Root Corrections for Dynamic Wake Models[1]

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Abstract

The solution for the induced flow of a rotor in axial flow has a long history of interest going back to Prandtl and Goldstein. The nominal inflow is computed in a rather straightforward manner for an infinite number of blades, but complicated root and tip corrections appear when there are a finite number of blades. Prandtl worked out an elegant, yet simple approximation for the **tip** correction that still has widespread applications. The purpose of this paper is to provide a similar **root** correction for the induced flow of rotors with a finite number of blades. Several alternative versions of the correction are provided, ranging from a simple computation in the spirit of Prandtl to a virtually exact numerical solution.

Nomenclature

a	polynomial coefficient, $1/b^2$
[**A**]	matrix of Galerkin coefficients
b	zero of root-correction polynomial
{**B**}	vector of Galerkin coefficients
c_0	homogenous coefficient
c_n	coefficients of circulation expansion
d, d_n	function coefficients
d_1, d_2	homogeneous constants, App. B
E	Prandtl tip-correction function
$f_k(\mu)$	velocity potential expansions
F(Q)	correction factor for forward problem
G(Q)	correction factor for inverse problem
$h_k(\mu)$	root-correction functions
h_A	final version of approximation, Eq. (19)
h_R	boundary correction (particular plus homogeneous)
H( )	root-correction function in either x or μ
i	imaginary number
j	summation index
k	harmonic number
m	summation index
n	exponent of circulation
n	harmonic number, App. B
p, p_n	elements of summation theorem
$P_j(x)$	Legendre polynomials
q	polynomial exponent
Q	number of blades
r	radial coordinate, m
R	blade radius, m
S(p)	summation result
S_o, S_e	summation for odd, even indices
s, t	polynomial exponents, $t \equiv q/2$
trig( )	either sin( ) or cos( )
v(r)	induced velocity at disk normal to vortex sheet, m/sec

[1] Presented at the 35th European Rotorcraft Forum, Hamburg, Germany, September 22-25, 2009.

$v_0(r)$	nominal induced velocity, no correction, $\Gamma/(2\cos\phi)$, m/sec
V	climb rate, m/sec
v_n, w_n	function coefficients
x	mapping coordinate, $\mu/\sqrt{1+\mu^2} = \cos(\phi)$
y	dynamic variable, App. B
z	axial coordinate, m
α, β	function coefficients
$\gamma(\mu)$	nondimensional normalized circulation, $\Gamma\Omega/V^2$
γ_b	nondimensional circulation per blade, $\Gamma_b\Omega/V^2$
γ_n	circulation expansion coefficients
$\Gamma(r)$	normalized nominal circulation, uncorrected, m^2/sec
$\Gamma_0(r)$	nominal circulation per blade, uncorrected, $(2\pi/Q)\Gamma(r)$, m^2/sec
$\Gamma_b(r)$	total circulation per blade, m^2/sec
δ	function coefficient
$\delta(\tau)$	Dirac delta function
ε	blade root cut-out, m
ζ	non-dimensional screw coordinate, $\zeta = \theta - \Omega z/V$
θ	angle of screw surface, rad
$\lambda(\mu)$	nondimensional induced velocity downstream, $2\nu(\mu)$
$\lambda_0(\mu)$	nondimensional nominal velocity, $2\nu_0(\mu)$
μ	radial coordinate, $\Omega r/V = \cot(\phi)$
μ_0	value of μ at blade tip, $\Omega R/V$
$\nu(\mu)$	nondimensional induced velocity at disk, $v(r)/V$
$\nu_0(\mu)$	uncorrected value of $\nu(\mu)$
τ	nondimensional time
ϕ	inflow angle, $\arctan(1/\mu)$
$\Phi(\mu,\zeta)$	velocity potential, normalized on V^2/Ω
Φ_j	admissible functions of either μ or x
Ω	rotor speed, rad/sec

Introduction

Betz and Prandtl, Ref. 1, found the optimum velocity distribution (i.e., for minimum power) for a rotor in axial flow. Although they were unable to find an exact solution for the circulation distribution that would result in such a velocity distribution, they **were** able to find this optimum circulation for a rotor with an **infinite** number of blades and then offer an approximate tip correction that would account for the effect of blade number. Although the Prandtl correction factor is based on a two-dimensional inflow model, it is quite accurate and is used extensively in rotorcraft analysis.

It fell to Goldstein, Ref. 2, to find the exact solution for the optimal circulation on a propeller with a **finite** number of blades. He treated both two-bladed and four-bladed rotors at various inflow angles. The results agree nicely with computations based on Prandtl's approximation, as shown in Fig. 1—taken from Ref. 2—where the condition chosen is $\mu_0 = 5.0$ which is a fairly high climb rate. For four blades, the two solutions are quite close, although there is a small discrepancy that occurs near the root of the blade. For the two-bladed rotor, this discrepancy is more pronounced. The behavior of the Prandtl approximation at small μ is nearly identical for all Q, but the Goldstein solution increasingly differs from the Prandtl solution (at small μ) as Q decreases.

For any number of blades, the correction that must be made to the Prandtl result is an **increase** in the optimum circulation for $\mu < 1$ (i.e., for inflow angles greater than 45º) and a decrease in circulation for $\mu > 1$. This result is counterintuitive because, for tip corrections, the increased inflow due to the ends of the vortex sheet implies that—to maintain constant inflow for optimum power—circulation must **decrease** at the tip. To the contrary, the Goldstein solution shows that—although outboard of a 45º inflow angle the circulation for minimum power **is** below nominal—inboard of a 45º inflow angle, circulation must be **increased** to maintain an optimum.

Makinen, Ref. 3, analyzed an optimum propeller by finite-state inflow methods. He found that dynamic inflow models successfully give the optimum circulation for rotors with a finite number of blades (including capture of the correct tip effects) but that they tend to follow the Prandtl solution rather than the Goldstein solution near the blade root, Fig. 2. This is because the assumption of an actuator-disk breaks down as the inflow angle approaches 90º. Garcia-Duffy, Ref. 4, looked further at these root effects by considering a propeller of infinite length. She discovered that the root correction has a defined shape that is similar for all blade numbers and inflow ratios. This result gave motivation to look for a simple root correction—similar in spirit to the Prandtl tip correction—that could enhance the accuracy of momentum or dynamic wake models for high-incidence inflow in real-time simulations and preliminary design applications. The purpose of this paper is to put forward such a root correction procedure.

Background

In this section, we derive the basic equations of root corrections. We will follow the general outline of Ref. 2 but proceed along what we believe is a more direct and compact approach. To begin, note that the pressure and velocity around a propeller in axial flow are governed by the following velocity potential:

$$\Phi(\mu,\zeta) = [\pi/Q - \zeta]\gamma(\mu) + \sum f_k(\mu)\,\mathrm{trig}(k\zeta), \quad 0 < \zeta < 2\pi/Q \tag{1}$$

where $\gamma(\mu)$ is some nominal circulation, μ is the nondimensional radial coordinate ($\mu=\Omega r/V$), ζ is the nondimensional axial screw coordinate, and Q is the number of blades. (Note that μ is also the cotangent of the inflow angle ϕ). The first term in Eq. (1) is the nominal velocity potential that is chosen to give a nondimensional velocity distribution $\lambda(\mu) = \gamma(\mu)/\cos(\phi)$. The second part of Eq. (1), involving $f_k(\mu)$, is a correction term that accounts for root effects. (The summation is taken over appropriate k's as will be defined later.)

This total velocity potential (nominal plus correction) must satisfy Laplace's equation in helical coordinates.

$$(\mu\partial/\partial\mu)^2\Phi + (1+\mu^2)\partial^2\Phi/\partial\zeta^2 = 0 \tag{2}$$

This implies that the correction functions $f_k(\mu)$ are related to a set of basic functions $h_k(\mu)$—governed by a differential equation that follows from Eqs. (1) and (2)—namely:

$$(\mu d/d\mu)^2[h_k(\mu)] - k^2(1+\mu^2)h_k(\mu) = -(\mu d/d\mu)^2[\gamma(\mu)] \tag{3}$$

(We will demonstrate the relationship between f_k and h_k later.) The resultant circulation per blade $\gamma_b(\mu)$ and the resultant velocity distribution in the wake $\lambda(\mu)$ can be given in terms of the total velocity potential:

$$\begin{aligned} \gamma_b(\mu) &= \Phi(\mu,0) - \Phi(\mu,2\pi/Q) \\ \lambda(\mu) &= -\sqrt{\frac{1+\mu^2}{\mu^2}}\left.\frac{\partial\Phi}{\partial\zeta}\right|_{\zeta=0} \end{aligned} \tag{4}$$

Therefore, once $h_k(\mu)$ and $f_k(\mu)$ are determined, the circulation and velocity can be found; and that is the focus of what is to follow.

