ON COMPUTING FLOQUET TRANSITION MATRICES OF ROTORCRAFT

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Summary

Stability analyses of rotorcraft systems require Floquet transition matrices (FTMs) which are the state transition matrices at the end of one period. The FTM of such an N dimensional system is computed either by the N-pass approach as an $N \times N$ matrix, by integrating the state equation N times, or by the single-pass approach as an $N^2 \times 1$ vector by integrating the modified state equation only once. There appear to be conflicting claims concerning the efficiency of different schemes of computing rotorcraft FTMs. Accordingly, both analytical and computer generated data are presented on comparative efficiency of four classes of methods—i) Runge-Kutta one step type, ii) Hamming's predictor-corrector multi-step type, iii) Bulirsch-Stoer extrapolation type and iv) hybrid or Variable-step Variable-order type, embodying the special features of one-step and multi-step methods, such as the Gear type, and the Shampine and Gordon type. Data with respect to single-pass and N-pass schemes are presented for four helicopter models except teetering—a rotor having one (N=4) to five (N=20) blades. Each rigid blade executes flapping and lead-lag motions. The analytical treatment provides a useful approximation to machine time in N-pass and single-pass and is economical to use. Though illustrated with reference to a specific scheme, it is adaptable for comparing different algorithms with respect to machine time. Data demonstrate that Hamming's fourth order predictor-corrector method in single-pass is the most economical with respect to three significant figure accuracy.

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1. Introduction

Stability of a lifting rotor system is generally analysed by finding the largest eigenvalue of the Floquet transition matrix (FTM)[1]. The crux of the problem boils down to generating the FTM, a process which is computationally lengthy. After all, the state equation, say of dimension N, has to be solved for discrete time values over one period, the solution being repeated N times for N initial states. This computational process is referred to here as the N-pass approach and is used in the solutions of many helicopter problems[1-3]. The use of the Floquet theory in such solutions is due to Peters and Hohenemser[1] who initiated the FTM concept via the N-pass approach.

An alternative to the N-pass is the single-pass approach in which the $N \times N$ FTM is computed only once as an $N^2 \times 1$ vector, the $N^2 \times N^2$ modified state matrix being identified with the initial state [1,0,0, ...0; 0,1,0,0.....0; 0,.....,0,0,1] No matter which algorithm is used, computational advantages of the single-pass over the N-pass come from three main sources. First, the number of function evaluations are reduced by N, a noteworthy feature for rotorcraft whose state matrices involve lengthy periodic functions. Although the time for function evaluations is longer in the single-pass, it is more than offset by the reduced number of evaluations. Second, only the original $N \times N$ state matrix is dealt with in the computer program, although the modified state matrix is of dimension $N^2 \times N^2$. Third, the same algorithm applicable for the N-pass with the $N \times 1$ state vector is directly applicable for the single-pass with the $N^2 \times 1$ modified state vector.

While computing the FTMs, the machine time saving through the single-pass approach is well attested[4-9]. Hammond [4,5] et al. use the $O(h^4)$ Runge-Kutta-Gill (RKG) method in N-pass and single-pass. Friedmann and Silverthorn[6] propose the $O(h^2)$ Hsu method in which the periodically varying coefficients between the two knots or azimuth discretizations are replaced by the trapezoidal constant parameter approximation. FTM data from this method in single-pass are compared with those from the $O(h^4)$ RKG method in N-pass. Further elaborations of the single-pass approach through the $O(h^2)$ Hsu method and the $O(h^4)$ RKG method are given by Friedmann, Hammond and Woo[7]. Von Kerczek and Davis[8] provide the FTM data of a periodic flow problem in single-pass using three $O(h^4)$ methods : Runge-Kutta-Classic[10], 'Adams-Moulton Method', and 'Second Derivative Method'. The last two methods are special methods in that they use multistep formulae, the usual combination with an appropriate predictor formula being replaced by a Gaussian elimination formulation at each knot. Although the third method is favoured in reference 8, generating a set of derivatives of a state matrix is likely to increase the machine time and to decrease the accuracy. The data concerning the use of these three methods for computing FTMs are not comprehensive enough to allow any general conclusions to be drawn, nor is the state matrix of the linear flow problem ($N = 5$) typical of rotors. Chen[9] compared the $O(h^2)$ Hsu method in single-pass with the $O(h^2)$ Runge-Kutta-Classic in N-pass, and the $O(h^4)$ RKG and Hamming's predictor-corrector method in N-pass with the $O(h^4)$ RKG method in single-pass. In reference 9 the $O(h^4)$ RKG method in single-pass is also referred to as the "Friedmann-Hammond-Woo method of order 4". Chen[9] also used an $O(h^2)$ spline function approximation which is shown to be competitive for single bladed cases and is still in developmental stages.

The emphasis thus far has been mostly on the computational advantages of the single-pass over the N-pass with reference to one or two existing or proposed algorithms. There also appear to be conflicting claims as to the reliability and machine time savings of different algorithms. There still is much research needed, *not* so much in devising special methods for non-stiff initial value problems, but rather, in comparing effectiveness of known methods[10-13]. The present study concerns such a comparison of methods used to compute rotorcraft FTMs. Compared to preceding related studies it is comprehensive in several respects :

1. An objective comparison of different methods is achieved through computational viability—machine time saving for a priori stipulated significant figure accuracy. An accuracy of three significant figures is considered adequate, which is generally maintained in most of the earlier studies with $O(h^4)$ type methods.

