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THE ANELASTIC COMPLIANT ROTOR
AN ANALYTIC AND EXPERIMENTAL INVESTIGATION

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Nomenclature

Symbol	Meaning
a	lift curve slope of blade element
a_T	lift curve slope of tab
c	blade chord
CF	centrifugal force
C_T	thrust coefficient, $T/\rho\pi R^2(\Omega R)^2$
I	moment of inertia
L_T	lift on tip tab
m	mass of tip weight
$M_{\beta t}$	tensile flapwise bending moment
$M_{\theta A}$	aerodynamic twisting moment
$M_{\theta I}$	inertia twisting moment
$M_{\theta t}$	tensile twisting moment
R	blade radius, subscript identifying blade root
r	radial distance, dimensional
S_T	tab area
T	thrust of one blade, subscript identifying blade tip or tab
t	tensile force
v	induced velocity
x	r/R, radial distance, non-dimensional
y	chordwise distance measured from the twist axis
y_A	chordwise distance of aerodynamic center from twist axis
y_I	chordwise distance of mass center from twist axis
y_T	chordwise distance between aerodynamic center of tip tab and twist axis
z_T	tip vertical distance above blade root plane
α	blade element angle of attack
β	twist axis slope
θ	blade pitch
θ_o	blade pitch at root
θ_T	blade pitch at tip
σ	solidity of one blade, $c/\pi R$
Ω	rotational speed, radians/sec
δ_T	tab setting
α_T	tab angle of attack

INTRODUCTION

Performance comparable to that achieved by fixed wing aircraft coupled with a vertical takeoff and landing and hover capability is a long-standing objective of the VTOL community. Proposals and prototypes with this goal in view have included varieties of powered lift aircraft, tilt-rotor and tilt-wing machines, compound rotorcraft, and airplanes with stoppable and retractable rotor blades.

The latter category has included various concepts involving highly flexible rotor blades capable of being rolled up on a root or tip spindle. One example is the so-called "sail rotor" consisting of leading and trailing edge catenary cables with cloth or plastic stretched between them. Success has been limited by the phenomenon of luffing, in which blade camber changes suddenly from positive to negative; and by high blade profile drag. Attempts have also been made to develop blades consisting of short rigid segments held together by cables. These have not met with success.

The concept here presented involves rotor blades fabricated from unidirectional Kevlar fabric impregnated with silicone rubber which serves as the upper and lower surface and carries tensile loads. The airfoil shape is maintained by suitable stitching between the upper and lower surfaces and a pressure difference between the interior and exterior of the blade generated by centrifugal pumping.

Such blades have essentially zero torsional and flapping rigidity and take their shape as a result of aerodynamic and inertia forces; hence the designation "anelastic compliant rotor."

ANALYSIS

The rotor blade has a tip weight of specified mass and polar moment of inertia, and a tip tab, as illustrated in Fig. 1. It is assumed that the blade proper has essentially zero torsional and flapping rigidity, and negligible mass.

The blade will be subject to twisting moments due to inertia, tensile, and aerodynamic forces; and flapwise bending moments due to tensile and aerodynamic forces.

TWISTING. The inertia twisting moment is designated $M_{\theta I}$. This has been called the "tennis racquet" moment and is the same moment which tends to drive propeller blades to flat pitch. It exists whenever the principal inertia axis of the tip weight does not lie in the plane of rotation. Its magnitude may be derived as follows:

Consider a mass element, dm, in the tip weight, a distance y from the twist axis, as illustrated in Fig. 2. There is a component of centrifugal force in the plane of rotation perpendicular to the twist axis acting on this mass element, of magnitude $CF \sin \eta$. Its moment arm about the twist axis is $y \sin \theta_T$ where θ_T is the tip pitch angle. The elemental inertia twisting moment is then

$$dM_{\theta I} = -dCF \sin \eta \; y \sin \theta_T \; .$$

From the plan view in Fig. 2 it is evident that

$$\tan \eta = \frac{y \cos \theta_T}{R}$$

Assuming all angles to be small and making the usual small angle approximations, i.e.,

$$\sin \eta = \tan \eta$$

$$\sin \theta_T = \theta_T$$

$$\cos \theta_T = 1$$

$$dM_{\theta I} = \frac{-dCF\ y^2}{R}$$

The integral is the moment of inertia of the tip weight about the twist axis. Then

$$(1) \quad M_{\theta I} = -\Omega^2\ \theta_T\ I\ .$$

If I_0 is the polar moment of inertia and y_I is the distance of the mass center of the tip weight from the twist axis, the inertia twisting moment is

$$M_{\theta I} = -\Omega^2\ \theta_T\ (I_0 + y_I{}^2\ m)\ .$$

The twisting moment due to centrifugal tensile force is actually an <u>un</u>twisting moment: it is the moment which causes a rope supporting a heavy weight to tend to unlay.

Fig. 3 illustrates a twisted blade and the tensile forces acting on a section of the blade, dr, distant r from the center of rotation. If the blade has positive twist, that is, if the root pitch setting is less than the tip pitch, tensile forces will produce a negative or nose down twisting moment. If the tensile force is t, and the blade chord is c, the tensile stress is t/c, and the force acting on an element dy long is (t/c)dy. The component of this force perpendicular to the chord is (t/c)dy sin Γ, and the moment arm is y. The elemental untwisting moment at any radius is then

$$dM_{\theta t} = -y\ \frac{t}{c}\ dy\ \sin \Gamma.$$

From the geometry shown at the bottom of Fig. 3, and making the usual small angle approximations,

$$\sin \Gamma = \tan \Gamma = y\ \frac{d\theta}{dr}$$

$$dM_{\theta t} = -\ \frac{t}{c}\ \frac{d\theta}{dr}\ y^2\ dy\ \ .$$

Integrating on y

$$M_{\theta t} = -\frac{t}{c}\frac{d\theta}{dr}\int_{-c/2}^{+c/2} y^2\, dy$$

$$= -\frac{t\, c^2}{12}\frac{d\theta}{dr}\ .$$

The tensile force, t, is essentially equal to the centrifugal force imposed by the tip weight, whence

$$(2)\quad M_{\theta t} = -\frac{m\ R\ \Omega^2\ c^2}{12}\left(\frac{d\theta}{dr}\right)\ .$$

If the twist of the blade can be approximated by a Taylor series, i.e.,

$$(3)\quad \theta_r = \theta_o + kr + nr^2$$

$$\frac{d\theta}{dr} = k + 2nr\ .$$

At the tip, $\theta = \theta_T$ and $r = R$

$$(4)\quad \theta_T = \theta_o + kR + nR^2$$

$$\left(\frac{d\theta}{dr}\right)_T = k + 2nR\ .$$

$$(5)\ M_{\theta t_T} = -\frac{m\ R\ \Omega^2\ c^2}{12}(k + 2nR)$$

At the root, $\left(\frac{d\theta}{dr}\right)_R = k$ and

$$(6)\quad M_{\theta t_R} = -\frac{m\ R\ \Omega^2\ c^2\ k}{12}\ .$$

The aerodynamic twisting moment due to the tip tab, $M_{\theta A_T}$, can be approximated by assuming uniform downwash. From Fig. 4

$$\alpha_T = \delta_T + \theta_T - \frac{v}{\Omega R}.$$ Then

$$L_T = \frac{1}{2}\rho\ (\Omega R)^2\ a_T\ (\delta_T + \theta_T - \frac{v}{\Omega R})\ S_T \text{ and}$$

$$M_{\theta_T} = -y_T\ L_T$$

For uniform downwash in hover

$$\frac{v}{\Omega R} = \sqrt{\frac{C_T}{2}}$$

$$(7) \quad M_{\theta A_T} = -\frac{1}{2} y_T \rho (\Omega R)^2 a_T \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right) S_T$$