First, consider the forward problem, "Given a specified circulation on the blades, $\gamma_b = (2\pi/Q)\gamma(\mu)$, find the resulting velocity distribution $\lambda(\mu)$." Since the circulation on the blades—and consequently $\gamma(\mu)$—are specified, the trig functions in Eq.(1) are taken as $\sin(k\zeta)$, where k = Q, 2Q, 3Q, etc., so that the specified $\gamma_b(\mu)$ is not affected. For convenience, $[\pi/Q - \zeta]$ is then expanded as a summation of sine terms over the same k's.

$$[\pi/Q - \varsigma] = \sum_{k=Q,2Q,3Q,\ldots}^{\infty} \frac{2}{k}\sin(k\zeta) \tag{5}$$

It follows that, in order to obtain the standard relation of Eq. (3) from Laplace's equation in Eq. (2), one must define $f_k(\mu) \equiv 2h_k(\mu)/k$. Once an appropriate solution for $h_k(\mu)$ is found from Eq. (3), the velocity in the far wake due to any specified circulation can be computed from Eq. (4) in terms of the $h_k(\mu)$ as follows:

$$\lambda(\mu) = \frac{\sqrt{1+\mu^2}}{\mu}\left[\gamma(\mu) - \sum_{k=Q,2Q,3Q,\ldots}^{\infty} 2h_k(\mu)\right] \tag{6}$$

(Note that the velocity in the far wake is twice the velocity at the rotor disk.) The correction in Eq. (6) involves the sum of $2h_k(\mu)$ over k = Q, 2Q, 3Q, etc.

Next, consider the inverse problem, "Given a velocity distribution $\lambda(\mu)$, find the applied circulation $\gamma_b(\mu)$ that would produce it." In order to preserve the desired velocity $\lambda(\mu)$ from Eq. (4), we take $\mathrm{trig}(k\zeta) = \cos(k\zeta)$ in the summation of Eq. (1) with k = Q/2, 3Q/2, 5Q/2, etc. This ensures that the derivative of Φ will be zero at the boundary. It is then convenient to expand $[\pi/Q - \zeta]$ in cosine terms summed over these same k's.

(7) $$[\pi / Q - \zeta] = \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots}^{\infty} \frac{2Q}{\pi k^2} \cos(k\zeta)$$

The nominal circulation to obtain the desired velocity distribution is

(8) $$\gamma(\mu) = \frac{\mu}{\sqrt{1+\mu^2}} \lambda(\mu)$$

and this becomes the forcing function for the correction terms in Eq.(3).

When Eq. (1) and Eq. (7) are placed into Eq. (2), it is clear that one must define the relation $f_k(\mu) \equiv 2Qh_k(\mu)/(\pi k^2)$ in order to obtain the standard form of Eq. (3). It follows that the total circulation distribution per blade is given from Eq. (4) as:

(9) $$\gamma_b(\mu) = \frac{2\pi}{Q}\left[\gamma(\mu) + \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots}^{\infty} \frac{2Q^2}{\pi^2 k^2} h_k(\mu)\right]$$

For the summation over k in Eq. (9), $\pi k/Q = \pi/2, 3\pi/2, 5\pi/2$, etc.

Both the forward problem (given circulation, find velocity) and the inverse problem (given velocity, find circulation) are governed by $h_k(\mu)$ from Eq.(3) with the appropriate k, as given by Eqs.(6) and (9). Notice that, in the forward problem, the velocity is decreased due to $h_k(\mu)$; while, in the inverse problem, the circulation is increased due to $h_k(\mu)$. Below is a table of typical blade numbers Q along with the values of k that would be needed to solve for either the forward or inverse problem in each case.

Q	k for forward problem	k for inverse problem
1	1, 2, 3, 4, 5, . . .	1/2, 3/2, 5/2, 7/2, 9/2, . . .
2	2, 4, 6, 8, 10, . . .	1, 3, 5, 7, 9, 11, . . .
3	3, 6, 9, 12, 15, . . .	3/2, 9/2, 15/2, 21/2, . . .
4	4, 8, 12, 16, 20, . .	2, 6, 10, 14, 18, . . .

For most applications (i.e., for forward problems with $Q \geq 2$), only integer values of k having $k \geq 2$ are required. Nevertheless, for completeness, here we will consider all integer multiples of 1/2 from k = 1/2 and on up.

Goldstein, Ref. 2, determined $\gamma_b(\mu)$ in closed form for some special conditions—namely: 1.) for the inverse problem only, 2.) for circulation $\gamma(\mu) = \mu^2/(1 + \mu^2)$, and 3.) for cases with Q = 2 or Q = 4. Here, we attempt a general solution for: 1.) both the forward and inverse problems, 2.) for general $\gamma(\mu)$, and 3.) for general Q. The Goldstein solution, although limited, will provide a good benchmark to ensure that our numerical approach is correct. The Goldstein procedure is provided in Appendix A, and examples of $h_k(\mu)$ from that methodology are given in Fig. 3 for k = 1, 2, 3. One can see from that figure that the general shape of the correction term is an initial peak near $\mu = 0.5$ with a zero near $\mu = 1.0$ followed by a secondary, negative peak and decay, just as we saw in Figs. 1-2.

Numerical Computation

Next, we formulate a numerical solution to the root-correction problem. Since both the forward and inverse problems are governed by $h_k(\mu)$ from Eq. (3), the goal is to find a numerical solution for $h_k(\mu)$. In particular, we are looking for a method that could be applied efficiently to dynamic conditions in which the circulation distribution is given numerically and may be changing as a function of time.

If one: 1.) takes the original differential equation in Eq. (3), 2.) expands $h_k(\mu)$ in a Galerkin series of admissible functions $c_m\Phi_m(\mu)$, 3.) multiplies by the comparison functions $\Phi_j(\mu)/\mu$, and 4.) integrates from zero to infinity, one obtains a numerical Galerkin scheme to solve for $h_k(\mu)$. Integration by parts takes the equations into a symmetric form,

(10) $$[\mathbf{A}]\ \{\mathbf{c}\} = \{\mathbf{B}\}$$

$$A_{jm} = k^2 \int_0^\infty \frac{1+\mu^2}{\mu} \Phi_j(\mu)\Phi_m(\mu)d\mu + \int_0^\infty \mu \left[\frac{d\Phi_j(\mu)}{d\mu}\right]\left[\frac{d\Phi_m(\mu)}{d\mu}\right]d\mu \tag{11}$$
$$B_j = -\int_0^\infty \mu \left[\frac{d\Phi_j(\mu)}{d\mu}\right]\left[\frac{d\gamma(\mu)}{d\mu}\right]d\mu = +\int_0^\infty \frac{d}{d\mu}\left[\mu \frac{d\Phi_j(\mu)}{d\mu}\right][\gamma(\mu)]d\mu$$

Note that the second version of B_j does not require that any derivatives of $\gamma(\mu)$ be taken.

Since the $\Phi_m(\mu)$ must be zero both at $\mu = 0$ and at $\mu = \infty$, the μ in the denominator of $\Phi_j(\mu)/\mu$ causes no problems in the integrals, but rather serves to weight more heavily the low-μ range (where the greater accuracy is required). The use of $\Phi_j(\mu)/\mu$ also serves to make the matrices symmetric. As a consequence, the formulation in Eqs. (10) and (11) is robust.

Next, a change of variable maps the domain and the integrals onto $0 < x < 1$

$$x = \frac{\mu}{\sqrt{1+\mu^2}} = \cos(\phi) \tag{12}$$

where ϕ is the angle of the vortex sheet leaving the blade (90º at the blade root). This change of variable allows admissible and comparison functions to be chosen on a more convenient interval, $0 < x < 1$. The integrals are consequently transformed to:

$$A_{jm} = k^2 \int_0^1 \frac{1}{x(1-x^2)^2} \Phi_j(x)\Phi_m(x)dx + \int_0^1 x(1-x^2)\left[\frac{d\Phi_j(x)}{dx}\right]\left[\frac{d\Phi_m(x)}{dx}\right]dx \tag{13}$$
$$B_j = -\int_0^1 x(1-x^2)\left[\frac{d\Phi_j(x)}{dx}\right]\left[\frac{d\gamma(x)}{dx}\right]dx = +\int_0^1 \frac{d}{dx}\left[x(1-x^2)\frac{d\Phi_j(x)}{dx}\right][\gamma(x)]dx$$

For test functions, we chose the combination of Legendre polynomials $P_j(2x-1)$ that have been applied to the p-version finite element method, Ref. 5,

$$\Phi_j = \frac{P_j(2x-1) - P_{j-2}(2x-1)}{\sqrt{2(2j-1)}} \qquad j = 2,3,4,\ldots \tag{14}$$

For $j \geq 2$, the polynomials are zero both at $x = 0$ and at $x = 1$. The derivatives of the $\Phi_j(x)$ are the Jacobi polynomials. The resultant matrices from the integrals in Eq. (13) can be found in closed form and are very well-conditioned. Figure 4 shows for $n = 2$ (with either $k = 1$ or $k = 8$) convergence of the method as the total number of terms in the Galerkin method is increased from 3 to 5 to 15. At 15 terms, the solution is virtually exact.

