# Validation of the SDRE Controller for Rotorcraft Trajectory Tracking Problems

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**Key words: Helicopter Flight Control, Trajectory Tracking, SDRE, Flight Simulation**

**Abstract**: This paper deals with the State-Dependent Riccati Equation (SDRE) technique for the design of helicopter nonlinear flight controller and the validation of designed SDRE controller using simulation environment. Since the SDRE controller requires a linear system-like structure for nonlinear motion equations, a State-Dependent Coefficient (SDC) factorization technique is developed in order to derive the conforming structure from a general nonlinear helicopter dynamic model. Also on-line numerical methods of solving the Algebraic Riccati Equation (ARE) are investigated to improve the numerical efficiency in design ing the SDRE controllers. The proposed method is applied to the various maneuvers and flight guidance logics are developed using maneuver attributes which consist of various mission task elements. Simulation environment is developed using three independent computers for validation the SDRE controller. Each PC is independently operated for flight control computer, artificial model helicopter and ground control station, respectively.

## 1. Introduction

The State-Dependent Riccati Equation (SDRE) technique is based on the Linear Quadratic Regulator (LQR) method and allows us to design the controllers for the nonlinear system with the same technique used in its linear counterparts[1,2,3]. The SDRE method basically differs from the LQR method in two points. First, the SDRE control technique demands a linear system-like structure with State-Dependent Coefficients (SDC) form of motion equations. But the general multi-variable nonlinear systems like helicopter do not always conform to this structure. Also, it is well known that the SDC form for multi-variable systems is not unique and there is no general method for its derivation. Second, the LQR method requires off-line solution of the ARE. However, the SDRE method needs the iterative solution of the ARE to calculate the feedback control in on-line because the state and control derivative matrices of derived SDC form are generally time-varying. Therefore an efficient numerical solution of the ARE is crucial for the system performance.

The main purpose of this paper is to validate the designed SDRE controller and to check the real-time applicability by using simulation environment. Langson et. al.[4] realized the experimental implementation of the SDRE controller for a third order nonlinear system. In its setup, one 75 MHz Pentium computer is used to simulate the plant dynamics, and another one of the same kind calculate the SDRE controller with 100Hz sampling rate. P. K. Menon et. al. [5] applied the SDRE controller to the missile control. To solve the ARE, various numerical methods and PC processors are used with sampling rate as high as 20 kHz. Above studies were made it possible to implement of SDRE in the real-time because comparatively simple models were used. However, the SDRE controller is computationally demanding high computing power when it is applied to complex dynamic models. In this regard, Kim et. al.[6] applied the indirect method to the various rotorcraft model to investigate the relative computational burden for its application to the design of nonlinear optimal controller.

This paper focuses on the design of the SDRE controllers using a complex level 2 rotorcraft model[7] and on its validation through a simulation environment. Main objectives are to access performance of the designed controllers and to identify the required computer resources for its real applications. For three purposes, flight guidance laws are designed by combining various maneuver attributes and important results are discussed to recommend the guide-line for the design and application of the rotorcraft SDRE controllers

## 2. SDRE Technique

The general infinite-horizon nonlinear regulator problem can be represented as:

$$\min \quad J(x,u,t) = \frac{1}{2}\int_{t_0}^{\infty} x(t)^T Qx(t) + u(t)^T Ru(t)dt \tag{1}$$

$$\text{s.t} \quad \dot{x} = f(x) + B(x)u \tag{2}$$

where $Q \geq 0, R > 0$

The SDRE technique requires the SDC form as shown in Eq. (3) with which the SDRE method finds controller gain by assuming the matrices, $A(x)$ and $B(x)$ , are locally constant and by solving the ARE.

$$\dot{x} = A(x)x + B(x)u \tag{3}$$

If $\{A(x), B(x)\}$ in Eq. (3) is a controllable pair, the procedure of the SDRE technique is summarized as follows.

( i ) Drive the SDC form as show in Eq. (3) using a SDC factorization.