Static equilibrium at the tip requires that the sum of the inertia, untwisting, and aerodynamic twisting moments be zero, i.e.,

$$(8) \quad M_{\theta I} + M_{\theta t_T} + M_{\theta A_T} = 0$$

Equations (1), (5), and (7) yield

$$-\Omega^2 \theta_T I - \frac{m R \Omega^2 c^2}{12} (k + 2nR)$$

$$-\frac{1}{2} y_T \rho \Omega^2 R^2 a_T S_T \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right) = 0$$

Dividing by $-\Omega^2 I$ gives

$$\theta_T + \frac{m R c^2}{12I} (k + 2nR) + y_T \frac{\rho R^2 a_T}{2I} S_T \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right) = 0$$

Defining $\gamma_T \equiv \frac{y_T \rho a_T R^2 S_T}{2I}$ and

$$\gamma \equiv \frac{m c^2}{I}$$

finally yields

$$(9) \quad \theta_T + \frac{\gamma R}{12} (k + 2nR) + \gamma_T \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right) = 0$$

The aerodynamic twisting moment due to the blade alone is simply

$$(10) \quad M_{\theta A} = y_A T$$

where y_A is the distance of the aerodynamic center of the blade element from the twist axis and T is the total thrust of one blade.

The total thrust can be approximated by reference to simple blade element theory. Neglecting the thrust of the tip tab,

$$T = \frac{1}{2} \rho ac \; \Omega^2 \int_0^R \left(\theta r^2 - \frac{vr}{\Omega} \right) dr$$

Defining $x \equiv \frac{r}{R}$

$$\sigma \equiv \frac{c}{\pi R}$$

$$C_T \equiv \frac{T}{\rho \pi R^2 \; (\Omega R)^2}$$

From (3)

$$\theta_x = \theta_o + kRx + nR^2x^2$$

$$T = \frac{1}{2} \rho ac \; \Omega^2 \; R^3 \int_0^R \left(\theta x^2 - \frac{v}{\Omega R} x \right) dx$$

$$C_T = \frac{1}{2} a\sigma \int_0^{1.0} \left(\theta x^2 - \frac{v}{\Omega R} x \right) dx$$

$$C_T = \frac{1}{2} a\sigma \int_0^{1.0} \left[\theta_o \; x^2 + kRx^3 + nR^2x^4 - \frac{v}{\Omega R} x \right] dx$$

$$(11) \quad C_T = \frac{1}{2} a\sigma \left[\frac{\theta_o}{3} + \frac{kR}{4} + \frac{nR^2}{5} - \frac{1}{2} \sqrt{\frac{C_T}{2}} \right]$$

For static equilibrium at the root the sum of all the twisting moments must be zero, i.e.,

$$(12) \quad M_{\theta I} + M_{\theta t_R} + M_{\theta A} + M_{\theta A_T} = 0$$

Equations (1), (6), (10), and (12) yield

$$\theta_T + \frac{YRk}{12} + y_A \frac{\rho \pi R^4 \; C_T}{I} + \gamma_T \left(\delta_T + \theta_T \sqrt{\frac{C_T}{2}} \right) = 0$$

There are four independent equations: (3), (7), (11), and (12). The independent variables are the coefficients of the twist schedule, n and k; blade tip pitch angle, θ_T; and the thrust coefficient, C_T. The physical parameters of interest are the blade root pitch setting, θ_o, tab setting, δ_T, and tip weight mass and moment, m and I.

FLAPWISE BENDING. Fig. 5 illustrates the blade shape under the influence of twisting and bending moments. The lower sketch is the curve of the twist axis in the plane containing the rotational axis. Its shape may be approximated by the Taylor series

$$(13) \quad z = z_T + j\,(R - r) + q\,(R - r)^2 \text{ or}$$

$$z = z_T + jR\,(1 - x) + qR^2\,(1 - x)^2$$

The slope of this curve at any radius is given by

$$\beta = \frac{dz}{dr} = -j - 2q\,(R - r)$$

The thrust developed by the blade must equal the vertical component of the tensile force at the root, as illustrated in Fig. 5, whence $T = t \sin \beta_0$.

For small β_0, which will generally be the case, $T = t\beta_0$ and

$$\beta_0 = \left(\frac{dz}{dr}\right)_0 = -j - 2qR$$

At the root $z = 0$ and $r = 0$. Then

$$(14) \quad j = -\frac{z_T}{R} - qR.$$

$$(15) \quad \beta_0 = \frac{z_T}{R} - qR. \text{ Hence,}$$

$$(16) \quad T = t\left(\frac{z_T}{R} - qR\right).$$

At the tip,

$$t \cos \beta_T = CF$$

β_T is small, however, so

$$(17) \quad t = CF = m\,R\,\Omega^2$$

Substituting for t in (16) and solving for q yields

$$q = \frac{z_T}{R^2} - \frac{T}{mR^2\Omega^2}$$

Substituting for q in (14)

$j = -\frac{2\ z_T}{R} + \frac{T}{mR\Omega^2}$. Then, from (13) and with $x = \frac{r}{R}$

$$(18)\quad z = z_T - \left(2\ z_T - \frac{T}{m\Omega^2}\right)(1 - x) + \left(z_T - \frac{T}{m\Omega^2}\right)(1 - x)^2$$

For static equilibrium in flapwise bending the sum of tensile and aerodynamic moments due to blade lift and tab lift must be zero at the root, thus

$$M_{\beta t} + M_{\beta A} = M_{\beta T} = 0$$

The tensile flapwise bending moment is

$$(19)\quad M_{\beta t} = -z_T\ m\ R\ \Omega^2$$

The aerodynamic moment is

$$M_{\beta A} = \int_0^R r\ d\ T$$

$$M_{\beta A} = \frac{1}{2}\ \rho ac\ \Omega^2 \int_0^R \left(\theta r^3 - \frac{vr^2}{\Omega}\right) dr$$

$$M_{\beta A} = \frac{1}{2}\ \rho ac\ \Omega^2\ R^4 \int_0^{1.0} \left(\theta x^3 - \frac{v}{\Omega R}\ x^2\right) dx$$

Substituting for θ from (3)

$$M_{\beta A} = \frac{1}{2}\ \rho ac\ \Omega^2\ R^4 \int_0^{1.0} \left(\theta_0 x^3 + kRx^4 + nR^2x^5 - \frac{v}{\Omega R}\ x^2\right) dx$$

Integrating

$$(20)\quad M_{\beta A} = \frac{1}{2}\ \rho ac\ \Omega^2\ R^4 \left(\frac{\theta_0}{4} + \frac{kR}{5} + \frac{nR^2}{6} \frac{1}{3} \sqrt{\frac{C_T}{2}}\right)$$

The flapwise bending moment due to the tip tab is

$$M_{\beta T} = L_T\ R$$

$$(21)\quad M_{\beta T} = \frac{1}{2} \rho \Omega^2 R^3 a_T S_T \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right)$$

Summing moments

$$- z_T m R \Omega^2 + \frac{1}{2} \rho a c \Omega^2 R^4 \left(\frac{\theta_o}{4} + \frac{kR}{5} + \frac{nR^2}{6} - \frac{1}{3}\sqrt{\frac{C_T}{2}}\right)$$

$$+ \frac{1}{2} \rho \Omega^2 R^3 a_T S_T \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right) = 0$$