2. The most viable method is determined both by single-pass and N-pass schemes with respect to a single-bladed (N = 4) model. It is further assessed through single-pass and N-pass computations with respect to three higher dimensional systems—a rotor having three (N = 12) to five blades (N = 20). Each rigid blade executes flapping and lead-lag motions. For reasons of checking numerical and programming errors, inter-blade coupling effects are intentionally suppressed so as to have the same N/4 repeating sets of eigenvalues of the single bladed model.

3. For each helicopter model machine-time data are generated by using and ignoring the sparseness of the state matrix so as to illustrate the sensitivity of machine-time data to programming efficiency.

4. The state of the art for non-stiff initial value problems is established[10-13] and the numerical methods are conventionally grouped into four categories[11]—i) one-step (Runge-Kutta type[11]) ii) multi-step (predictor-corrector type[11]), iii) extrapolation (Bulirsch-Stoer type[11,12]), and iv) hybrid or variable-step variable-order (Gear type[11,12] and Gordon and Shampine type[11,13]). Widely used methods in engineering from each of these categories are selected for comparative testing.

5. An analytical formulation is suggested which provides useful approximations to the observed machine-time results and which is economical to use. Although the estimated machine-time data are based on Hamming's predictor-corrector algorithm, the formulation is adaptable for comparative testing of different algorithms with respect to machine time.

2. Data Genesis

Data including machine times concern FTMs, and damping levels which are the real parts of logarithms of characteristic exponents. All the eigenvalues of FTMs or the characteristic multipliers are computed using the subroutine of reference 14. These data are generated with respect to four helicopter models except teetering—a rotor having one (N = 4) to five (N = 20) rigid blades. Each blade executes flapping and lead-lag motions as treated in reference 15 for the single bladed case and in reference 16 for the multibladed case. Inter-blade coupling effects due to dynamic inflow[17] etc. are intentionally suppressed so as to have N/4 repeating pairs of eigenvalues of the single-bladed model. The absence of repeating pairs indicates presence of numerical or programming errors. Following Lambert[11], numerical methods for initial value problems are classified into four groups, as noted in table 1 (and table 2, column 1), which also includes the six methods selected for final comparison.

The first group refers to one-step methods among which Runge-Kutta type methods are the best known. The literature concerning Runge-Kutta type methods is extensive[10-13], e.g. error estimates with step-doubling[18,19] and modifications due to Gill, Fehlberg[11,12] Verner[12,14] and others[10-13]. However in rotorcraft solutions, Gill's version through the IBM package DRKGS is probably the most widely used[2,4-7,9,16,17]. In table 1 only two Runge-Kutta type schemes due to Gill, and Fehlberg are included. The one due to Verner[12,14] is found to be almost identical to these two with respect to machine time and accuracy. The second group refers to multi-step methods among which the ABM method[11] (Adams-Bashforth predictor with Adams-Moulton corrector) and the Hamming predictor-corrector (Hamming) method[1-3,9,10] are widely used. The difference between the ABM and Hamming does not seem to be of much significance with respect to machine time and accuracy, and only the $O(h^4)$ Hamming method based on IBM package DHPCG is used here. The third group refers to extrapolation methods among which the Bulirsch—Stoer scheme is well appraised in the literature[11,18] for cases involving high accuracy and cheap function evaluations For this scheme well tested computer packages are given in references 12 and 14. The package used in the present study is from reference 14 called DREBS. The fourth group refers to the variable-step variable-order (VSVO) schemes[10-13] which received increased attention in the numerical analysis literature. Since VSVO schemes are of recent origin and they are extensively covered in the literature two schemes—one due to Gear[12] and another due to Gordon and Shampine[13]—are selected.

3. Approximate Estimation of Execution Time

An exact analytical estimation of the time taken for the computation of a FTM by the N-pass and single-pass approaches is impossible. There are several factors which are not amenable to simple treatment ; these can be classified into three broad categories :

1. programming details such as branching, loops, information flow between subprograms and book-keeping operations.

2 difference between integer and floating point operations.

3. an exact a priori count of steps or discretizations for the stipulated tolerance.

Of these, the first category is the most difficult to treat. As for the second, although it is possible to include an exact count of integer mode operations, it will complicate the expressions. In the analysis to follow, all the integer mode operations are neglected since the bulk of arithmetic operartons is in floating point and since any arithmetic operation takes considerably shorter time in the integer mode than in floating point. Finally, the third factor can not be estimated. But it can be controlled in that the step-size remains essentially unaltered by appropriate combinations of step-size and tolerance. In general, sophisticated computer codes for initial value problems choose the step-size automatically in such a way that an estimate of the local truncation error is less than the specified tolerance. It must be noted, however, that in most of the widely used computer packages the truncation error is estimated at each step whether or not this information is used to alter the step-size.

In the sequel the computation time is estimated for the $O(h^4)$ Hamming method only. A formulation for estimating machine times for other methods could be developed on similar lines. The accent here is on explaining the observed trends rather than on precise estimation. The analysis shows in quantitative terms the factors going into the superiority of the single-pass approach over the N-pass. However, in view of the three factors mentioned above, the actual computation times are expected to be somewhat greater.