The above Galerkin method can be used to solve for all $h_k(\mu)$ and thus obtain the proper root correction for all cases. To apply this methodology to a particular problem, at each time step in the solution one would: 1.) find the nominal $\gamma(\mu)$ for that condition, 2.) form A_j and B_j from Eq. (13), 3.) solve for the appropriate set of $h_k(\mu)$ due to $\gamma(\mu)$ (where k would take on values of Q, 2Q, 3Q, etc. for the forward problem or Q/2, 3Q/2, 5Q/2, etc, for the inverse problem), and 4.) modify either the induced flow by the summation in Eq. (6) or the circulation by the summation in Eq. (9) as appropriate. Since tip and root corrections are virtually instantaneous, no time delay would be needed in such an application.

Practical Solutions

Despite the efficacy of the direct method described above, it is often advantageous to have a simpler and more easily implemented method available. In this section, we offer such a procedure. To begin, Goldstein shows in Ref. 2 that $h_k(\mu)$ can be expanded in a series the first term of which is

$$h_k(\mu) = \left(\frac{1}{k^2}\right)\left(\frac{1}{1+\mu^2}\right)\left(\mu\frac{d}{d\mu}\right)^2[\gamma(\mu)] \equiv \frac{1}{k^2}H(\mu) \tag{15}$$

Although the series (for which this is the first term) does not converge for $k/n \leq 1$, the first term nonetheless has a reasonable shape that seems qualitatively to approximate $h_k(\mu)$ over a wide range of k and over a large class of $\gamma(\mu)$.

For example, consider the family of circulation distributions of the following form (where n=2 is the case treated by Goldstein).

$$\gamma(\mu) = \left[\frac{\mu}{\sqrt{1+\mu^2}}\right]^n \equiv x^n \tag{16}$$

For this family of distributions, one can write the resultant H(μ)—or alternatively H(x)—in closed form from Eq. (15) as

$$H(x) = n^2 x^n \left(1-x^2\right)^2 \left(1-x^2/b^2\right); \; b = \sqrt{\frac{n}{n+2}} \tag{17}$$

Thus, for all values of n, H(x) from Eq. (17) has zeroes at x = 0, x = b, and x = 1; it has a positive peak in the range $x \leq b$; and it has a negative, smaller peak in the range x > b. This behavior follows the general form of the exact solution.

By comparing Eq.(17) with numerical results, we have been able to determine a magnitude-correction factor that gives a reasonable approximation for $h_k(\mu)$—valid over both a broad range of k and a broad class of circulation distributions.

$$h_k(\mu) = \left[\frac{k^2+2.0}{\left(k^2+0.5\right)\left(k^2+3.5\right)}\right] \frac{(\mu d/d\mu)^2[\gamma(\mu)]}{1+\mu^2} \tag{18}$$

Equation (18) provides a valuable computational tool for use in root corrections. It is quantitatively accurate for k > 2 and qualitatively accurate for k =2 for all n. Thus, it is an excellent approximation for the forward problem with rotors having three or more blades (or the inverse problems with rotors having four or more blades); and it is a reasonable approximation for the forward problem with rotors having two blades (or the inverse problem for rotors having three blades).

In the application of this approach, note that (μd/dμ) = (rd/dr) where r is the radial position. It follows that $(\mu d/d\mu)^2\gamma(\mu) = r d\gamma/dr + r^2 d^2\gamma/dr^2$ which can be readily computed for any general circulation distribution. Furthermore, since the factor $1+\mu^2 = 1 + (\Omega r/V)^2$ is in the denominator, the correction quickly dies out as r moves away from the root (i.e., as the inflow angle becomes less than 45º).

For example, given a helicopter in climb with inflow, ΩR/V = 10, the resultant denominator in Eq. (13) would be $[1 + (10r/R)^2]$. Since the maximum $h_k(\mu)$ occurs at about μ = 0.4 (which is r/R = .04)—and since the zero cross-over is at about μ = 1.0 (which is r/R = .10)—the bulk of the root correction is relegated to the region of the inner 10% of the rotor radius, much of which would be inboard of the blade, itself (depending on the root cut-out). Furthermore, since a typical maximum value of $h_k(\mu)$—for the region μ > 1.0—is h_k = .015, the correction for r/R > 0.1 would be less than 3%. Thus, the correction need only be computed over a small range of blade stations.

Higher-Order Approximation

If the above solution is not accurate enough for a particular application, there is an alternate procedure that we have developed in which a more general functional form is used for arbitrary values of k and n in which x^n is replaced by x^q and $(1-x^2)^2$ by $(1-x^2)^s$

$$H(x) = F(k,n) x^q \left(1-x^2\right)^s \left(1-x^2/b^2\right) \equiv h_A \tag{19}$$

where F, b, q, and s depend on k and n. To utilize this form, one must map a given circulation distribution γ(μ) onto x, (i.e., 0 < x < 1), and then fit that distribution with a polynomial in x^n where n = 1, 2, 3, . . .

$$\gamma(\mu) = \sum_{n=1,2,3,\ldots}^{\infty} \gamma_n x^n \tag{20}$$

The root correction then appears in the form of a summation over n = 1, 2, 3, . . .

(21) $$h_k(\mu) = \sum_{n=1,2,3,\ldots}^{\infty} \gamma_n F(k,n) x^q \left(1-x^2\right)^s \left(1 - x^2/b^2\right)$$

where $x = \mu/\sqrt{1+\mu^2}$. Of course, for blades with a root cut-out ε and a finite length, R, γ(μ) is assumed to be zero outside of the interval [Ωε/V < μ < ΩR/V)].

The parameters for Eq. (21) have been determined by an optimal fit of the numerical results from the Galerkin/Jacobi method over a broad range of data from n = 0 to n = 4 and from k = 0.5 to k = 8. The parameters for the polynomial are represented as follows.

(22) $$q = n - \frac{7.09n}{(1+0.062n)} \cdot \frac{\left(k^2+3.84\right)}{\left(k^4+16.5k^2+29.3\right)}$$

(23) $$s = 2 - \frac{8v_n}{8+k^2}$$
$$v_1 = 1.00,\ v_2 = 1.40,\ v_n = 1.52 \quad n \geq 3$$

(24) $$\frac{1}{b^2} \equiv a = 1 + \frac{2}{n} - \frac{4w_n}{n\left(4+k^2\right)}$$
$$w_1 = 7/16,\ w_2 = 7/8,\ w_n = 1.0 \quad n \geq 3$$

The weighting function F(k, n) is approximated as given below.

(25) $$F(k,n) = \frac{n^2}{k^2+d}; \quad \text{where } d = \min(d_1, d_2)$$
$$d_1 = \frac{\alpha + \beta k}{1+\delta k}; \quad \beta = \delta(7.25 + 5.75n)$$

For $n = 1$:	$\alpha = 0.610$,	$\delta = 0.510$,	$d_2 = 10.0$
For $n = 2$:	$\alpha = 1.32$,	$\delta = 0.935$,	$d_2 = 16.0$
For $n \geq 3$:	$\alpha = 1.66n - 1.32$,	$\delta = 1.0$,	$d_2 = 3n + 10.0$

Note that the min(d_1, d_2) is d_1 for small k; but is d_2 at larger k.

The above approximation was determined by utilization of 1.) q to match the peak location, 2.) s to match the ratio of maximum to minimum, and 3.) F(k, n) to match the maximum peak of the numerical result. The parameter values were then approximated by a set of appropriate functions. The solution in Eqs. (19) - (25) is valid for k > 0.5, and so is applicable to all cases with the exception of the inverse problem for one-bladed rotors. For one-bladed rotors—or for increased accuracy at very small μ for other cases—boundary corrections can be made. We treat these corrections in the next section.

Boundary Corrections

The assumed solution in Eq. (19) emerges from x = 0 as x^q and ends near x = 1 as $(1-x)^s$. However, the behavior of the exact solution close to the root can be more complicated than this and can involve fractional powers and log terms. These boundary terms have only minor effect on the results at most conditions, but can have significant effects for the case of k = 0.5 which emerges as $h_k = x^{1/2}$ for small x. Our approach here is to find closed-form solutions (near x = 0) and then blend that with the approximate solution above so as to model properly the boundary behavior whenever such behavior is important. For simplicity, this boundary behavior is studied in terms of the variable x at the root with the assumption x << 1.