Dividing by $m R^2 \Omega^2$

$$(22)\quad \frac{z_T}{R} = \frac{1}{2} \rho a c R^2 \left(\frac{\theta_o}{4} + \frac{kR}{5} + \frac{nR^2}{6} - \frac{1}{3}\sqrt{\frac{C_T}{2}}\right)$$

$$+ \frac{1}{2} \rho a_T S_T R \left(\delta_T + \theta_T - \sqrt{\frac{C_T}{2}}\right)$$

Equation (13) may be re-written as

$$(23)\quad \frac{z}{R} = \frac{z_T}{R} + j (1 - x) + qR (1 - x)^2$$

<u>RESULTS</u>. The twisting and flap-wise bending equations were solved numerically. Fig. 6 presents the minimum tip mass ratio as a function of tip inertia moment ratio for three root pitch settings. Tip mass ratio is the mass of the tip weight divided by the mass of a conventional blade having the same dimensions as the anelastic rotor blade. Tip inertia moment ratio is the moment of inertia of the tip mass expressed as a distance between two point masses each equal to one half the total tip mass, divided by the blade chord. The criterion for determining the minimum tip mass ratio was that no part of the blade be stalled. Thus, the region to the left is the stall region, and the region to the right is the no-stall region. It may be noted that beyond a moment ratio of approximately 1.5 the minimum tip mass ratio is essentially independent of the inertia moment.

Fig. 7 illustrates blade twist for a particular root pitch setting and inertia moment ratio, and several tip mass ratios.

It is apparent that the twist is negative and non-linear, and that increasing the tip mass reduces the non-linearity.

Fig. 8 shows the variation of thrust coefficient with root pitch setting for an inertia moment ratio of 2.0 and the minimum tip mass ratio which ensures no stall at the maximum root pitch setting. The variation of C_T with θ_o is essentially linear. Collective effectiveness,

$$\frac{\partial C_T}{\partial \theta_o} = .00011 \text{ per degree per blade.}$$

If a tip tab is added blade twist and thrust coefficient change with tab deflection. The results of tab deflection presented in the following figures are for a tab with an area equal to 5 percent of the blade area and with its aerodynamic center located one chord length behind the blade twist axis. Positive tab deflection is taken trailing edge down, resulting in a negative pitching moment.

The variation of thrust coefficient with tip tab deflection is shown in Fig. 9. The variation of C_T with δ_T is essentially linear, and $\frac{\partial C_T}{\partial \delta_T} = .000034$ per degree.

The effect of tab deflection on blade twist is presented in Fig. 10. The principle effects are to change the pitch of the blade tip, and the non-linearity of the twist.

Fig. 11 shows the effect of tip mass ratio on flap-wise bending. It is evident that there is little bending, and the nominal coning angle, as defined by tip elevation with respect to root height, is independent of tip mass ratio. Fig. 12, which is Fig. 11 plotted to a uniform scale, graphically illustrates these conclusions.

The effect of changes in root pitch setting on the nominal coning angle is presented in Fig. 13. It is seen to be linear and slight.

$$\frac{\partial \beta_o}{\partial \theta_o} = .037$$

The effect of tip tab deflection on nominal coning angle is shown in Fig. 14. This, too, is linear, and

$$\frac{\partial \beta_o}{\partial \delta_T} = .008$$

DYNAMIC STABILITY. For the purpose of this analysis, the blade will be assumed to be in a condition of static equilibrium and the effects of a small change in blade tip pitch will be considered. It will be assumed that such small changes do not change the shape of the blade significantly. Consequently,

$$(24) \quad \Delta\theta = \Delta\theta_T \, x$$

The entire blade will then be subjected to the twisting and flapping moments derived in the static analysis. The aerodynamic moments will, however, be altered due to the pitching and flapping velocities of the aerodynamic center of the blade element. Also, if the mass center of the tip weight does not coincide with the twist axis there will be a twisting moment due to flapping accelerations, thus:

$$(25) \quad M_{\theta I_{\ddot{\beta}}} = -y_I \, m\Delta\ddot{z}_T \text{ where}$$

y_I is the distance between the mass center and the twist axis.

The twisting equation of motion following a small change in tip pitch may then be written as follows, where the subscript o refers to the equilibrium value and ΔM a small change

$$(26)\quad \left(M_{\theta I_o} + \Delta M_{\theta I}\right) + \left(M_{\theta t_o} + \Delta M_{\theta t}\right) + \left(M_{\theta A_o} + \Delta M_{\theta A}\right)$$

$$+ \left(M_{\theta T_o} + \Delta M_{\theta T}\right) + M_{\theta I \ddot{\beta}} = I \frac{d^2}{dt^2} \left(\theta_{T_o} + \Delta\theta_T\right)$$

For static equilibrium

$$M_{\theta I_o} + M_{\theta t_o} + M_{\theta A_o} + M_{\theta T_o} = 0. \quad \text{Then}$$

$$(27)\quad \Delta M_{\theta I} + \Delta M_{\theta t} + \Delta M_{\theta A} + \Delta M_{\theta T} + M_{\theta I \ddot{\beta}} = I \Delta \ddot{\theta}_T$$

where

$$\Delta M_{\theta I} = -\Omega^2 I \Delta\theta_T$$

$$\Delta M_{\theta t} = - \frac{m\ R\ \Omega^2\ c^2}{12} \frac{d\ (\Delta\theta)}{dr}$$

$$= - \frac{m\ \Omega^2\ c^2}{12} \Delta\theta_T$$

$$\Delta M_{\theta A} = y_A\ \Delta T$$

$$\Delta M_{\theta T} = -\ y_T\ \Delta L_T$$

The twisting equation of motion then becomes

$$- \Omega^2\ I\ \Delta\theta_T - \frac{m\ \Omega^2\ c^2}{12} \Delta\theta_T + y_A\ \Delta T$$

$$- y_T\ \Delta L_T - y_I\ m\Delta\ddot{z}_T - I\ \Delta\ddot{\theta}_T = 0$$

Dividing by I, changing signs, and collecting terms

$$(28)\quad \left[1 + \frac{m\ c^2}{12 I}\right] \Omega^2\ \Delta\theta_T - \frac{y_A}{I} \Delta T + \frac{y_T}{I} \Delta L_T$$

$$+ \Delta\ddot{\theta}_T + \frac{y_I}{I} m\Delta\ddot{z}_T = 0$$

It is now necessary to evaluate T and L_T. These changes are caused by changes in section and tip tab angle of attack due to a change in tip pitch and pitching and flapping velocities. These effects are illustrated in Fig. 15. For the blade section

$$\Delta\alpha = \Delta\theta - \frac{\Delta\dot{z}}{\Omega r} - \frac{y_A\ \dot{\theta}}{\Omega r}$$

$$\Delta T = \frac{1}{2}\ \rho a c\ \Omega^2\ R^3 \int_0^{1.0} \left[\Delta\theta x^2 - \frac{\Delta\dot{z}x}{\Omega R} - \frac{y_A\ \dot{\theta}x}{\Omega R}\right] dx$$

$\Delta\theta = \Delta\theta_T x$, and from (23)