Consider the state equation for the $N \times 1$ state vector $X(t)$:

$$\dot{X} = A(t)X \tag{1}$$

with initial state $X(t_o) = X_o$. By-passing the details of starting values, Hamming's predictor-corrector sequence runs as follows[9,10] :

Predictor : $$P_{j+1} = X_{j-3} + \tfrac{4}{3}h\,(2\dot{X}_j - \dot{X}_{j-1} + 2\dot{X}_{j-2}) \tag{2}$$

Modifier : $$M_{j+1} = P_{j+1} - \tfrac{112}{121}(P_j - C_j) \tag{3}$$

$$\dot{M}_{j+1} = A(t_{j+1})M_{j+1} \tag{4}$$

Corrector : $$C_{j+1} = \tfrac{1}{8}[9X_j - X_{j-2} + 3h(\dot{M}_{j+1} + 2\dot{X}_j - \dot{X}_{j-1})] \tag{5}$$

Final value : $$X_{j+1} = C_{j+1} + \tfrac{9}{121}(P_{j+1} - C_{j+1}) \tag{6}$$

Local truncation error in $X_{j+1} = \tfrac{9}{121}(P_{j+1} - C_{j+1})$ (7)

Control of accuracy and adjustment of step-size is done by generating the following test value :

$$\sum_{i=1}^{N} a_i \left| p_{j+1,i} - c_{j+1,i} \right| \tag{8}$$

where the coefficients a^i $(i = 1,2,\ldots,N)$ are specified error weights and $p_{j+1,i}$ and $c_{j+1,i}$ are the i-th components of P_{j+1} and C_{j+1} .

Following Ralston and Wilf[20], if we denote by τ the time required to compute $A(t)X$ (that is one function evaluation), then the approximate time to compute (2) to (6) and X_{j+1} is given by

$$2\tau + N(16\alpha + 5\mu) \tag{9}$$

where α is the time for one addition and μ is the time for one multiplication. Further the time required to compute (7) and (8) is given by[20]

$$N(2\alpha + \mu) \quad . \tag{10}$$

Ignoring all the book-keeping operations in the flow, we arrive at the estimated time per step of the integration as[20]

$$2\tau + N(18\alpha + 6\mu)$$

If n denotes the number of blades and σ, the total number of steps taken, then we obtain the execution times as follows :

N-pass : $(N=4n)$

$$\text{Total time} = 4n\sigma[2\tau_N + 4n(18a+6\mu)]$$
$$= \sigma[8n\tau_N + 96n^2(3a+\mu)] \quad (12)$$

Single-pass : $(N=16n^2)$

$$\text{Total time} = \sigma[2\tau_S + 96n^2(3a+\mu)]. \quad (13)$$

In equations (12) and (13), τ_N and τ_S are times for a single function evaluation in N-pass and single-pass approaches respectively.

It is possible to estimate τ_N and τ_S analytically for special cases. However, they are problem-dependent and hence, are of no general utility. In the present study they are taken directly from computer experiments (table 4), and are then introduced into (12) and (13) to obtain the execution times for 1,3,4 and 5 bladed rotors (table 5 and figures 1 and 2)

For the system treated, matrix A(t) is banded, the larger the number of blades and less the inter-blade coupling effects, the greater is the sparseness in the state matrix. Results exploiting this sparseness are also included in tables 3 and 5 and in figures 3 and 4, as elaborated in the next section.

4. Discussion of Data

The data are presented here with reference to two a priori criteria : i) economy as assessed through the C.P.U. time for program execution (execution time) and ii) accuracy as assessed through the number of significant figures computed from the formula[21]

$$\frac{|x-x^*|}{|x|} \leq \tfrac{1}{2}\, 10^{1-r} \quad (14)$$

In relation (14), x^* is the computed value, x, the reference value, and r, the number of significant figures. Values obtained through the $O(h^8)$ Bulirsh-Stoer scheme[9] with a starting step-size of $2\pi/100$ are taken as exact, since this scheme is known for its high accuracy[9,10,11]. The execution times are routine computer data. Similar data pertaining to C.P.U. time for completion are not reproduced here since they qualitatively confirm the comparisons established through execution-time data.

The six numerical schemes identified in table 1 are taken up again in table 2 which also includes characteristic exponents, significant figures, modulus error, execution time etc. More than 90% of the total execution time to compute damping levels is for FTM computations. Spot checks with respect to randomly selected elements of FTM provide the same significant figure accuracy as observed through damping values. Since it is more realistic to compare data with respect to unique damping levels of engineering interest, the significant figure accuracy, as in reference 9, is computed with respect to damping. Considerable trial and error is expended for selecting as large a tolerance value as possible to achieve an accuracy of atleast three significant figures. All the algorithms have built-in mechanisms of

altering the step-size in response to the stipulated tolerance. However these mechanisms of automatic step-size control vary from algorithm to algorithm, being based on "heuristic tuning" of different error control criteria, for details see Lambert[11]. Though quantitatively, the results to be presented here may require some correction due to using non-optimal combinations of step-size and tolerance for the required accuracy, the established comparative trends of viability of different methods should remain valid.

From the data in table 2, it is seen that all the six methods provide an accuracy of atleast three significant figures. Regarding execution time, the Hamming method takes the least amount and the Bulirsch-Stoer method, the highest. The next "best" method is due to Gordon and Shampine, followed by the three Runge-Kutta methods due to Gill, Fehlberg and Verner (not shown in table 2). Observe that the Gordon-Shampine method, inspite of its overhead costs of self-starting, automatic selection of step-size and order, is competitive to Runge-Kutta type methods. This is due to increased number of function evaluations in Runge-Kutta type methods, whereas the Gordon-Shampine method is basically a predictors corrector type method as far as the number of function evaluations per step is concerned. The Gear method is not found to be competitive.