( ii ) Solve the State-Dependent Algebraic Riccati Equation (SDRE)

$$PA(x) + A^T(x)P - PB(x)R^{-1}B^T(x)P + Q = 0 \tag{4}$$

to obtain $P(t) \geq 0$, where $P$ is function of $x$

( iii ) Construct the nonlinear feedback controller

$$u = -Kx(t) = -R^{-1}B^T Px(t) \tag{5}$$

On the contrast to the LQR solution, the SDRE technique requires the SDC factorization and the solution of state-dependent ARE at each time of control engagement.

## 3. Numerical Methods for the SDRE Controller Design

In the previous research [8], A. Bogdanov et. al. proposed a numerical SDC factorization method for nonlinear helicopter dynamic model. This paper uses the modified version of the SDC factorization proposed by the present author[3], by considering both system equilibrium states and kinematical offset in forcing functions. The nonlinear motion equation at an arbitrary time instant and that at an equilibrium condition can be written as following.

$$\dot{x} = F(x,u) \tag{8}$$

$$\dot{x}_0 = F(x_0,u_0) = \text{const} \tag{9}$$

where $x_0, u_0$ denote equilibrium states and controls, respectively.

Even in an equilibrium state, the value of forcing function, $F(x_0,u_0)$ , is generally nonzero. The consideration on the state variation from the equilibrium condition and the Taylor series expansion of motion equation around the equilibrium states and controls enable us to derive the motion equation for the perturbed states and controls as shown in Eq. (12).

$$x = x_0 + \tilde{x}, \quad u = u_0 + \tilde{u} \tag{10}$$

$$\begin{aligned} \dot{x} &= \dot{x}_0 + \dot{\tilde{x}} \\ &= F(x,u) \end{aligned} \tag{11}$$

$$\dot{\tilde{x}} = F(x,u) - F(x_0,u_0) \tag{12}$$

Then, the SDC factorization can be carried out by using the following steps.

( i ) Control derivative matrix

$$B(x) = B(\tilde{x}) = \frac{\partial \tilde{F}}{\partial \tilde{u}}, \quad \text{where} \quad \tilde{F}(x,u) = F(x,u) - F(x_0,u_0) \tag{13}$$

( ii ) State derivative matrix

$$\tilde{f}(\tilde{x}) = A(\tilde{x})\tilde{x}, \quad \text{where} \quad \tilde{f}(\tilde{x}) = \tilde{F}(x,u) - B(x)\tilde{u} \tag{14}$$

As mentioned in previous section, the SDRE technique requires the derivation of the SDC form of motion equations and the solution of the state-dependent ARE. There are the causes of high computation burden. Many numerical methods to solve the ARE are being developed but the selection from the existing one is not simple since the efficiency and applicability of each algorithms depend on numerous factors.

H.T Bank and K. Ito proposed a hybrid method to solve the ARE for a system with a large size [9]. In their method, the initial feedback gain $K$ is obtained from the limiting solution of Chandrasekhar system and the result is refined through the Newton-Kleinman algorithm. Since the system dynamics of the rotorcraft are inherently unstable, the initial feedback system can be unstable. The solution of the Chandrasekhar system provides a stable feedback gain matrix with suitable chosen weighting matrices even for an unstable system.

The Chandrasekhar system for initial gain can be written as following and the suitable time integrator such as Runge-Kutta algorithm can be used for resolving the initial value problem.

$$\begin{bmatrix} \dot{K}(t) \\ \dot{L}(t) \end{bmatrix} = \begin{bmatrix} -B^T L^T(t) L(t) \\ -L(t)[A - BK(t)] \end{bmatrix} \quad with \begin{bmatrix} K(0) = 0 \\ L(0) = C = Q^{1/2} \end{bmatrix} \tag{15}$$

The state-dependent ARE shown in Eq. (4) can be solved by applying the Newton-Kleinman algorithm for following transformed system.

$$\begin{aligned} &S = A - BK \\ &S^T P + PS + K^T RK + Q = 0 \end{aligned} \tag{16}$$

If we obtain a stable pair $\{A - BK_0, B\}$ by solving Chandrasekhar system, the control gain matrix is iteratively obtained by applying the Newton-Kleinman algorithm using the following equations $(i = 1, \cdots, i_{\max})$.

