an actual calculation, much computation time can be saved by making use of several recurrence relations that can be derived. Furthermore, a lot of zero-matrices and symmetric matrices occur, so that considerable savings may be realized by identifying these matrices beforehand.

From (11) it follows:

$$[\ H_k\ ] = [\ \alpha_k\ ].[\ H_{k-1}\ ] \quad (k=1\ \ldots\ K) \qquad (25)$$

which sequence may be started from:

$$[\ H_0\ ] = [\ I\ ],\ \text{the unit matrix} \qquad (26)$$

From (20):

$$[\ \partial H_k/\partial\alpha_i\ ] = [\ \alpha_k\ ].[\ \partial H_{k-1}/\partial\alpha_i\ ] \quad (i<k) \qquad (27)$$

$$[\ \partial H_k/\partial\alpha_i\ ] = [\ \alpha_k'].[\ H_{k-1}\ ] \quad (i=k) \qquad (28)$$

$$[\ \partial H_k/\partial\alpha_i\ ] = [\ 0\ ] \quad (i>k) \qquad (29)$$

From (23):

$$(a_{ij})_{k,rot} = (a_{ji})_{k,rot} \qquad (30)$$

$$(a_{ij})_{k,rot} = 0 \quad (i>k \text{ or } j>k) \qquad (31)$$

From (19):

$$\left(\partial R_k/\partial \alpha_i\right)_{cg} = \left(\partial R_{k-1}/\partial \alpha_i\right)_{cg} + \left(X_k\right)_{cg} \cdot \left[\partial H_k/\partial \alpha_i\right] + \left(\left(e_{k-1}\right) - \left(X_{k-1}\right)_{cg}\right) \cdot \left[\partial H_{k-1}/\partial \alpha_i\right] \quad (i<k) \qquad (32)$$

$$\left(\partial R_k/\partial \alpha_i\right)_{cg} = \left(X_k\right)_{cg} \cdot \left[\partial H_k/\partial \alpha_i\right] \quad (i=k) \qquad (33)$$

$$\left(\partial R_k/\partial \alpha_i\right)_{cg} = \left(0\right) \quad (i>k) \qquad (34)$$

From (18):

$$(a_{ij})_{k,transl} = (a_{ji})_{k,transl} \qquad (35)$$

$$(a_{ij})_{k,transl} = 0 \quad (i>k \text{ or } j>k) \qquad (36)$$

The resulting computational scheme is shown schematically in fig.2.

9. Further extensions of the theory

Two modifications of the theory are straightforward (although not shown for lack of space):

1. The method is easily extended to configurations consisting of chains of segments where branching of the chain occurs. This is just a matter of properly reorganizing the computational scheme.

2. Savings in computing time are possible when some of the generalized coordinates describe small deviations. In that case partial linearizations are possible, and not all of the a_{ij} have to be calculated during each timestep.

10. Conclusions

A method has been developed for the numerical solution of non-linear dynamic problems. The method only requires the geometrical and inertial data of the system to be specified. Following this, the motion of the system can be simulated immediately, without any further analytical preliminaries.

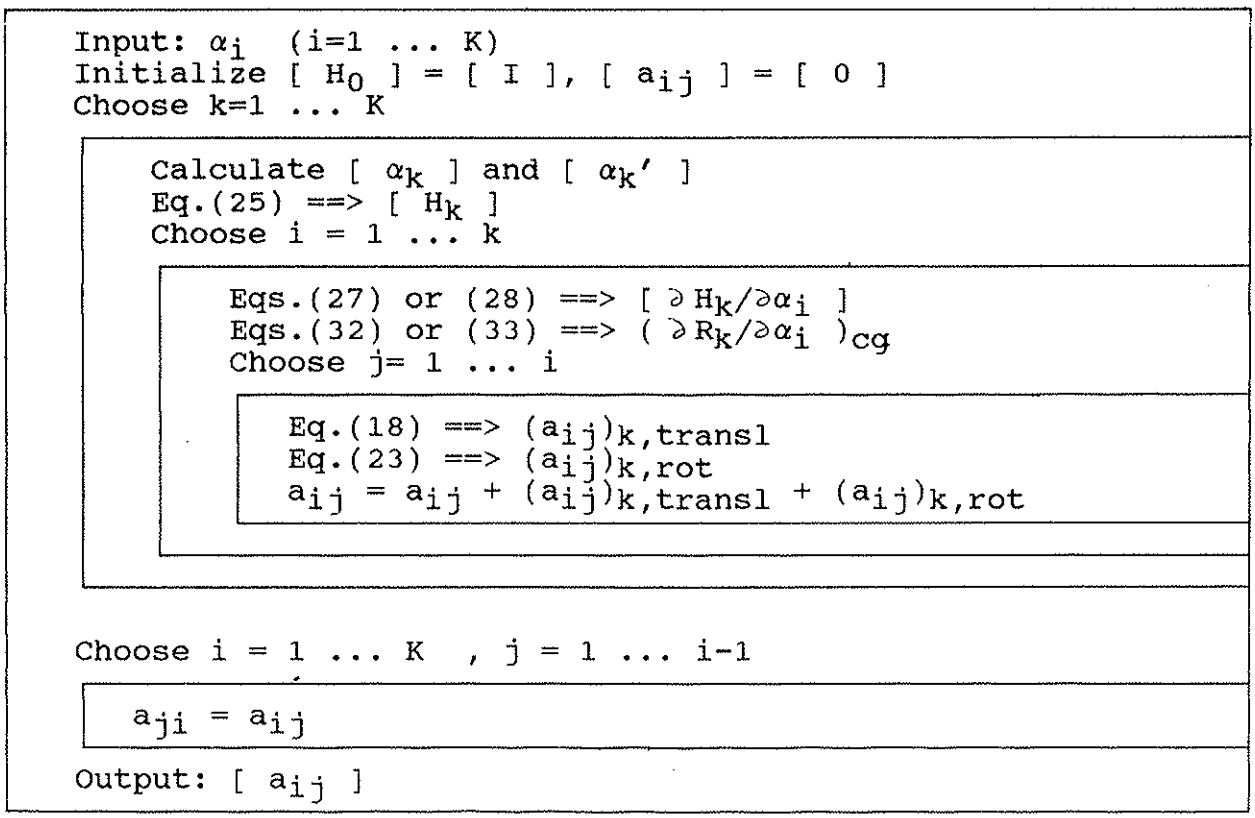

Fig.2: Summary of computational scheme acc. to chap.8

11. References

1. M.P.Gibbons, G.T.S.Done: Automatic generation of helicopter rotor aeroelastic equations of motion, Vertica vol.8, no.3, 1984.

2. M.H.Patel, G.T.S.Done: Experience with a new approach to rotor aeroelasticity, Vertica vol.9, no.3, 1985.

3. G.T.S.Done, P.T.W.Juggins, M.H.Patel: Further experience with a new approach to helicopter aeroelasticity, vertica vol.12, no.4, 1988.

4. S.Hanagud, S.Sarkar: Kane's methods in rotor blade dynamics, 14th European Rotorcraft Forum, sep.1988, Milan.

5. P.P.Friedmann: Recent trends in rotary-wing aeroelasticity, Vertica vol.11, no.1/2, 1987.

6. Th. van Holten: Hamiltonian mechanics as a possible alternative for deriving aeroelastic equations, paper 6.8, 13th European Rotorcraft Forum, Arles 1987.