$\Delta\dot{z} = \Delta\dot{z}_T\ x^2$. Then

$$\Delta T = \frac{1}{2}\ \rho a c\ \Omega^2\ R^3 \int_0^{1.0} \left[\Delta\theta_T\ x^3 - \Delta\dot{z}_T\ \frac{x^3}{\Omega R} - \frac{y_A\ \Delta\dot{\theta}_T\ x^2}{\Omega R}\right] dx$$

Defining $\gamma = \frac{\rho a c\ R^4}{I}$

$$\Delta T = \frac{I\ \gamma\ \Omega^2}{2R}\left[\frac{\Delta\ \theta_T}{4} - \frac{\Delta\dot{z}_T}{4\Omega R} - \frac{y_A\ \Delta\dot{\theta}_T}{3\Omega R}\right]$$

The change in the tip tab angle of attack due to tip pitching and flapping velocities, and assuming negligible change in downwash velocity, may be determined from Fig. 16. It is

$$\Delta\alpha_T = \frac{y_T\ \Delta\dot{\theta}_T - \Delta\dot{z}_T}{\Omega R} \text{ and}$$

$$\Delta L_T = \frac{1}{2}\ \rho\,(\Omega R)^2 a_T\ S_T\left(y_T\ \Delta\dot{\theta}_T - \Delta\dot{z}_T\right)$$

The entire pitching equation of motion then becomes

$$\left[1 + \frac{mc^2}{12I}\right]\Omega^2\ \Delta\theta_T - y_A\ \frac{\gamma\Omega^2}{2R}\left[\frac{\Delta\theta_T}{4} - \frac{\Delta\dot{z}_T}{4\Omega R} - \frac{y_A\ \Delta\dot{\theta}_T}{3\Omega R}\right]$$

$$+\ y_T\ \frac{\rho\,(\Omega R)^2 a_T\ S_T}{2I}\left(y_T\ \Delta\dot{\theta}_T - \Delta\dot{z}_T\right) + \Delta\ddot{\theta}_T + \frac{y_I}{I}\ m\Delta\ddot{z}_T = 0$$

Collecting terms

$$(29)\quad \left[1 + \frac{mc^2}{12I} - \frac{y_A\ \gamma}{8R}\right] \Omega^2\ \Delta\theta_T + \left[\frac{{y_T}^2\ \rho\ R^2 a_T\ S_T}{2I} + \frac{{y_A}^2\ \gamma}{\Omega 6R^2}\right] \Omega^2 \Delta\dot{\theta}_T$$

$$+\ \Delta\ddot{\theta}_T + \left[\frac{y_A\ \gamma}{\Omega 8R^2} - \frac{y_T\ \rho\ R^2 a_T\ S_T}{2I}\right] \Omega^2 \Delta\dot{z}_T + \frac{y_I\ m}{I}\ \Delta\ddot{z}_T = 0$$

The flapping equation of motion following a small change in tip pitch may be written as follows:

$$(30)\quad \left(M_{\beta t_o} + \Delta M_{\beta t}\right) + \left(M_{\beta A_o} + \Delta M_{\beta A}\right) + \left(M_{\beta T_o} + \Delta M_{\beta T}\right) =$$

$$R\ m\ \frac{d^2}{dt^2}\left(z_{T_o} + \Delta_{z_T}\right)$$

For static equilibrium

$$M_{\beta t_o} + M_{\beta A_o} + M_{\beta T_o} = 0.\quad \text{Then}$$

$$(31)\quad \Delta M_{\beta t} + \Delta M_{\beta A} + \Delta M_{\beta T} = R\ m\ \Delta\ddot{z}_T$$

$$\Delta M_{\beta t} = -\ m\ R\ \Omega^2\ \Delta z_T$$

$$\Delta M_{\beta A} = \frac{1}{2}\ \rho a c\ \Omega^2\ R^4 \int_0^{1.0} \left[\Delta\theta x^3 - \frac{\Delta\dot{z} x^2}{\Omega R} - \frac{y_A\ \dot{\theta}\ x^2}{\Omega R}\right] dx$$

$$= \frac{I\ \gamma\ \Omega^2}{2} \int_0^{1.0} \left[\Delta\theta_T\ x^4 - \frac{\Delta\dot{z}_T\ x^4}{\Omega R} - \frac{y_A\ \dot{\theta}_T\ x^3}{\Omega R}\right] dx$$

$$= \frac{I\ \gamma\ \Omega^2}{2}\left[\frac{\Delta\theta_T}{5} - \frac{\Delta\dot{z}_T}{5\Omega R} - \frac{y_A\ \dot{\theta}_T}{4\Omega R}\right]$$

$$\Delta M_{\beta T} = R\ \Delta\ L_T$$

$$= \frac{1}{2}\ \rho\ \Omega^2 R^3\ a_T\ S_T \left(y_T\ \Delta\dot{\theta}_T - \Delta\dot{z}_T\right)$$

After dividing by -I, the entire flapping equation of motion becomes

$$\frac{m\ R\ \Omega^2\ \Delta z_T}{I} - \frac{\gamma\ \Omega^2}{2}\left[\frac{\Delta\theta_T}{5} - \frac{\Delta\dot{z}_T}{5\Omega R} - \frac{y_A\ \dot{\theta}_T}{4\Omega R}\right]$$

$$- \frac{\rho\ \Omega^2 R^3\ a_T\ S_T}{2I}\left(y_T\ \Delta\dot{\theta}_T - \Delta\dot{z}_T\right) + \frac{R\ m}{I}\ \Delta\ddot{z}_T = 0$$

Collecting terms

$$(32)\quad \frac{m\ R}{I}\ \Omega^2\ \Delta z_T + \left[\frac{\gamma}{10R\Omega} + \frac{\rho\ R^3\ a_T\ S_T}{2I}\right]\Omega^2\Delta\dot{z}_T$$

$$+ \frac{R\ m}{I}\ \Delta\ddot{z}_T - \frac{\gamma}{10}\ \Omega^2\ \Delta\theta_T + \left[\frac{\gamma\ y_A}{8R\Omega} - \frac{\rho\ R^3\ a_T\ S_T\ y_T}{2I}\right]\Omega^2\Delta\dot{\theta}_T = 0$$

The characteristic equation may be found in the usual way and is of the form

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \text{ where}$$

$$\lambda = n \pm i\omega$$

For dynamic stability Routh's Discriminant, given by

$$R = D\ (BC - AD) - B^2\ E$$

must be positive. R = 0 therefore establishes a stability boundary. In Fig. 17 Routh's Discriminant is plotted as a function of the location of the mass center of the tip mass with respect to the blade twist axis, expressed as a fraction of the chord, for the anelastic blade without a tip tab. For stability, the required location of the mass center is seen to be .025C forward of the twist axis, which is located essentially at mid-chord.

The effects of the addition of a tip tab on dynamic stability are presented in Fig. 18, which shows the stability boundaries for several center of gravity locations as functions of tab size and the location of the tab aerodynamic center with respect to the blade aerodynamic center. It is interesting to note that the addition of a tab is stabilizing if the tab or its moment arm are small, but an increase in either tab area or moment arm length makes it necessary to move the tip mass center of gravity forward to retain stability.

The characteristic equation indicates that there may be two oscillatory modes. Accordingly, the "shape" of the modes was investigated for a blade with and without a tab. The results are summarized in Table I.