$$S_i = A - BK_i \tag{17}$$

$$S_i^T P_i + P_i S_i = -Q - K_i^T RK_i \tag{18}$$

$$\lim_{i \to \infty} P_i = P \tag{19}$$

$$\text{where,} \quad K_{i+1} = R^{-1} B^T P_i$$

## 4. SDRE Controller Design for the Helicopter Flight Guidance

The rotorcraft flight dynamic model used in this study is based on the previous research in Ref. [7]. Since the flap and inflow states are the unobservable states that can not measured or estimated, these states are considered as hidden or unobservable states during the controller design. Therefore, the rotorcraft model for controller design is defined by using the rigid body states as shown in Eq.(20) and the rotor states corresponding to flap motion and inflow are estimated by using the rotor trim solution. However, the level 2 simulation model developed in Ref. [7] is used as the validation model.

$$x_R = [u, v, w, p, q, r, f, q, y, x_E, y_N, h] \tag{20}$$

In this paper, flight trajectories for the validation of the SDRE controller are defined by combining various maneuver attributes. A maneuver attribute is one of the flight elements with which a maneuvering flight can be totally defined. As an example, an acceleration maneuver from a trimmed rectilinear flight can be splitted into an initial trimmed rectilinear flight, an acceleration flight from maneuver entry to maneuver exit, and a final stabilized flight after finishing the maneuver. Then, these three maneuver stages consist maneuver attributes required for the description of an acceleration maneuver. If we have optimal control solution for each maneuver attributes, the flight guidance law can be designed by combining these solutions for a specific guidance mission. For these purposes, we consider various maneuver attributes related to bob-up, side step, turn, slalom, hurdle hop, acceleration, and deceleration maneuvers. In real applications, the maneuver attributes required for a specific mission task element should be identified prior to the application of the designed flight controller.

The each maneuver attributes can be expressed as the sum of states at maneuver entry and its variation during the maneuver.

$$x(t) = x(t_{entry}) + \Delta x(t) \tag{21}$$

$$x(t) = x(t_{entry}) + \int_{t_{entry}}^{t} \Delta \dot{x}(t)\, dt \tag{22}$$

In this paper trigonometric functions are utilized to define various maneuver[6,11] because of its smoothness and simplicity. The height, lateral-position and forward flight speed change during bob-up, side step, acceleration and deceleration are described as the following formula.

$$\Delta x(\bar{t}) = \frac{(\Delta x)_{\max}}{16.0}\left[8 + \cos(3p\bar{t} - 9\cos(p\bar{t})\right] \tag{23}$$

The height variation in hurdle hop maneuver and the lateral position change in slalom maneuver are formulated using Eq. (24) and (25), respectively.

$$\Delta x(\bar{t}) = \frac{(\Delta x)_{\max}}{46.8}\left[32 + \sin(2p\bar{t}) - 20\sin(4p\bar{t}) + 2\sin(8p\bar{t})\right] \tag{24}$$

$$\Delta x(\bar{t}) = 64(\Delta x)_{\max}[1 - 3\bar{t} + 3\bar{t}^{\,2} - \bar{t}^{\,3}]\bar{t}^{\,3} \tag{25}$$

where, $\bar{t} = (t - t_{entry})/(t_{finish} - t_{entry}) \quad 0 \le \bar{t} \le 1$

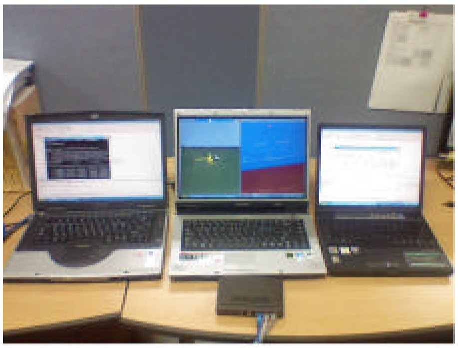
The time, $t_{entry}$ and $t_{finish}$, denote the maneuver entry and finish times, respectively and $(\Delta x)_{\max}$ are the maximum variation in states variable.