Equation (3) can be written as an expansion in x for small x. Then both the particular and homogenous solutions can be found in closed form via a straightforward application of the theory of differential equations. Occasionally, secular terms appear in the expansion which result in xln(x) terms; and thus there are special cases to the particular solution, as shown below.

(26) $$k \neq n,\ k \neq n+2 \qquad h_k = \frac{x^n}{k^2-n^2}\left[n^2 - \frac{x^2 k^2\left(3n^2+n\right)}{k^2-(n+2)^2}\right]$$

(27) $$k = n+2 \qquad h_k = \frac{x^n}{k^2-n^2}\left[n^2 - \frac{1}{2}x^2\ln(x)\left(n^2+2n\right)(3n+2)\right]$$

(28) $$k = n \qquad h_k = -x^n \ln(x)\left[\frac{n}{2} + \frac{n^2x^2}{4}(3n+2)(3n+4)\right] + \frac{n}{4(n+1)}x^{n+2}$$

The homogeneous solution is given by the following, which is independent of n,

(29) $$h_k = c_0 x^k\left[1 + x^2\frac{\left(3k^2+2k\right)}{4(k+1)}\right]$$

where c_0 is a constant that can be determined by matching the homogeneous inner solution with the particular outer solution.

Although a matched asymptotic expansion to obtain c_0 is beyond the scope of this paper, by comparison of the sum of the particular and homogeneous solutions above with the numerical results of the Galerkin Method, we have: 1.) verified that the numerical solution gives the proper root behavior and 2.) found an approximation for the values of c_0 in the homogeneous solution of Eq. (29), as shown below.

(30)

For $n \neq k$

$$k = \frac{1}{2} \qquad c_0 = \frac{0.20(n+6)}{(n+k)}$$

$$k = 1 \qquad c_0 = \frac{1.60(n-1/2)}{\left(n^2-k^2\right)}$$

$$k = \frac{3}{2} \qquad c_0 = \frac{1.15(n+1/2)}{\left(n^2-k^2\right)}$$

$$k \geq 2 \qquad c_0 = \frac{k^2+1}{\left(n^2-k^2\right)n^{k-2}}$$

For $n = k$

all k's $$c_0 = \frac{-17(5n-4)}{150}$$

Note that, for n > k, it is the homogeneous solution that dominates in the small-x region; whereas, for n < k, it is the particular solution that dominates. For n = k, the secular terms involving ln(x) dominate.

Although the tip behavior is not as crucial to accuracy as is the root behavior, it is nonetheless interesting to look at its form. For the tip solution, only particular terms are involved; and the expansion can be given in terms of 1-x.

(31) $$h_k = -\frac{8n}{k^2}(1-x)^2\left[1-(1-x)^2\left(2n+3+\frac{16}{k^2}\right)\right]$$

We have verified that this expansion matches the numerical solution from the Galerkin method for the region $0.98 < x < 1.0$ ($5 < \mu < \infty$).

To blend the closed-form solution in Eq.(19)—i.e., $h_A(x)$—with the sum of the particular and homogeneous solutions in Eqs. (26-30)—i.e., $h_R(x)$—we multiply $h_R(x)$ by a function that brings it to the peak of $h_A(x)$ and with zero slope.

(32) $$h(x) = h_R(x)\left[1 + \frac{x_m h_{Am} h'_m}{h_{Rm}^2}\left(\frac{x}{x_m} - \frac{x^2}{x_m^2}\right) + \frac{2\left(h_{Am}-h_{Rm}\right)}{h_{Rm}}\left(\frac{x}{x_m} - \frac{1}{2}\frac{x^2}{x_m^2}\right)\right] \qquad x < x_m$$

$$h_{Am} \equiv h_A(x_m),\ h_{Rm} \equiv h_R(x_m),\ h'_m \equiv \frac{dh_R}{dx}(x_m)$$

This solution is only for $x \leq x_m$ after which one transitions to the original solution in Eq.(19).

(33) $$h(x) = h_A(x) \qquad x \geq x_m$$

The parameter x_m is the location of the first positive peak of $h_A(x)$ and is given by the smaller positive root of the quartic equation:

(34) $$(1+s+t)x^4 - [(s+t)a+1+t]x^2 + at = 0$$

where $a \equiv 1/b^2$ is from Eq. (24); and s and t are from Eqs. (22) and (23) with $t \equiv q/2$. This can be solved for x_m^2 by the quadratic equation, and then the value of x_m follows immediately. With this root boundary correction, the approximation in Eq. (32) is valid even for one-bladed rotors. However, for maximum accuracy, one can always revert to the Galerkin procedure.

Numerical Results

Figure 5 compares the virtually exact Galerkin method (Polys) with both the simplest approximation (Quick) from Eq. (18), and the higher-order approximation (Complex) from Eqs. (19, 22-25). For illustration, the following family of distributions is chosen as typical inputs.

(35) $$\gamma(\mu) = x^n = \left[\frac{\mu}{\sqrt{1+\mu^2}}\right]^n$$

Results are plotted on $0 < x < 1$, which the entire range $0 < \mu < \infty$. Figure 5 shows that the simple method is **qualitatively** accurate at all values of n for $k \geq 2$ and is **quantitatively** accurate at all n's for $k > 2$. Thus, the simple method should be sufficient either in applications to the forward problem (for rotors with two or more blades) or in applications of the inverse problem (for rotors with three or more blades). Figure 5 also demonstrates that the higher-order approximation is **qualitatively** accurate for all cases and **quantitatively** accurate for all cases except $k = 1/2$ with $n \geq 3$. Figure 6 shows the improvement in the approximate functions that can be obtained when the boundary corrections in Eqs. (26) - (31) are blended according to the algorithm in Eqs. (32) - (34). Because Fig. 6 deals only with a small-x correction (there is no correction past the first peak of the curve), results are plotted on $0 < x < 0.5$ (i.e., $0 < \mu < 0.577$). The curves labeled “Polys” are the virtually exact solution from the Galerkin procedure with the Legendre polynomials. The curves labeled “H_a” are the improved approximation based on Eq.(19) but without the root correction. The curves labeled “root correction” are the small-x asymptotic expansions. Finally, the curves labeled "H_{shift}" are the results with the blending of the other two results according to Eq. (32). One can see that this procedure greatly improves the small-x behavior of the approximation for all values of k and n.

When deciding whether or not to utilize one of the more accurate approximations, however, one must weigh the gains against the losses. The higher-order approximations involve mapping the instantaneous circulation onto $0 < x < 1$ and then fitting it with a power series in x^n. After this, one must solve for a family of $h_k(\mu)$ for every power of n, and then sum all effects over n and k. This is much more involved than simple utilization of Eq. (18). At some point, if additional accuracy is desired, one might also consider the possibility of the direct Galerkin solution to obtain $h_k(x)$; but a summation over k is still required. With the simplest approach, on the other hand, one needs only first and second derivatives of the existing circulation distribution; and even the k-summation can be eliminated, as shown in the section to follow.

Applications

We now utilize the above methodologies in some typical applications. Because there is always a summation over k (either for the forward or inverse problem), the simple method of Eq. (18) has another special advantage. In particular, since the functional derivatives taken over μ (or over r) do not depend on k, the correction factors in the summations can be summed *a priori* for use in any application. For example, in the forward problem, we need the summation of $2h_k(\mu)$ in Eq. (6) which involves

(36) $$\sum_{k=Q,2Q,3Q,...}^{\infty} \frac{2(k^2+2)}{(k^2+1/2)(k^2+7/2)} = \sum_{k=Q,2Q,3Q,...}^{\infty} \left[\frac{1}{k^2+1/2} + \frac{1}{k^2+7/2}\right]$$

where k takes on multiples of Q. Appendix B gives the derivation of closed-form expressions that can then be used to find the summations. To utilize these theorems, we put Eq. (36) into the form of the theorems in Appendix B:

(37)
$$\sum_{k=Q,2Q,3Q,\ldots}^{\infty} 2h_k(\mu) = F(Q)\frac{(\mu d/d\mu)^2[\gamma(\mu)]}{1+\mu^2}$$

(38)
$$F(Q) = \frac{1}{Q^2}\sum_{k=Q,2Q,3Q,\ldots}^{\infty}\left[\frac{1}{n^2+p_1^2}+\frac{1}{n^2+p_2^2}\right]$$

(39)
$$p_1^2 = \frac{1}{2Q^2} \qquad p_2^2 = \frac{7}{2Q^2}$$

where n = 1, 2, 3, . . .∞. The theorems of Appendix B yield a closed-form expression for this correction factor.

(40)
$$F(Q) = -\frac{8}{7}+\frac{\pi}{2Q^2}\left[\frac{1+\cosh(2\pi p_1)}{p_1\sinh(2\pi p_1)}+\frac{1+\cosh(2\pi p_2)}{p_2\sinh(2\pi p_2)}\right]$$

We note that Eq. (40) is well-approximated by a simple rational function of Q.