Of particular significance is the execution-time data of table 2 in single-pass and N-pass approaches. The substantial saving through single-pass is clearly seen. It is consistent with the physics of the problem since lifting rotors do involve lengthy periodic functions. For the single-bladed case ($N = 4$) the saving through single-pass is close to 59% in the Hamming schemes and to 53% in the RKG scheme. Similarly, the saving through the two VSVO schemes is about 40%. Further elaborations of the single-pass approach with Hamming's is discussed in tables 3 to 5. The data pertaining to "Full" indicate that sparseness of the state matrix is not taken into account in function evaluations, whereas, data pertaining to "Sparse" indicate that only non-zero elements are included tn the function evaluation.

As seen from table 3, the higher the system dimension, the greater is the saving through the single-pass approach. This saving increases from 59% (17 seconds compared to 7) for the single-bladed case to about 71% (160 seconds compared to 551) for the five-bladed case, without exploiting sparseness. When sparseness is exploited, a token of efficiency in programming, the saving for the five-bladed case is close to 79% (i.e. 99 seconds compared to 482). It is worth observing the significant saving both in N-pass and single-pass by exploiting sparseness, for the five-bladed case, 482 seconds compared to 551 (13% in N-pass; and 99 seconds compared to 160 (38%) in single-pass. It is mentioned in passing that sparseness decreases with the inclusion of inter-blade coupling effects such as dynamic inflow feedback, etc.[16,17].

Data in tables 4 and 5 concern an analytical formulation of estimating execution time. As stated earlier, the time for one function evaluation is obtained as computer data which in conjunction with formulas (12) and (13) give the execution time to compute the FTM. In single-pass the time for one function evaluation is higher since the modified state matrix is of dimension $N^2 \times N^2$, whereas in N-pass it is of dimension $N \times N$. For example, for the five-bladed model the times for one function evaluation are 0.124 and 0.378 seconds respectively for the N-pass and single-pass approaches without exploiting sparseness. When sparseness is exploited the time for one function evaluation is 0.148 seconds in single pass—a reduction of approximately 2.6 times (0.378 compared to 0.148); and it is 0.094 seconds in N-pass—a reduction of approximately 1.3 times (0.124 compared to 0.094). As expected sparseness has more pronounced effect in single-pass than in N-pass.

Given the simplifications made in deriving equations (12) and (13), data in table 5 correlate reasonably well with the data in table 3, as graphically presented in Figure 1. Figure 2 concerns the ratio of machine times for FTM as obtained through computer (last but one column in table 3) and from formulas (12) and (13) (last but one column in table 5). Note that concerning data presented in Figures 1 and 2 sparseness of the state matrix is not exploited. Similar sets of data, when sparseness of the matrix is exploited, are graphically preseated in Figures 3 and 4.

5. Concluding Remarks

The intent of the preceding study is to establish comparative trends concerning the viability of different numerical methods to compute rotorcraft FTMs. Main assumptions and stipulations of this study are : i) Double precision arithmetic on IBM 360/44 is adequate to provide at least an accuracy of four significant figures ii) The C.P.U. time for execution is a rational basis of comparing different methods with respect to saving in machine time iii) While computing eigenvalues according to reference 14, the computational errors are equally distributed with respect to all the numerical methods iv) The selected computer packages are equally efficient with respect to all the methods v) The reference values agree with the "exact" at least up to four significant figures vi) It is rational to compare different methods with built-in step-size control by selecting by trial and error the largest tolerance value to achieve three significant figures accuracy.

Subject to the correctness of the above assumptions and stipulations, the data demonstrate the following : 1) Hamming's predictor-corrector method in single-pass is the most viable with respect to three significant figure accuracy 2) The analytical formulation reveals the advantage of the single-pass approach over the N-pass, provides useful approximations to machine-time data, and is an economical and feasible approach of comparing different methods with respect to machine time.

References

1. Peters, D.A and Hohenemser, K.H., 'Application of the Floquet Transition Matrix to Problems of Lifting Rotor Stability,' **Journal of the American Helicopter Society**, Vol 16, No. 2, April 1971, pp. 25-33.

2. Gaonkar, G.H , Hohenemser, K.H. and Yin, S.K., 'Random Gust Response Statistics for Coupled Torsion-Flap Blade Vibrations,' **Journal of Aircraft**, Vol. 9, No. 10, October 1972, pp. 726-729.

3. Schrage, Daniel P., Effects of Structural Parameters on the Flap-Lag Forced Response, Doctoral Dissertation, Washington University, May 1978.

4. Hammond, C.E., 'An Application of Floquet Theory to Prediction of Mechanical Instability,' **Journal of the American Helicopter Society**, Vol. 19, No. 4, October 1974, pp. 14-23.

5. Kaza, K.R.V. and Hammond, C.E., 'An Investigation of Flap-lag Stability of Wind Turbine Rotors in the Presence of Velocity Gradients and Helicopter Rotors in Forward Flight,' **17th AIAA/ASME/SAE Structures, Structural Dynamics and Materials Conference**, Valley Forge, Pennsylvania, May 5–7, 1976.

6. Friedmann, P. and Silverthorn, L.J., 'Aeroelastic Stability of Periodic Systems with Application to Rotor Blade Flutter,' **AIAA Journal**, Vol. 12, No. 11, November 1974, pp. 1559-1565.

7. Friedmann, P., Hammond, C.E. and Woo, Tze-Hsin, 'Efficient Numerical Treatment of Periodic Systems with Application to Stability Problems,' **International Journal for Numerical Methods in Engineering**, Vol. 11, 1977, pp. 1117-1136.