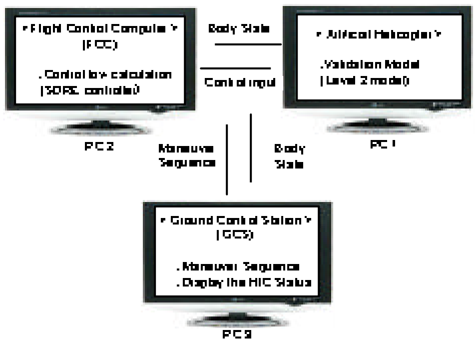
A turning flight is described by different maneuver phases as like entry phase, constant turning rate phase, exit phase. Ref[6,11] describes each maneuver phase using piecewise polynomials as

$$\Delta \dot{y}(t) = \begin{cases} (\Delta \dot{y})_{\max}(-2\bar{t}^{3} + 3\bar{t}^{2}), & t_{entry} \le t \le t_{steady} \\ (\Delta \dot{y})_{\max}, & t_{steady} \le t \le t_{exit} \\ (\Delta \dot{y})_{\max}(-2\tilde{t}^{3} + 3\tilde{t}^{2}), & t_{exit} \le t \le t_{finish} \end{cases} \tag{26}$$

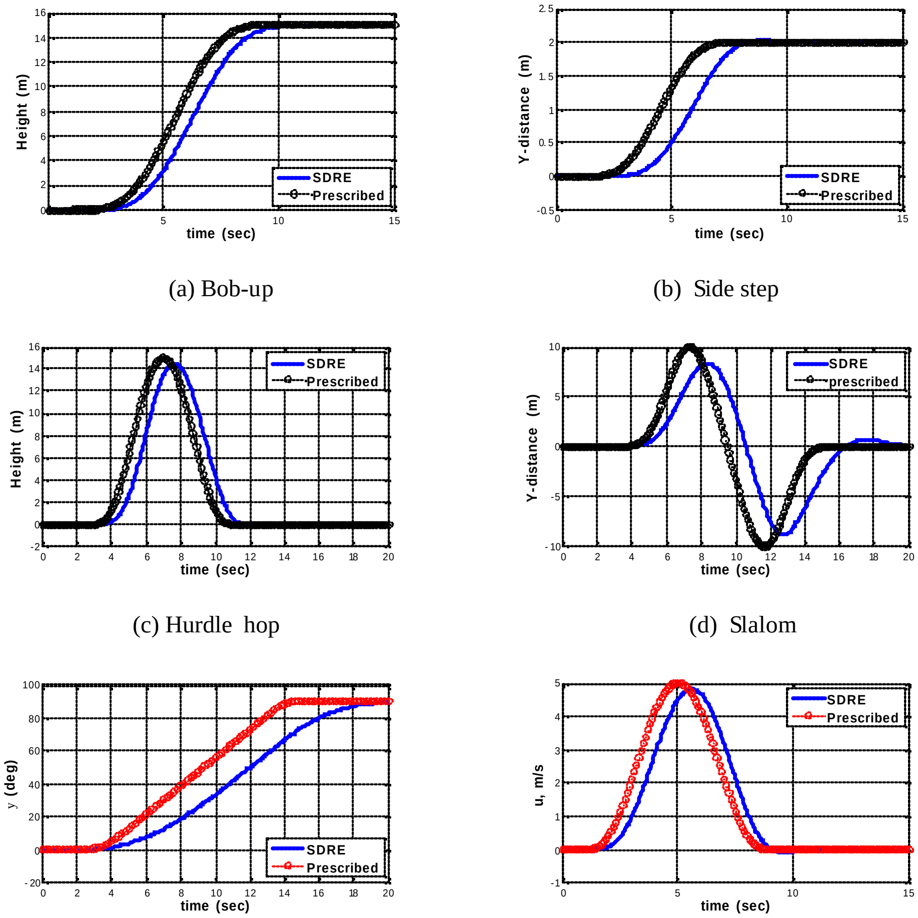
$$\begin{aligned} &(\Delta \dot{y})_{\max} = (\Delta y_{\max}) / (t_{finish} + t_{exit} - t_{steady} - t_{entry}) \\ &\bar{t} = (t - t_{entry}) / (t_{steady} - t_{entry}) \\ &\tilde{t} = 1 - (t - t_{exit}) / (t_{finish} - t_{exit}) \end{aligned} \tag{27}$$

Therefore, each maneuver attributes for the turning flight are completely defined by time, $t_{entry}, t_{steady}, t_{exit}, t_{finish}$, and the maximum heading change $(\Delta y)_{\max}$.





maneuver start its maneuver from steady level flight at 60 knots. Figure 1 compares calculated trajectories to the prescribed one. The SDRE controller shows some delay in the response but the results generally trace the prescribed trajectory well.