TABLE I

OSCILLATORY MODES

CONDITIONS	MODE	$t_{1/2}$ (sec)	$N_{1/2}$ (cycle)	$\frac{\omega}{\Omega}$	$\frac{\theta}{\beta}$	δ (deg)
NO TAB						
N = 600 RPM	1	.237	2.1	.89	1.04	47
WT RATIO = 1	2	.723	7.6	1.05	.66	83
MOM RATIO = 2						
C.G./c = .05						
$\theta_o = 15^o$						
WITH TAB						
N = 600 RPM	1	.159	1.6	.99	.173	38
WT RATIO = 1	2	2.58	24.3	.94	1.55	50
MOM RATIO = 2						
TAB SIZE = .05						
TAB LENGTH = 1						
CG = 0						
$\theta_o = 15^o$						

$t_{1/2}$ = Time to damp to one half amplitude

$N_{1/2}$ = Oscillations to damp to one half amplitude

ω/Ω = Ratio of oscillation frequency to rotational speed

θ/β = Ratio of maximum pitching displacement to flapping displacement

δ = Phase angle between θ_{max} and β_{max}

It is evident that flapping and pitching oscillations are essentially 1 per rev for all modes. In the case of the blade with no tab, in the first mode the maximum angular displacements in pitching and in flapping are of the same magnitude, and one leads the other by 47°. The motion is damped to one half after two revolutions.

In the second mode the flapping displacements are approximately 50% greater than the pitching displacements and the motion is somewhat more lightly damped, decreasing to one half after about eight revolutions. The motions are also roughly 90° out of phase. This is somewhat comparable to the Phugoid oscillations of a fixed wing airplane.

In the case of the blade with a tip tab, the first mode represents a very nearly pure flapping motion which is highly damped. In the second mode pitching displacements are approximately 50% greater than flapping displacements and the motion is very lightly damped. Although the phase angle is 50°, the motion again crudely mimics a Phugoid oscillation.

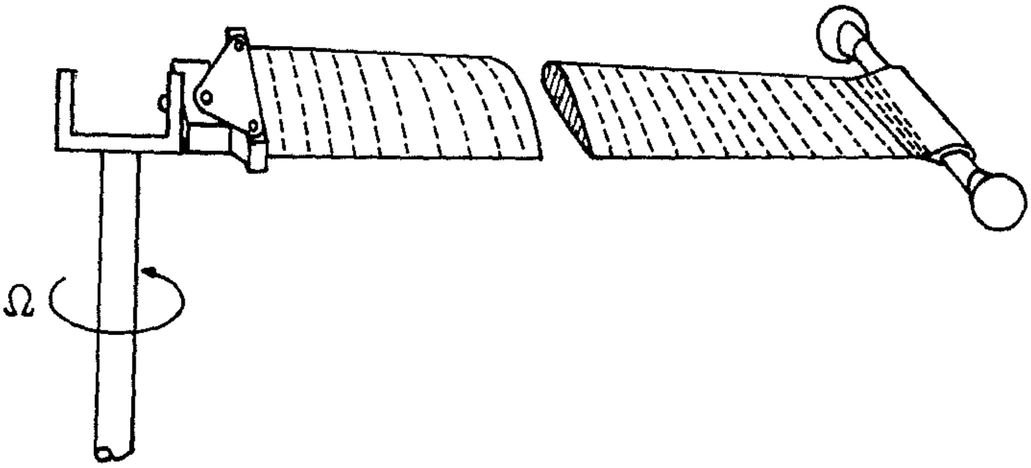
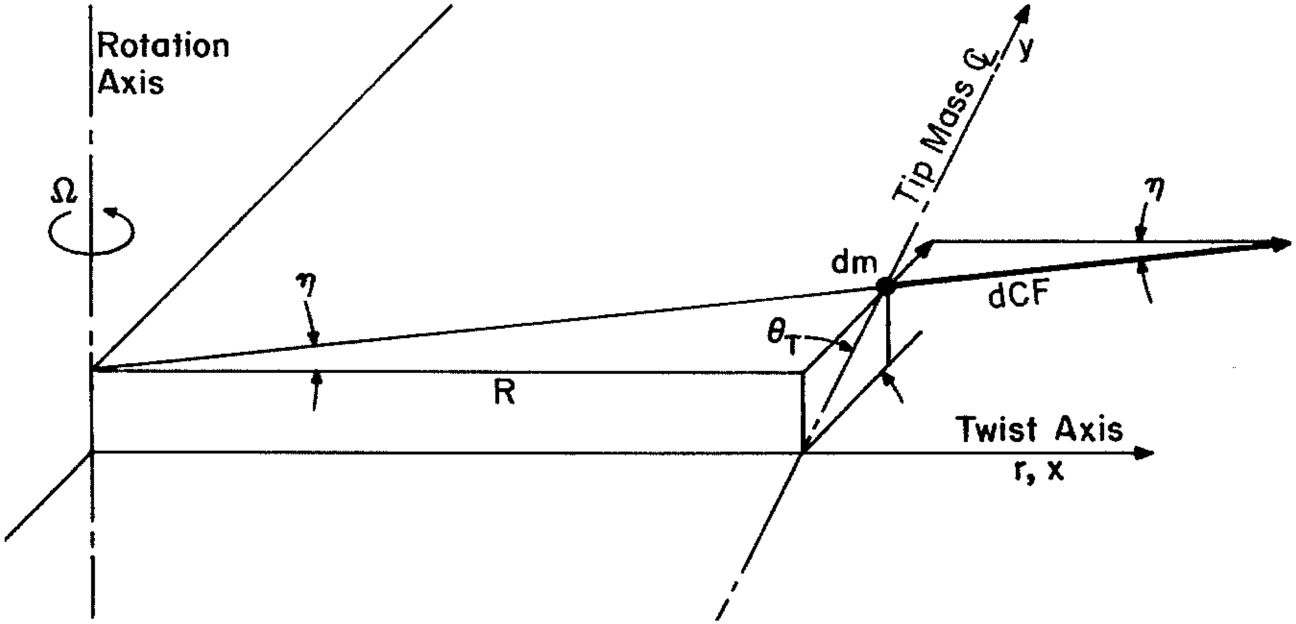
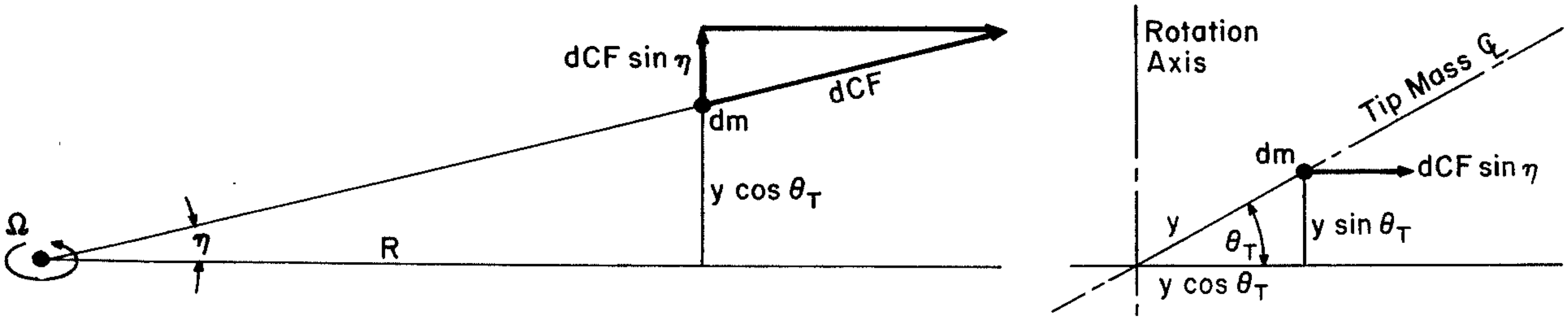