8. Von Kerczek, C. and Davis, S. H., 'Calculation of Transition Matrices,' **AIAA Journal**, Vol. 13, No. 10, October 1975, pp. 1400-1403.

9. Chen, Berhord David, Comparing Methods for Computing Period Transformation Matrices to Determine Helicopter Stability, M.S. Paper, Southern Illinois University, Edwardsville, Illinois, August 1978.

10. Lapidus, L. and Seinfeld, J.H., Numerical Solution of Ordinary Differential Equations, Academic Press, New York, 1971, Chapters 2, 4 and 5.

11. Lambert, J.D., 'The Initial Value Problems for Ordinary Differential Equations,' Chapter IV.1, The State of the Art in Numerical Analysis, Edited by D. Jacobs, Academic Press, 1977, pp. 451-491.

12. Gear, C.W., Numerical Initial Value Problems in Ordinary Differential Equations, Prentice-Hall, Inc., 1971, Chapter 9.

13. Shampine, L.F. and Gordon, M.K., Computer Solution of Ordinary Differential Equations, W.H. Freeman and Company, 1975.

14. 'Eigenvalue Subroutine EIGRF, Bulirsch-Stoer Subroutine DREBS, and Runge-Kutta-Verner Subroutine DVERK', International Mathematical and Statistical Libraries, sixth floor, GNB Building, 7500 Bellaire Boulevard, Houston, Texas 77036.

15. Peters D.A., 'Flap-Lag Stability of Helicopter Rotor Blades in Forward Flight,' **Journal of the American Helicopter Society**, October 1975, pp. 2-13.

16. Gaonkar, G.H. and Peters, D.A., 'Flap-Lag Stability with Dynamic Inflow by the Method of Multiblade Coordinates', **20th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference**, St. Louis, Missouri, April 4-6, 1979, paper no. 79-0729.

17. Peters, D.A. and Gaonkar, G.H., 'Theoretical Flap-Lag Damping with Various Dynamic Inflow Models,' **35th Annual National Forum of the American Helicopter Society**, Washington, D.C., May 21-23, 1979, paper no. 79-20.

18. Sedgwick, A.E., Hull, T.E., Enright, B.M. and Fellen, B.M., 'Comparing Numerical Methods for Ordinary Differential Equations,' **SIAM Journal of Numerical Analysis** Vol. 9, No. 4, Dec. 1972.

19. Schuessler, Richard B., 'A Numerical Study of Ordinary Differential Equations with Periodic Coefficients,' M. S. Thesis, University of Rolla, 1974.

20. Ralston, A. and Wilf, H., Mathematical Methods for Digital Computers, Vol. I, Wiley, New York, 1960, p. 108.

21. Conte, S.D. and de Boor, Carl, Elementary Numerical Analysis, McGraw-Hill Book Company, New York, 1972, p. 10.

22. Forsythe, G.E., Malcolm, M.A. and Moler, C.B., Computer Methods for Mathematical Computations, Prentice-Hall, Englewood Cliffs, 1977, Chapter 6.

TABLE 1

Selected Methods and Grouping

Name	Generic group	Order in terms of step-size h	Remarks
1. Runge-Kutta-Gill	Single–step Runge-Kutta type	4	IBM–SSP package DRKGS
2. Runge-Kutta-Fehlberg	,, ,,	4	Following references 12 and 22
3. Hamming predictor-corrector	Multistep	4	IBM–SSP package DHPCG
4. Bulirsch-Stoer	Extrapolation	4	Following reference 9 for $O(h^8)$ and IMSL package DREBS for $O(h^4)$ from reference 14
5. Gear	VSVO (Variable-Step Variable-Order)	automatic order selection	Following reference 12 (for most of the calculations the automatically selected order was 7).
6. Gordon-Shampine	,, ,,	,, ,,	Following reference 13 (for most of the calculations the automatically selected order was 9).

TABLE 2

Viability of different methods with respect to a single bladed model

Method	Eigenvalues	$\frac{\lvert x-x^* \rvert}{\lvert x \rvert}$ †	Significant figures††	Modulus errors	CPU time in seconds‡	
					N–pass	Single pass
Bulirsch-Stoer[9] $O(h^8)$	−0.003531 ± i 0 389068 −0.308186 ± i 0.107021	— —	— —	— —	—	—
Runge-Kutta-Gill $O(h^4)$	−0.003519 ± i 0.388323 −0.308026 ± i 0.106284	0.0034 0.00052	3 3	0.00075 0 00039	36	17
Runge-Kutta-Fehlberg $O(h^4)$	−0.003519 ± i 0.388323 −0.308026 ± i 0.106284	0.0034 0.00052	3 3	0.00075 0.00039	36	17
Hamming $O(h^4)$	−0.003519 ± i 0.388323 −0.308026 ± i 0.106284	0.0034 0.00052	3 3	0.00075 0.00039	17	7
Bulirsch-Stoer ‡‡$O(h^4)$	−0.003519 ± i 0.389037 −0 308611 ± i 0.107318	0.0014 0.0014	3 3	0 00003 0.00050	442	293
Gear‡‡	−0.003516 ± i 0.389381 −0.308968 ± i 0.107751	0.0042 0.0025	3 3	0.00031 0.00098	89	54
Gordon-Shampine	−0.003519 ± i 0.388323 −0.308026 ± i 0.107284	0.0034 0.00052	3 3	0.00075 0.00039	27	16