(41)
$$F(Q) = \frac{3.3Q+6.1}{Q^3+1.8Q^2+2.0Q}$$

This is the final form for the correction factor to be applied to Eq. (37).

For the inverse problem, the summation in Eq. (9) implies a correction factor G(Q) that must similarly multiply $[1+\mu^2]^{-1}(\mu d/d\mu)^2\,[\gamma(\mu)]$,

(42)
$$\gamma_b(\mu) = \frac{2\pi}{6}\left[\gamma(\mu)+G(Q)\left(\mu\frac{d}{d\mu}\right)^2\gamma(\mu)\right]$$

where

(43)
$$G(Q) = \frac{Q^2}{\pi^2}\sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots}^{\infty}\left[\frac{1}{k^2(k^2+1/2)}+\frac{1}{k^2(k^2+7/2)}\right]$$

where k takes odd multiples of Q/2. Application of Appendix B yields a closed form for G(Q).

(44)
$$\begin{aligned}G(Q) &= \frac{8}{7}-\frac{4}{p_1\pi}\frac{1+\cosh(2\pi p_1)}{\sinh(2\pi p_1)}+\frac{2}{p_1\pi}\frac{1+\cosh(\pi p_1)}{\sinh(\pi p_1)}\\ &\quad -\frac{4}{7p_2\pi}\frac{1+\cosh(2\pi p_2)}{\sinh(2\pi p_2)}+\frac{2}{7p_2\pi}\frac{1+\cosh(\pi p_2)}{\sinh(\pi p_2)}\\ p_1^2 &= \frac{2}{Q^2},\quad p_2^2 = \frac{14}{Q^2}\end{aligned}$$

This G(Q) also has a simple approximation in terms of a rational function.

(45)
$$G(Q) = \frac{3.3}{Q^2+Q+2.8}$$

The closed-form expressions for F(Q) and G(Q) allow for efficient application of the formula for root corrections of either forward or inverse problems. It is interesting that F(Q) and G(Q) have the same large-Q behavior, $F = G = 3.3/Q^2$.

The total correction can be computed for either the forward or the inverse problem in a straightforward manner. In particular, combination of Eqs. (6) and (9) with the closed-form Eq. (18) gives solutions that can be cast dimensionally. For the forward problem with Q blades—and with a nominal velocity at the rotor disk of $v_0(r)$—the corrected velocity field at the rotor disk can be given in terms of the circulation per blade Γ_b as:

(46) $$v(r) = v_0(r) - \frac{Q}{4\pi}[F(Q)\sin\phi]\left[\frac{d\Gamma_b}{dr} + r\frac{d^2\Gamma_b}{dr^2}\right]$$

For the inverse problem—if $\Gamma_0(r)$ is the nominal total circulation per blade that gives the desired velocity field without root effects—the corrected circulation distribution per blade is given by:

(47) $$\Gamma_b(r) = \Gamma_0(r) + G(Q)\sin^2\phi\left[r\frac{d\Gamma_0}{dr} + r^2\frac{d^2\Gamma_0}{dr^2}\right]$$

F(Q) and G(Q) give the effect of blade number, and the $\sin(\phi)$ or $\sin^2(\phi)$ causes the effect to die out away from the root.

It is interesting to consider how the Prandtl tip effect compares to the root effect and how Prandtl's correction could be included with Eq. (46) or (47). To apply the Prandtl method simultaneously with the root correction formulae (for either the forward or inverse problem), one defines the Prandtl correction function as:

(48) $$E = \frac{2}{\pi}\arccos\left[\exp\left(-\frac{Q(R-r)}{2\sin\phi}\right)\right]$$

Equation (48) represents a two-dimensional approximation for the circulation required to obtain a uniform inflow—the optimum distribution in two dimensions. Prandtl shows in Ref. 1 that the formula can also be used to obtain a correction for the flow at the tip of a three-dimensional rotor. Thus, Eq. (48) has been used for many years as a practical tool.

Some dynamic inflow models already include tip-loss effects implicitly in the formulation. However, when tip effects are not included in the nominal inflow, the Prandtl function can be used in conjunction with the root correction function to obtain the entire solution to either the forward or inverse problem. For the forward problem, the application takes the form:

(49) $$v(r) = v_0(r) - \frac{Q}{4\pi}[F(Q)\sin\phi]\left[\frac{d\Gamma_b}{dr} + r\frac{d^2\Gamma_b}{dr^2}\right] + Q\left(\frac{1}{E} - 1\right)\left[\frac{\Gamma_b(r)}{4\pi}\right]$$

If the blade circulation $\Gamma_b(r)$ does not approach zero at the root or the tip, the inflow approaches infinity at that end, which is in accordance with the physics of a concentrated tip or root vortex.

For the inverse problem, the Prandtl correction takes the simple form:

(50) $$\Gamma_b = \left[\Gamma_0(r) + G(Q)\sin^2(\phi)\left(r\frac{d\Gamma_0}{dr} + r^2\frac{d^2\Gamma_0}{dr^2}\right)\right]E$$

The E insures that the circulation has the correct tip behavior—in that it approaches zero at the blade tip in such a way as to give the desired induced flow. It is interesting (and not surprising) that both the root correction and the tip correction explicitly involve both the blade number Q and the angle of the vortex sheet ϕ.

Summary and Conclusions

Several alternative solution methods are offered for finding a correction term for root effects in rotors. These are intended for use as corrections to either dynamic wake models or momentum models—in the same spirit as the Prandtl tip-loss correction. The methods apply either to the forward problem (given circulation, find induced flow) or to the inverse problem (given induced flow, find circulation). They are:

1.) The simplest approach in Eqs. (46-47)—valid for the forward problem in rotors with two or more blades (or for inverse problems in rotors with three or more blades)—involves simple derivatives of the applied circulation multiplied by either the factor F(Q) for the forward problem or by G(Q) for the inverse problem, Eqs. (41) and (45). This method is applied in a straightforward manner to the circulation on the blade Γ_b or Γ_0.

2.) The general form of Eq. (19) is more accurate than the first method and is applicable for either: 1.) forward problems and rotors with one or more blades or 2.) inverse problems and rotors with two or more blades. It involves a more refined fit of the root correction function for individual harmonic numbers k. In this solution methodology, one maps the circulation onto the x domain, where $x = \mu / \sqrt{1+\mu^2}$, and then fits the circulation with a polynomial in x, Eq.

(20). Then the correction functions follow from Eq. (19), with the coefficients from Eqs. (22) - (25). The summation in Eq. (21) gives individual $h_k(\mu)$ each of which is then used in either the forward correction, Eq. (6), or the inverse correction, Eq. (9).

3.) The boundary correction from Eqs. (26) - (30) can be blended with the $h_k(\mu)$—from the second approach—and this extends the applicability to rotors with any number of blades for either the forward or inverse problem. This is an engineering approximation.

4.) The Galerkin procedure gives a highly accurate solution in terms of Legendre polynomials, Eqs. (10), (13), and (14). Since this approach is virtually exact, it can be used either as the method of choice or as a benchmark for the other solutions.

Numerical results verify the accuracy and region of applicability for each method.