Figure 1



ISOMETRIC VIEW



PLAN VIEW

END VIEW

Figure 2

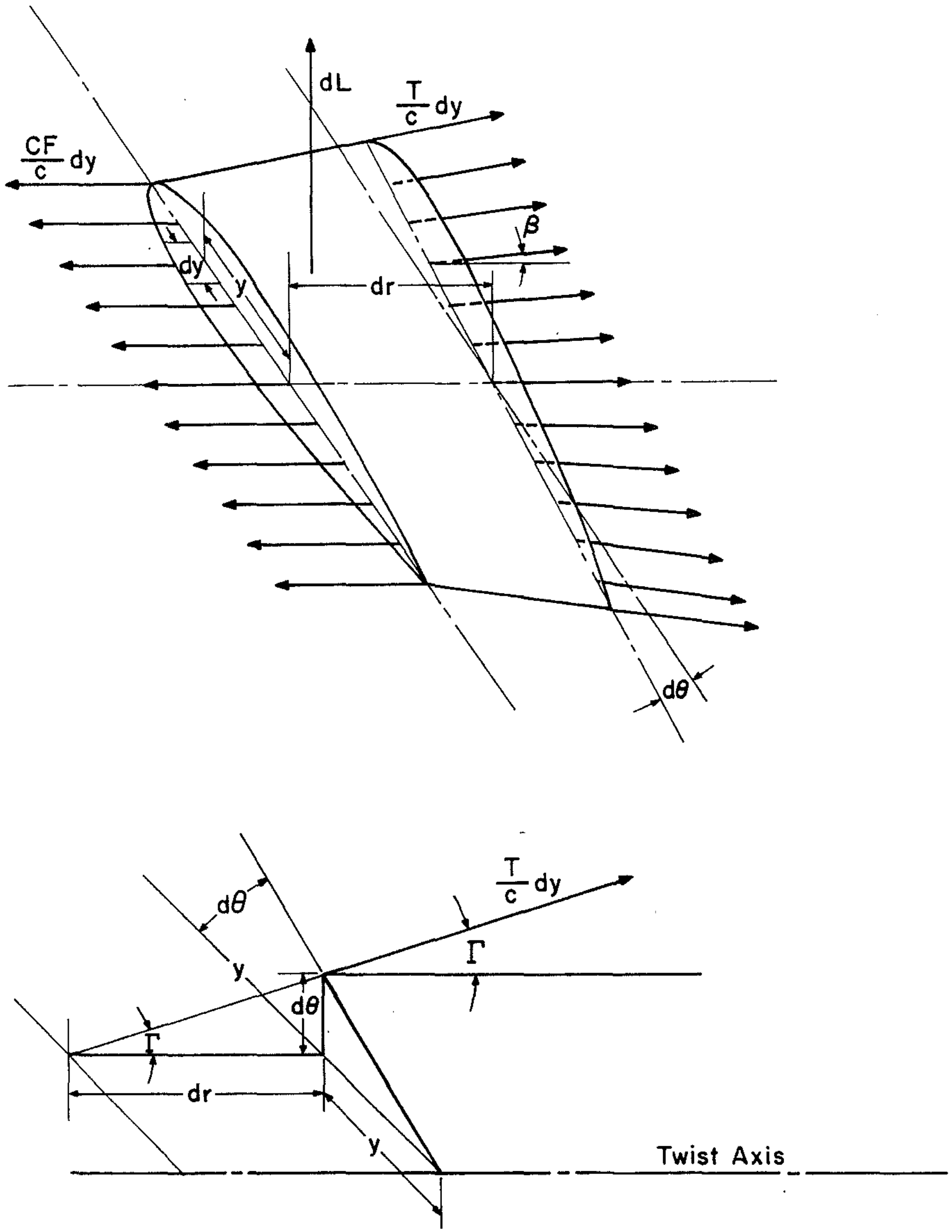
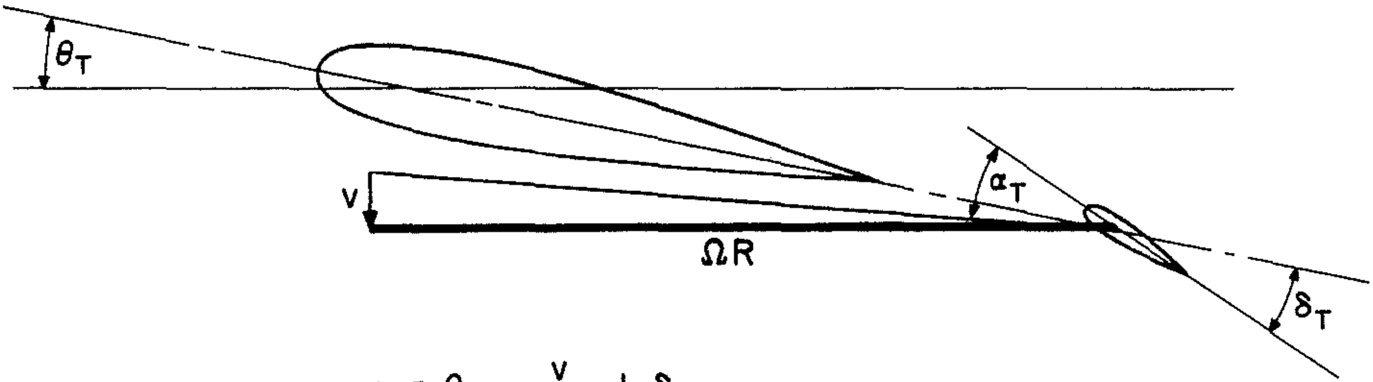

dL
$\frac{T}{c}dy$
$\frac{CF}{c}dy$
dy
y
dr
β
dθ
$\frac{T}{c}dy$
dθ
y
Γ
dθ
Γ
dr
y
Twist Axis