† With respect to real part of eigenvalues (damping levels)

†† With x and x* representing respectively the reference and calculated values, the number of significant figures (r) is calculated from the formula $\lvert x-x^* \rvert / \lvert x \rvert \leq \frac{1}{2} \cdot 10^{1-r}$

‡ Computation of FTM only

‡‡ Results correspond to N - pass. Single - pass results are marginally better

TABLE 3

Comparison of N-pass and single-pass approaches using Hamming's predictor - corrector algorithm

(From Computer)

Number of blades	CPU time in seconds				Single-pass / N - pass	
	N-pass		single-pass			
	Full	Sparse	Full	Sparse	Full	Sparse
1	17	*	7	*	0.412	*
3	168	152	54	40	0.322	0.263
4	317	281	98	66	0.310	0.235
5	551	482	160	99	0.291	0.205

* No appreciable sparseness

TABLE 4

Time for a single function evaluation in seconds
(From Computer)

Number of blades	N-pass		Single-pass	
	Full	Sparse	Full	Sparse
1	0.128	0.0124	0.0156	0.0144
3	0.0496	0.0476	0.1136	0.0668
4	0.0728	0.0672	0.2152	0.1008
5	0.1240	0.0940	0.3780	0.1480

TABLE 5

Comparison of N-pass and Single-pass approaches using Hamming's predictor-corrector algorithm
(From analysis)

Number of blades	CPU time in seconds				$\frac{\text{Single-pass}}{\text{N - pass}}$	
	N-pass		single-pass			
	Full	Sparse	Full	Sparse	Full	Sparse
1 (N=4)	10.32	*	3.20	*	0.310	*
3	126.65	121.86	30.33	20.98	0.240	0.172
4	246.50	228.58	56.58	33.70	0.230	0.147
5 (N=20)	517.15	397.15	96.75	50.75	0.187	0.128

* No appreciable sparseness

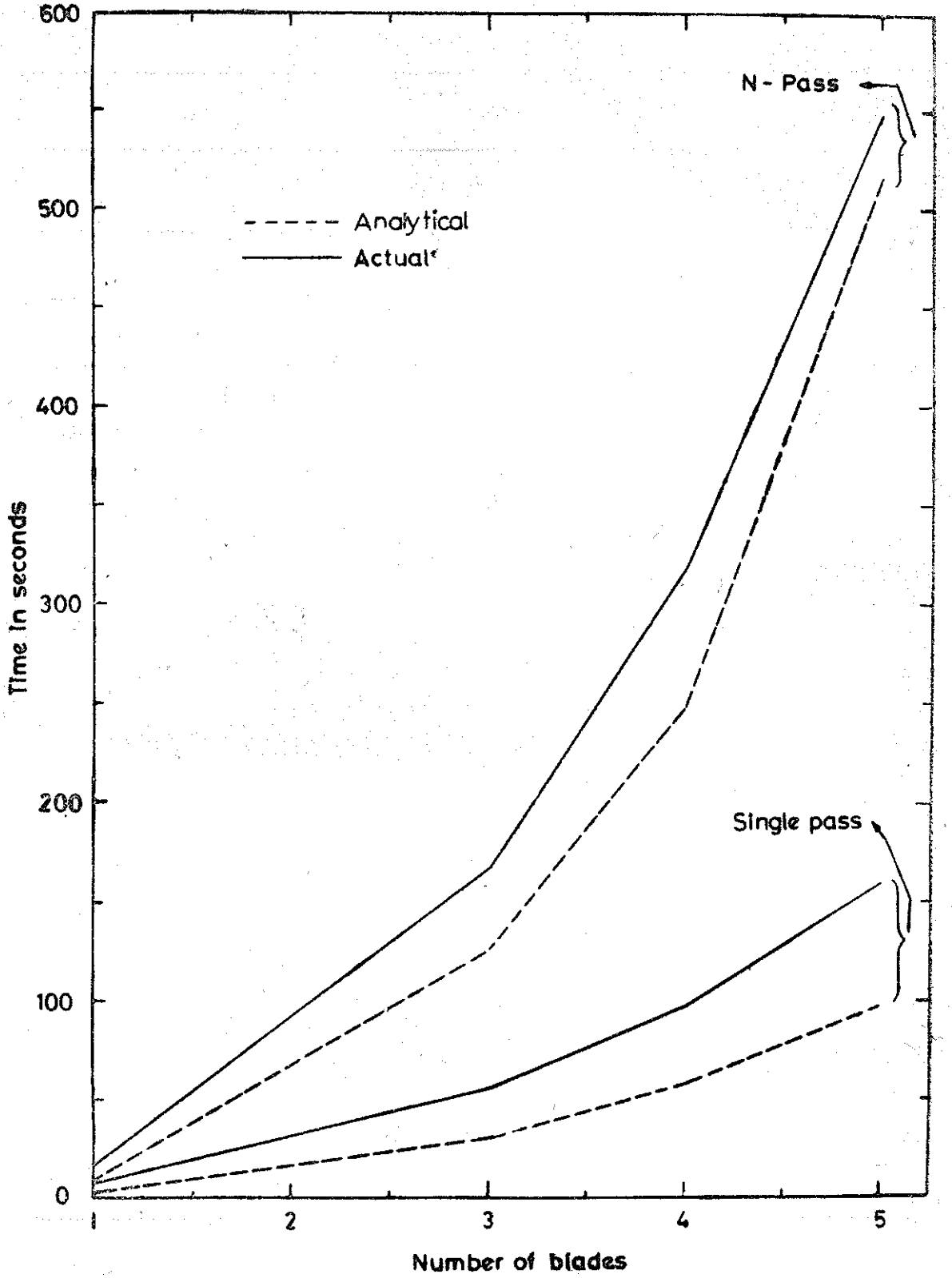

Fig. 1. CPU TIME ON IBM 360/44 FOR COMPUTING FTM
(Sparseness of state matrix not exploited)