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Figures

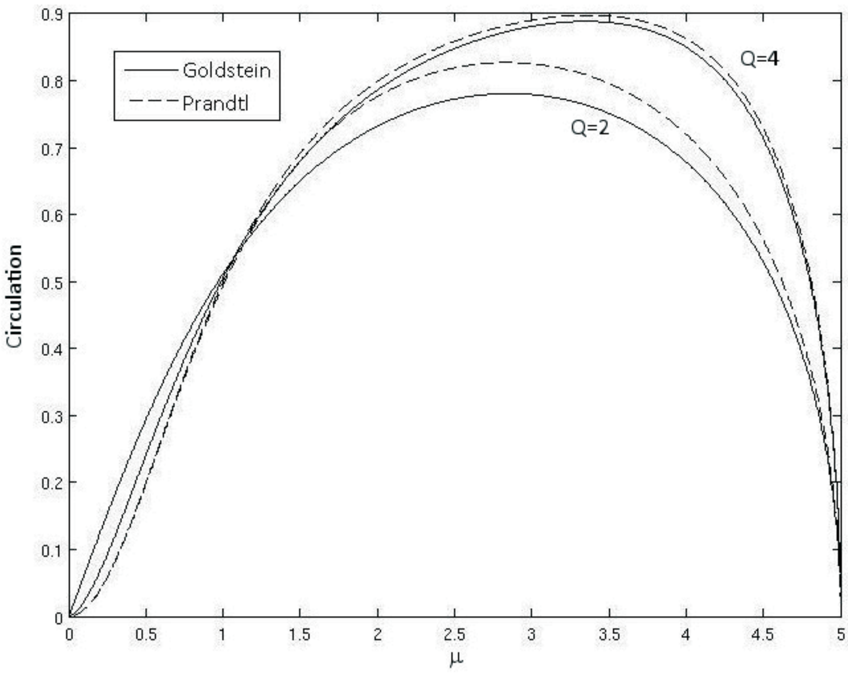


Figure 1. Optimum Circulation for 2-Bladed and 4-Bladed Rotors, Ref. 2

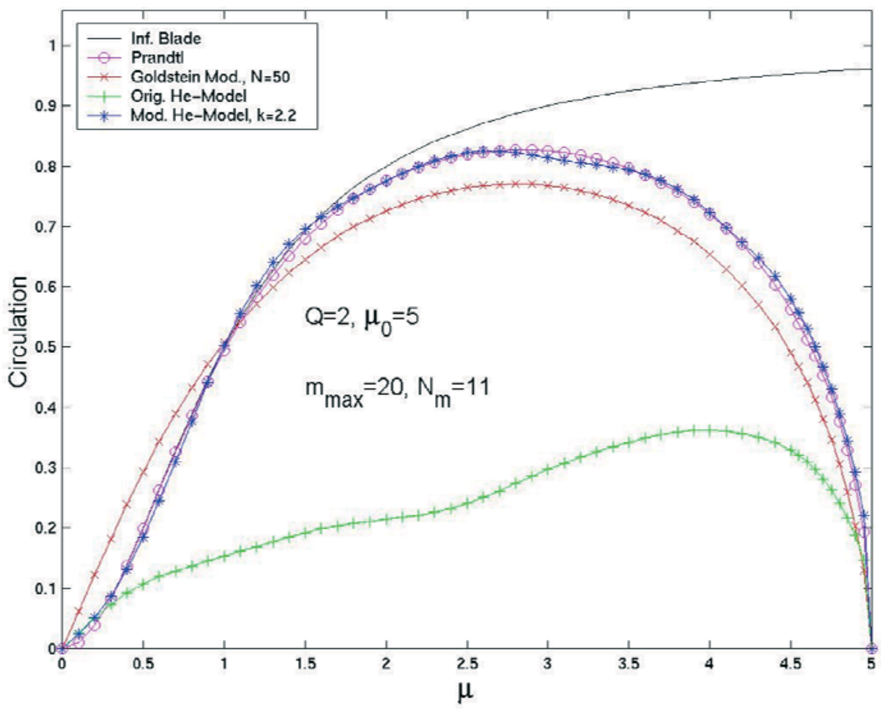


Figure 2. Optimum Circulation by Finite-State Methods, $\mu_0 = 5$

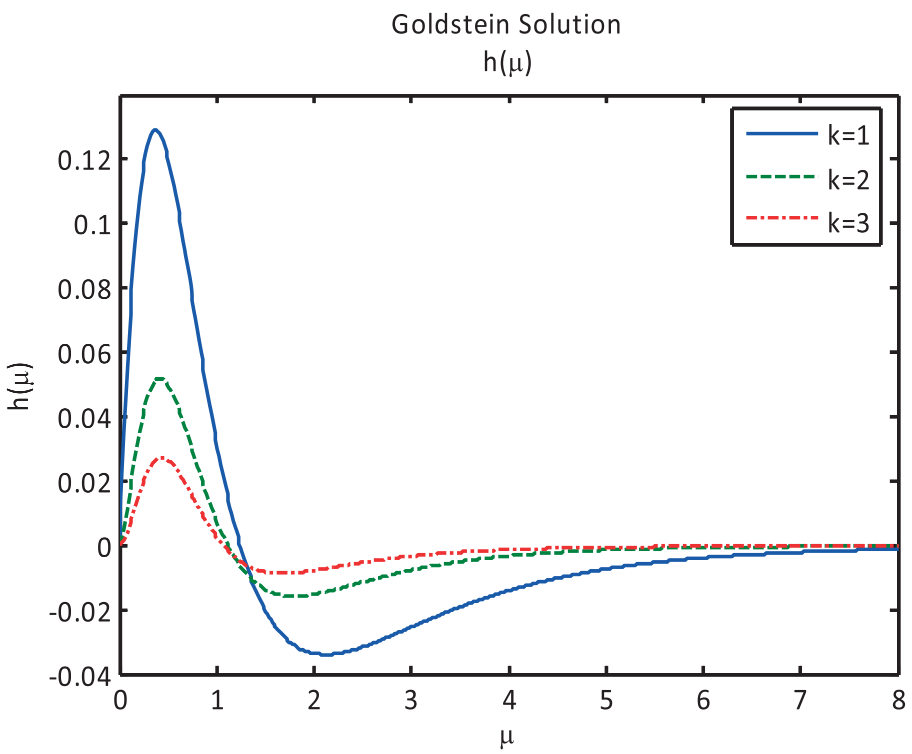
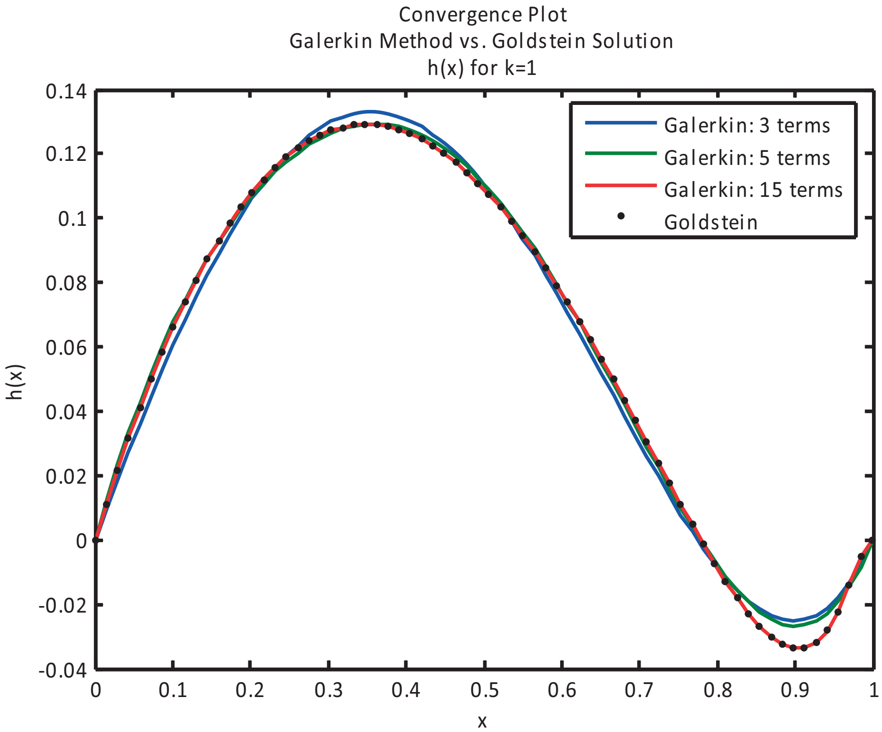