Figure 3



$$\alpha_T = \theta_T - \frac{V}{\Omega R} + \delta_T$$

Figure 4

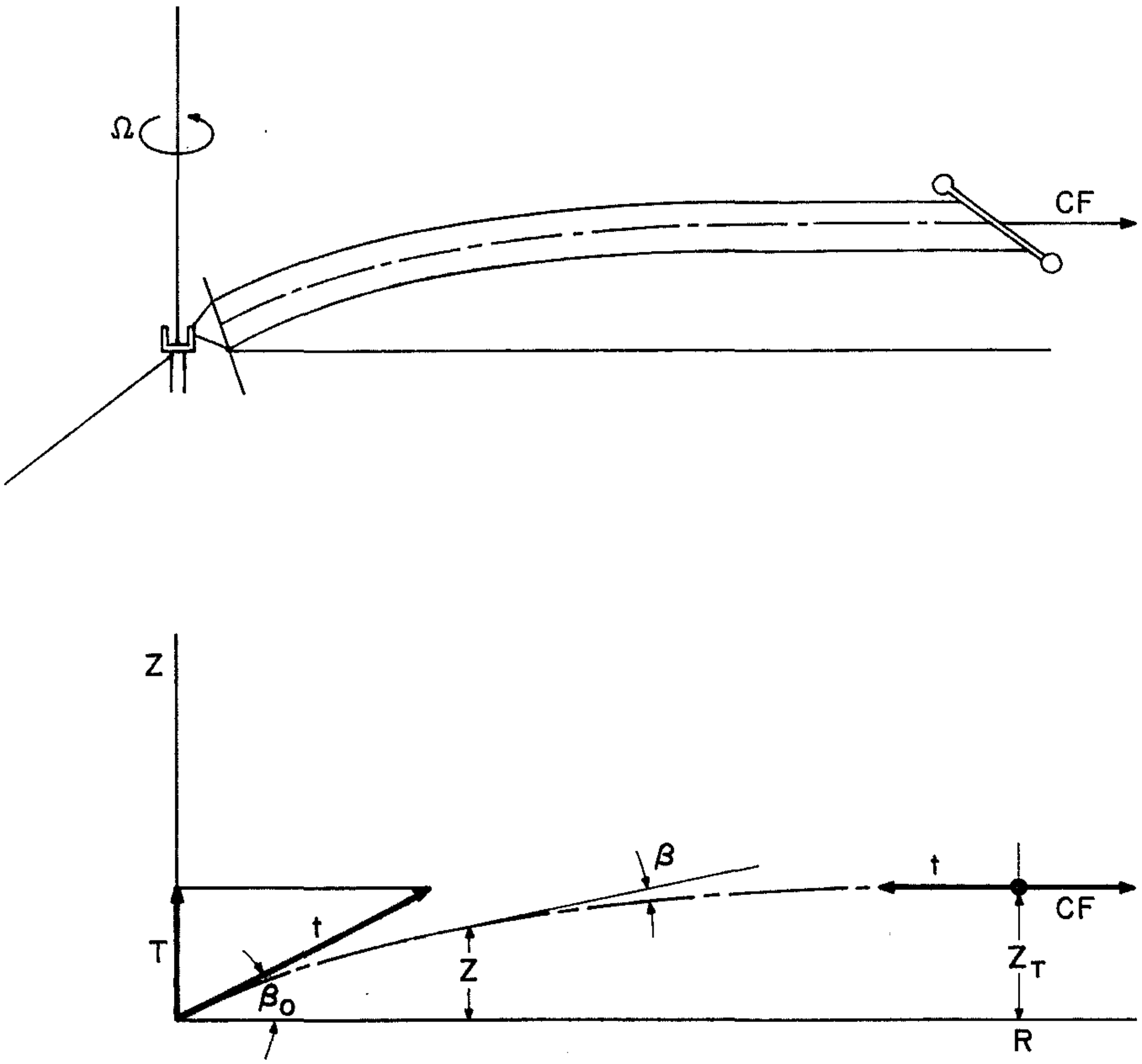


Figure 5

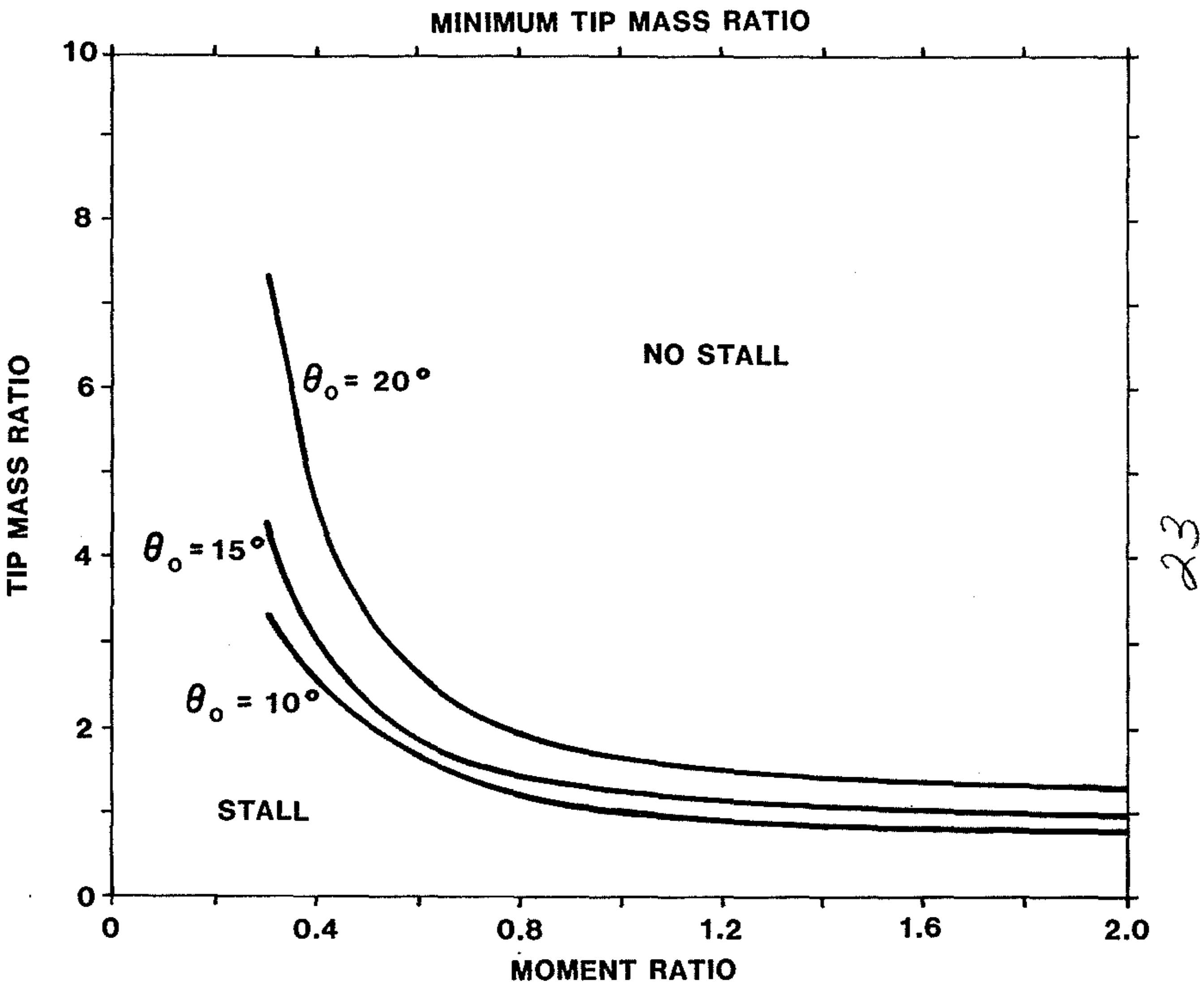

MINIMUM TIP MASS RATIO
TIP MASS RATIO
MOMENT RATIO
10
8
6
4
2
0
0
0.4
0.8
1.2
1.6
2.0
NO STALL
STALL
θo = 20°
θo = 15°
θo = 10°