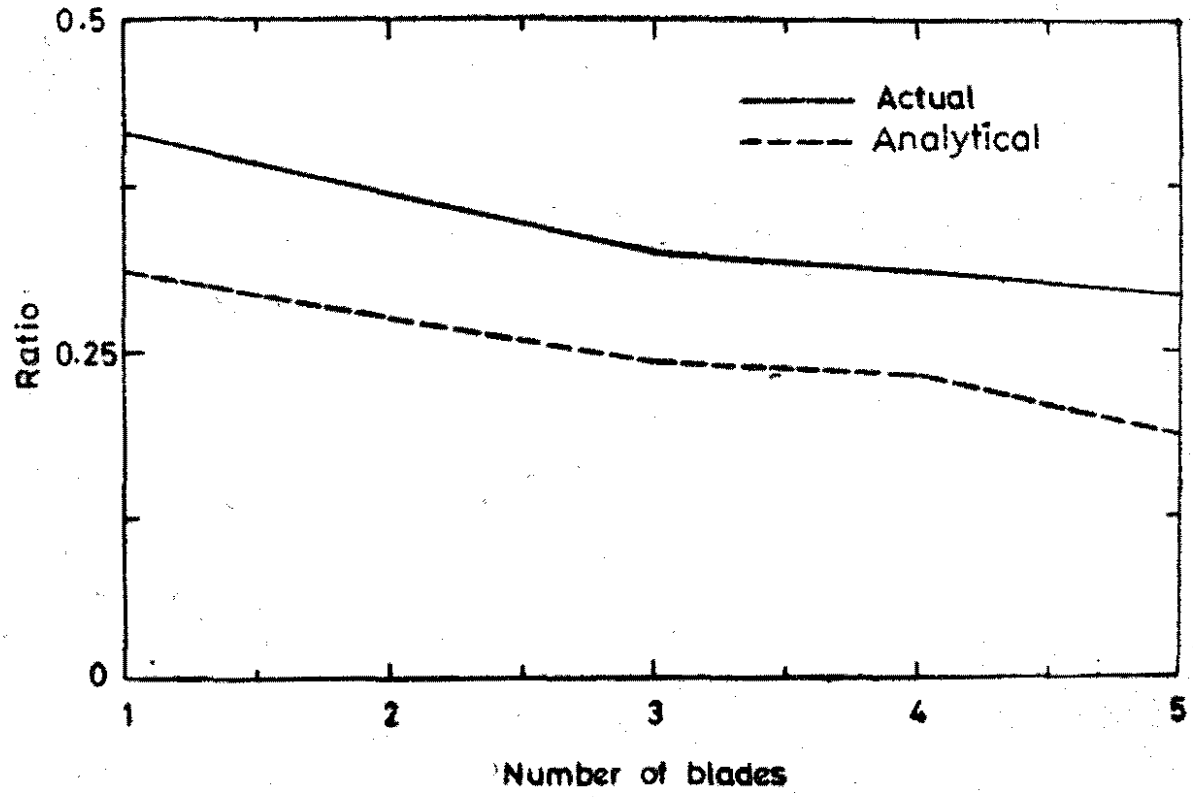

Fig. 2. RATIO OF CPU TIMES SINGLE PASS/N-PASS
(Sparseness of state matrix not exploited)