(a) Bob-up (b) Side step

(c) Hurdle hop (d) Slalom

(e) Turn (f) Acceleration/Deceleration

Figure 3. SDRE controller results with prescribed trajectory

To validate the application of the SDRE controller, two mission segments are defined as shown in Table 1 and Table 2.

Table 1. Flight guidance procedure Ⅰ

| Maneuver Sequence | Simulation time (sec) | Maneuver Description | Entry Time (sec) | Exit Time (sec) |
|---|---|---|---|---|
| Bob-up | 0 ~ 12 | Δh=15 m | 2 | 10 |
| Side step | 12 ~ 27 | Δy= 5m | 14 | 25 |
| Bob-down | 27 ~ 39 | Δh=-15m | 29 | 37 |

Table 2. Flight guidance procedure Ⅱ

| Maneuver Sequence | Simulation time (sec) | Maneuver Description | Entry Time (sec) | Exit Time (sec) |
|---|---|---|---|---|
| Bob-up | 0 ~ 12 | Δh=15 m | 2 | 10 |
| Acceleration | 12 ~ 32 | Δu=30 knots | 13 | 30 |
| Turn | 32 ~ 62 | $\Delta y$ =360 deg | 34 | 55 |

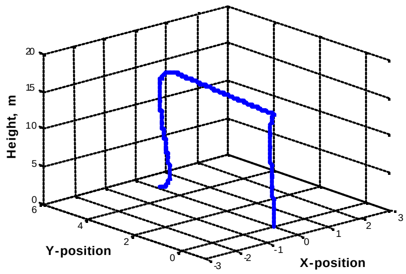



Figure 4. Maneuver trajectory for flight guidance Ⅰ

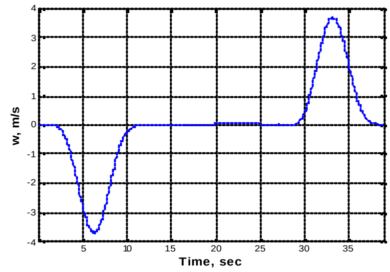



Figure 5. Vertical velocity variation

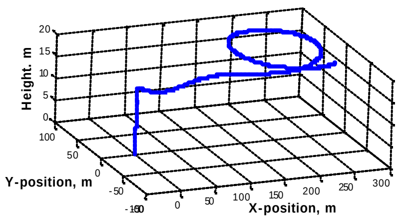



Figure 6. Maneuver trajectory for flight guidance Ⅱ

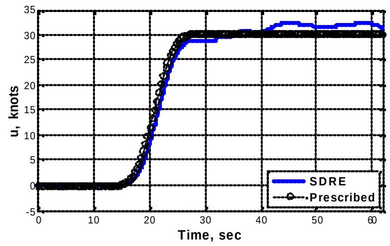



Figure 7. Forward speed variation

Figure 4-5 present the calculated maneuver trajectory in 3 dimensional space and vertical velocity variation for the flight guidance procedure Ⅰ. Figure 6-7 show the results for the flight guidance procedure Ⅱ. The simulation results show the controlled trajectory trace the prescribed guidance well and indicate the successful implementation of the rotorcraft SDRE controller.

## 6. Conclusion

This paper has studied the numerical method of the nonlinear SDRE technique and validated the designed SDRE controller using a simulation environment. The results show the SDRE controller can be applied to the rotorcraft guidance and control. And the real-time applicability of the SDRE controller highly depends on the model complexity for the SDC factorization and on the efficient solution of the state-dependent ARE.

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