Figure 3. Correction Function for k = 1, 2, 3 by Goldstein Method



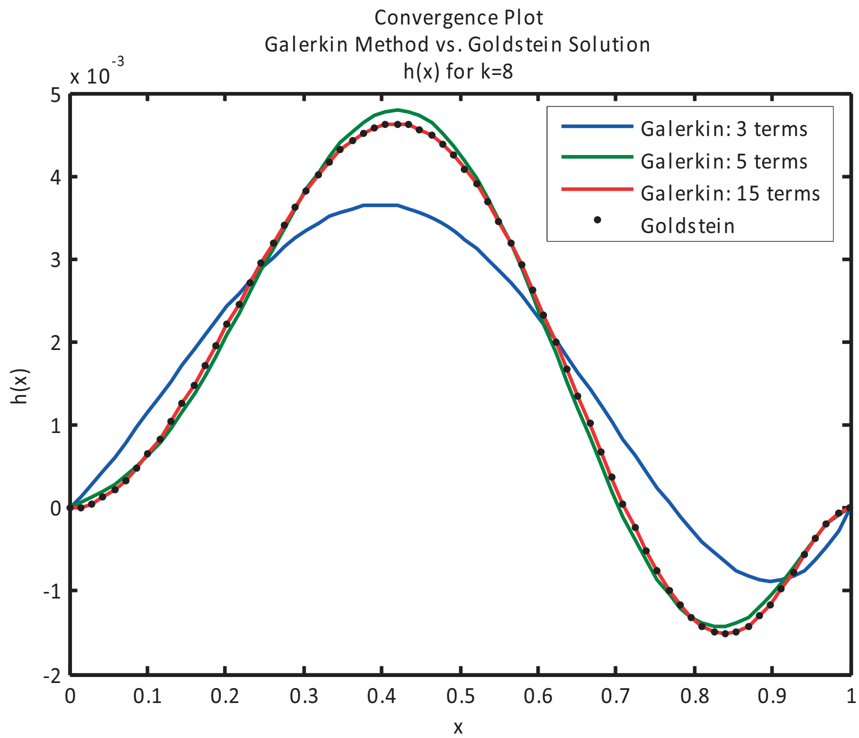


Figure 4. Convergence of Galerkin Method for k = 1 and k = 8, Goldstein Case

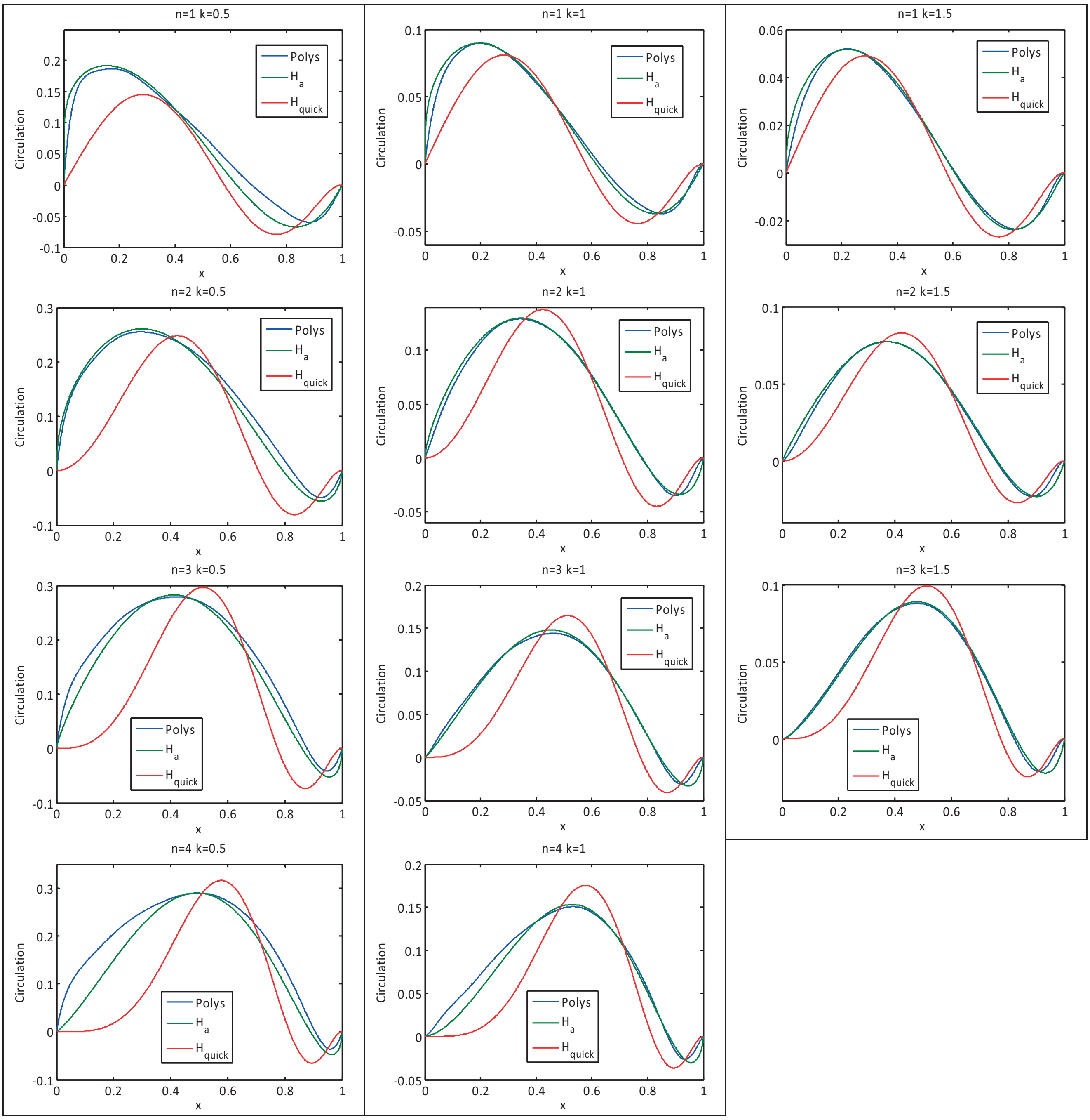


Figure 5. Approximations to Root Correction for Various Cases

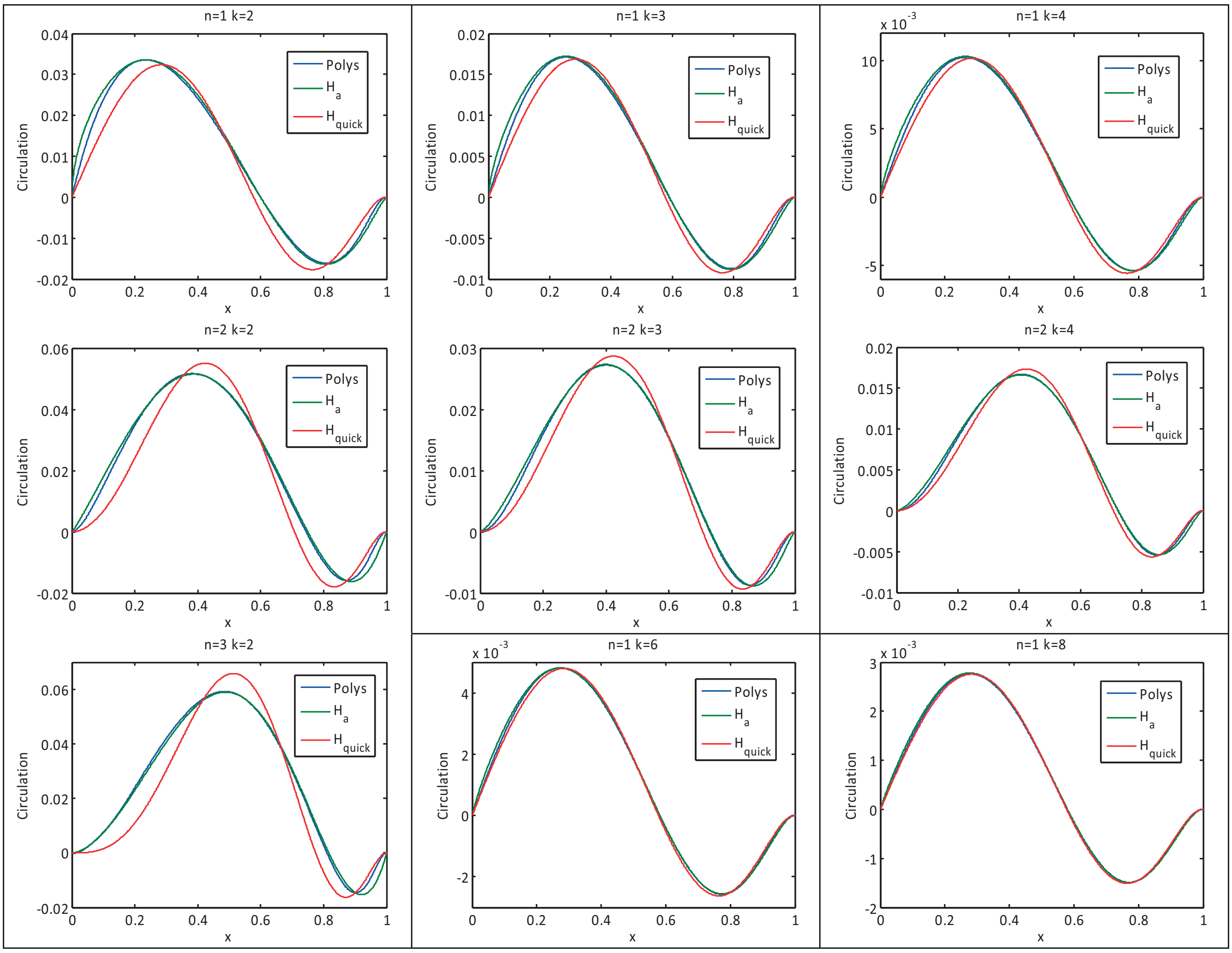


Figure 5. (continued)