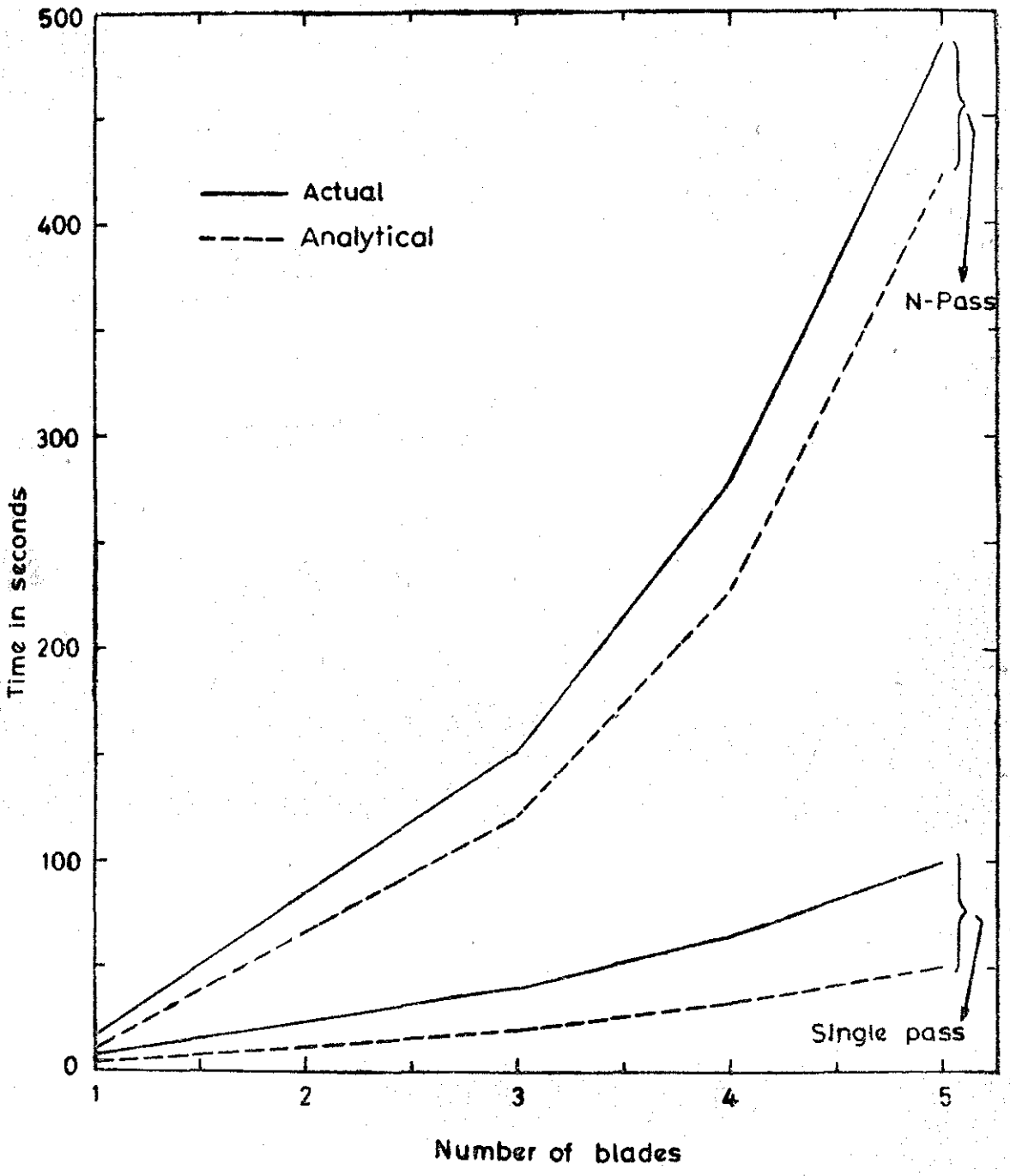

Fig. 3. CPU TIME ON IBM 360/44 FOR COMPUTING FTM
(Sparseness of state matrix exploited)

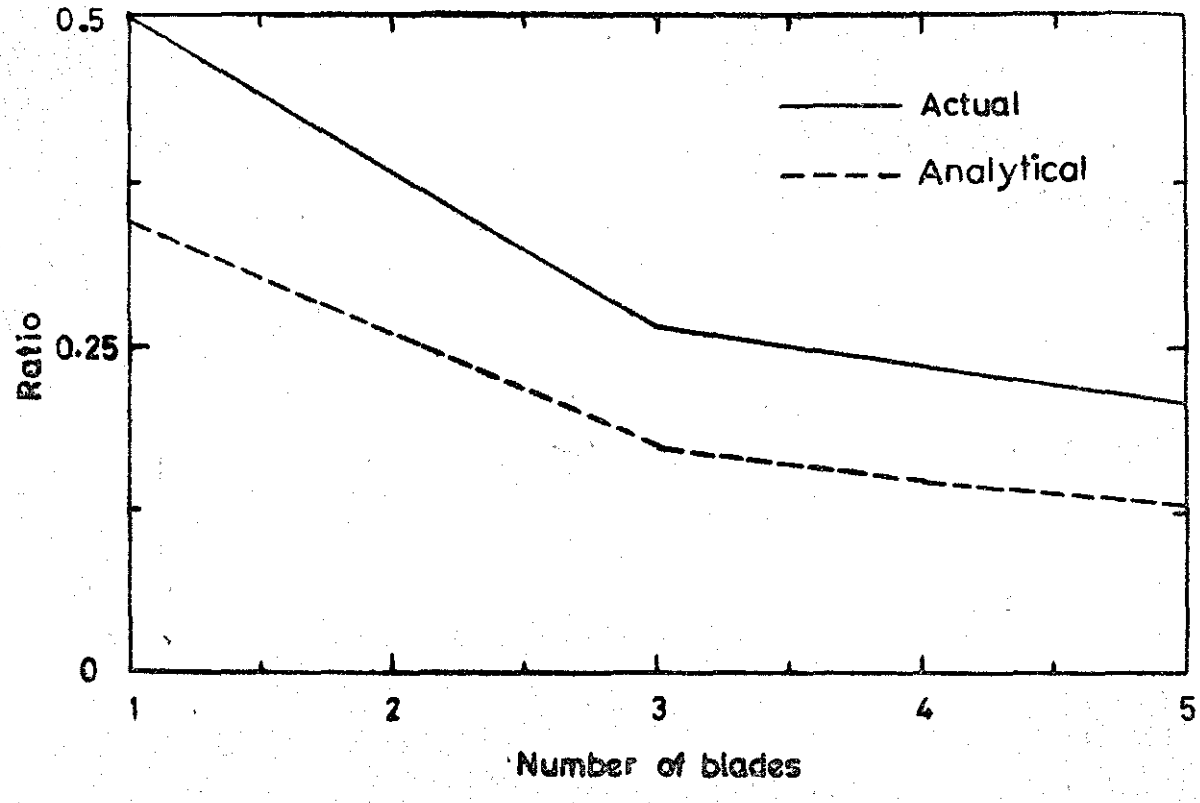

Fig. 4. RATIO CPU TIMES SINGLE PASS/N-PASS
(Sparseness of state matrix exploited)