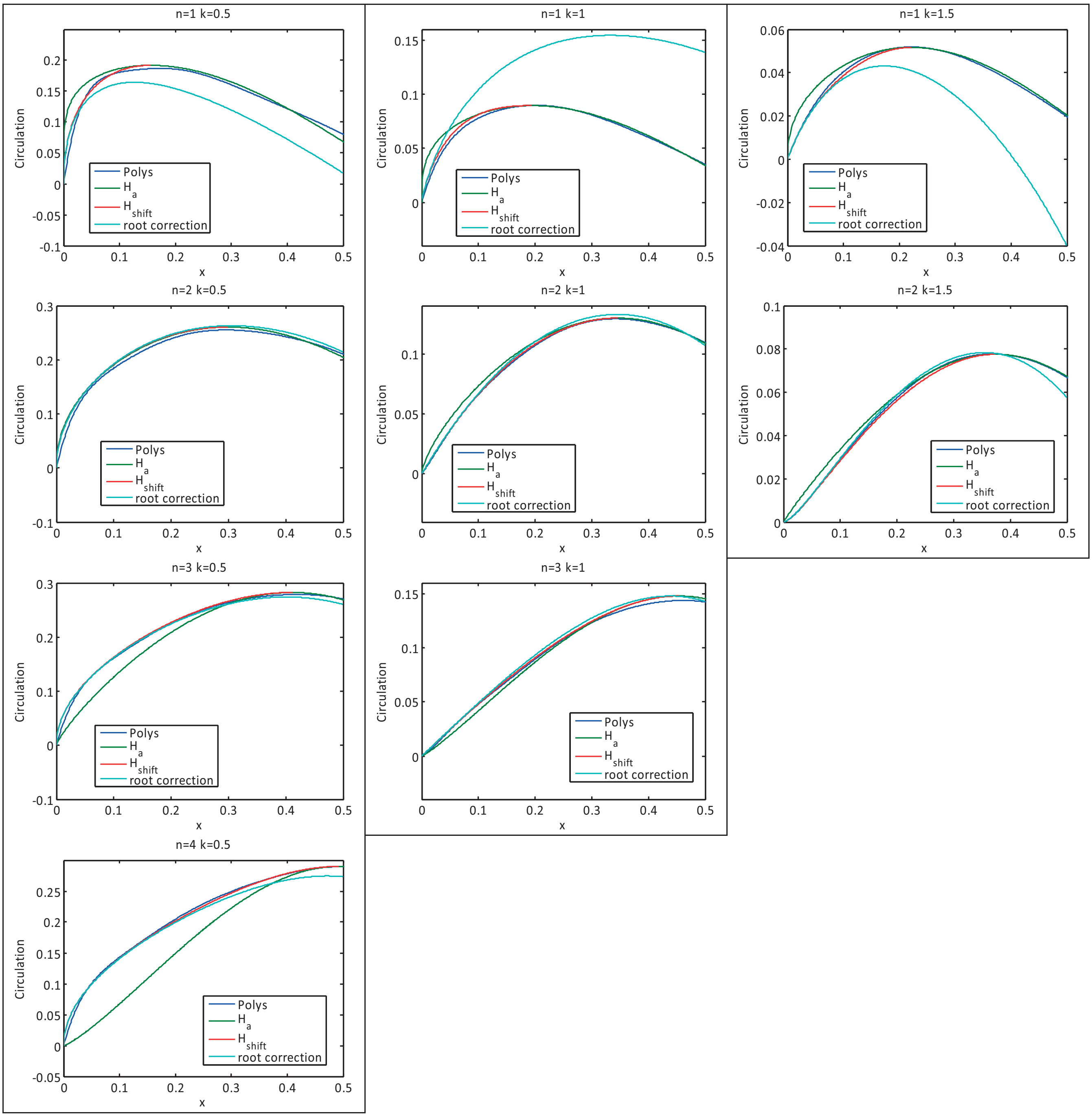


Figure 6. Boundary Terms in Root Correction

Acknowledgment

This work was sponsored by the U.S. Army Research Office, Grant No. DAAD19-01-1-0697, Tom Doligalski, Technical Monitor. Many thanks to Loren Ahaus, graduate research assistant, for his help with editing the text and figures.

Appendices

Appendix A: Goldstein Solution

In this Appendix, we summarize the numerical procedure outlined by Goldstein, which we use for comparisons with our numerical procedures. Goldstein, Ref. 1, solved for $\gamma_b(\mu)$ in closed form for the special conditions of: 1.) the inverse problem, 2.) $\gamma(\mu) = \mu^2/(1 + \mu^2)$, and 3.) cases of Q = 2 and Q = 4, while leaving a framework to solve for any Q. The Goldstein solution, although limited, provides a good benchmark to ensure that our numerical schemes are correct.

While Goldstein found a formal solution to the aforementioned differential equation, that solution is not amenable to numerical evaluation; and it is necessary to use three separate equations—in combination—to obtain an accurate solution for arbitrary values of k. The formal solution can be found based on Goldstein's Eq. (15) of Section 3.1 [p. 444] where our k is denoted by $\nu = 2m+1$ in the Goldstein notation. (Since Goldstein's solution [p. 444] is restricted to the inverse problem for Q = 2, only odd integers occur for k, thus the 2m+1.) Although this solution is in closed form in terms of Bessel functions, the issue yet arises as to how to evaluate those Bessel functions. Goldstein uses Eq. (14) of Section 3.1 [p. 444] and Eq. (2) of Section 3.2 [p. 445] to compute the appropriate functions for Q = 2. For Q =4, however, these series diverge; and so he uses Eq. (1) of Section 4.2 [p. 455]. In order to relate Goldstein's results to the present work, note that $h_k(\mu)$ of this paper corresponds to $F_k(\mu)$ in Goldstein's notation—with the formula for F_k found from Goldstein's Eq. (8) in Section 3.2 [p. 446].

Appendix B: Closed-Form Summation

Here we derive a closed-form expression for the following summation.

(B1)
$$S(p) = \sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{n^2 + p^2}$$

The key to finding this sum is to recognize that it is the solution to a dynamics problem. To be precise, consider the following dynamic system driven by an impulse train:

(B2)
$$\frac{d^2 y}{d\tau^2} - p^2 y = -\sum_{m=-\infty}^{+\infty} \delta(t - 2m\pi)$$

where $\delta(\tau)$ is the Dirac delta function. The periodic solution for $y(\tau)$ is given in closed form as:

(B3)
$$y(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \frac{\exp(in\tau)}{n^2 + p^2}$$

which further implies that

(B4)
$$2\pi y(0) = \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + p^2}$$

This relation can be used to solve for the summation of interest.

(B5)
$$S(p) = \sum_{n=1}^{\infty} \frac{1}{n^2 + p^2} = \pi\, y(0) - \frac{1}{2p^2}$$

The next step is to solve Eq. (B2) directly for $y(\tau)$ by noting that, between impulses, the solution must be of the form:

(B6)
$$y(\tau) = d_1 \cosh(p\tau) + d_2 \sinh(p\tau)$$

The conditions on d_1 and d_2 are that $y(\tau)$ be periodic with $y(0) = y(2\pi)$ and that the slope must jump by negative unity across the impulse, $dy/d\tau\,(0) = dy/d\tau\,(2\pi) - 1$. From that, one can find the constants for the solution on $0 < t < 2\pi$:

(B7) $$d_1 = \frac{1+\cosh(2\pi p)}{2p\sinh(2\pi p)};\ \ d_2 = -\frac{1}{2p}$$

Since y(0) = d_1, the summation over n = 1, 2, 3, . . ∞ [in Eq. (B5)] follows immediately:

(B8) $$S(p) = \sum_{n=1}^{\infty} \frac{1}{n^2 + p^2} = \frac{\pi}{2} \frac{[1+\cosh(2\pi p)]}{p\sinh(2\pi p)} - \frac{1}{2p^2}$$

For the degenerate case of the formula (i.e., p = 0), a limit can be taken in Eq. (B8) to show that the summation reduces to

(B9) $$S(0) = \pi^2 / 6$$

which can be verified by the use of the Riemann zeta function, Ref. (6).

A corollary can be made to the above either for the sum of n= 1,3,5,… [which we will call $S_o(p)$] or for the sum of n = 2, 4, 6, . . ∞ [which we will call $S_e(p)$]. Consider $S_e(p)$ which is the sum over n = 2,4,6. If we replace n by 2m in the series, it will change from the sum of n = 2, 4, 6 to a sum over m =1, 2, 3, . . ∞. If we next factor out 4 from the denominator, then we immediately obtain:

(B10) $$S_e(p) = \frac{1}{4} S(p/2)$$

Next, since $S_e(p) + S_o(p) = S(p)$, it follows that:

(B11) $$S_o(p) = S(p) - \frac{1}{4} S(p/2) = \frac{\pi}{2} \frac{1+\cosh(2\pi p)}{p\sinh(2\pi p)} + \frac{\pi}{4} \frac{1+\cosh(\pi p)}{p\sinh(\pi p)}$$

For the degenerate case (p = 0), Eq. (B11) implies that the odd terms in $1/n^2$ will sum to:

(B12) $$S_o(0) = S(0) - \frac{1}{4} S(0) = \frac{\pi^2}{8}$$

This is verified by the formula found in Ref. 7.

Thus, we have formulas for all three summations that can be used in the development of the root correction.