Figure 6

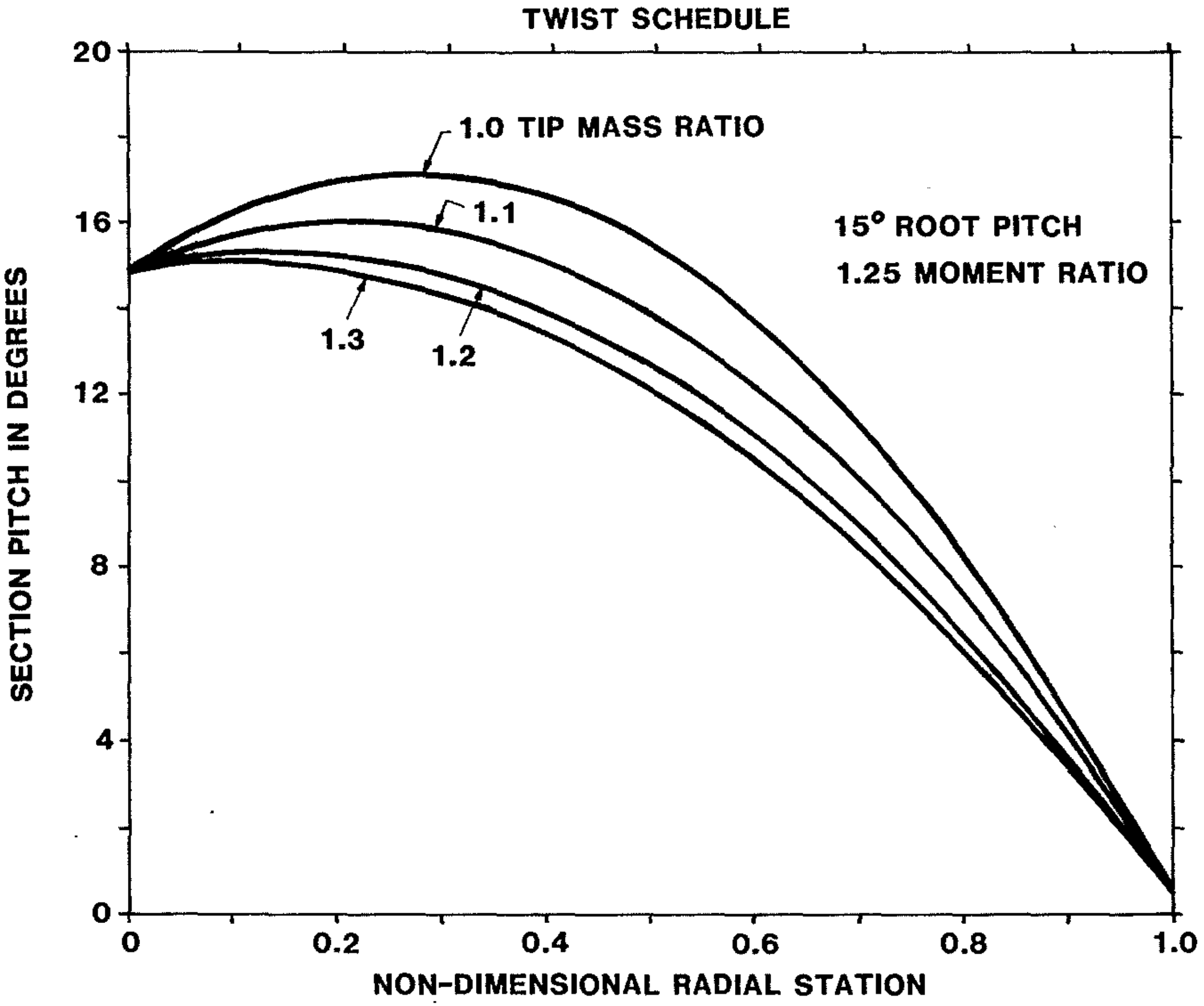

TWIST SCHEDULE
SECTION PITCH IN DEGREES
20
16
12
8
4
0
0.2
0.4
0.6
0.8
1.0
NON-DIMENSIONAL RADIAL STATION
1.0 TIP MASS RATIO
1.1
1.2
1.3
15° ROOT PITCH
1.25 MOMENT RATIO

Figure 7

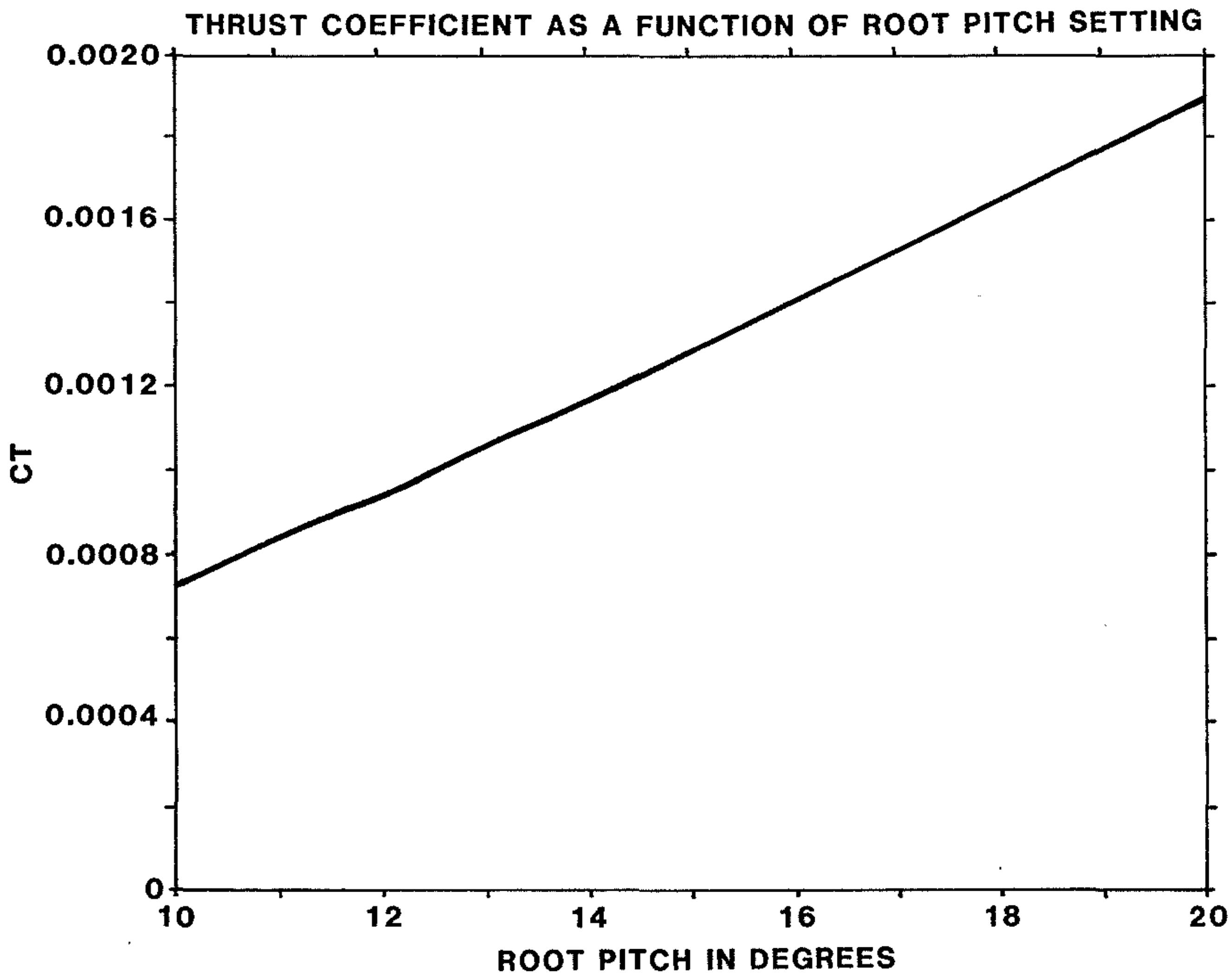

THRUST COEFFICIENT AS A FUNCTION OF ROOT PITCH SETTING
CT
0.0020
0.0016
0.0012
0.0008
0.0004
0
10
12
14
16
18
20
ROOT PITCH IN DEGREES

Figure 8

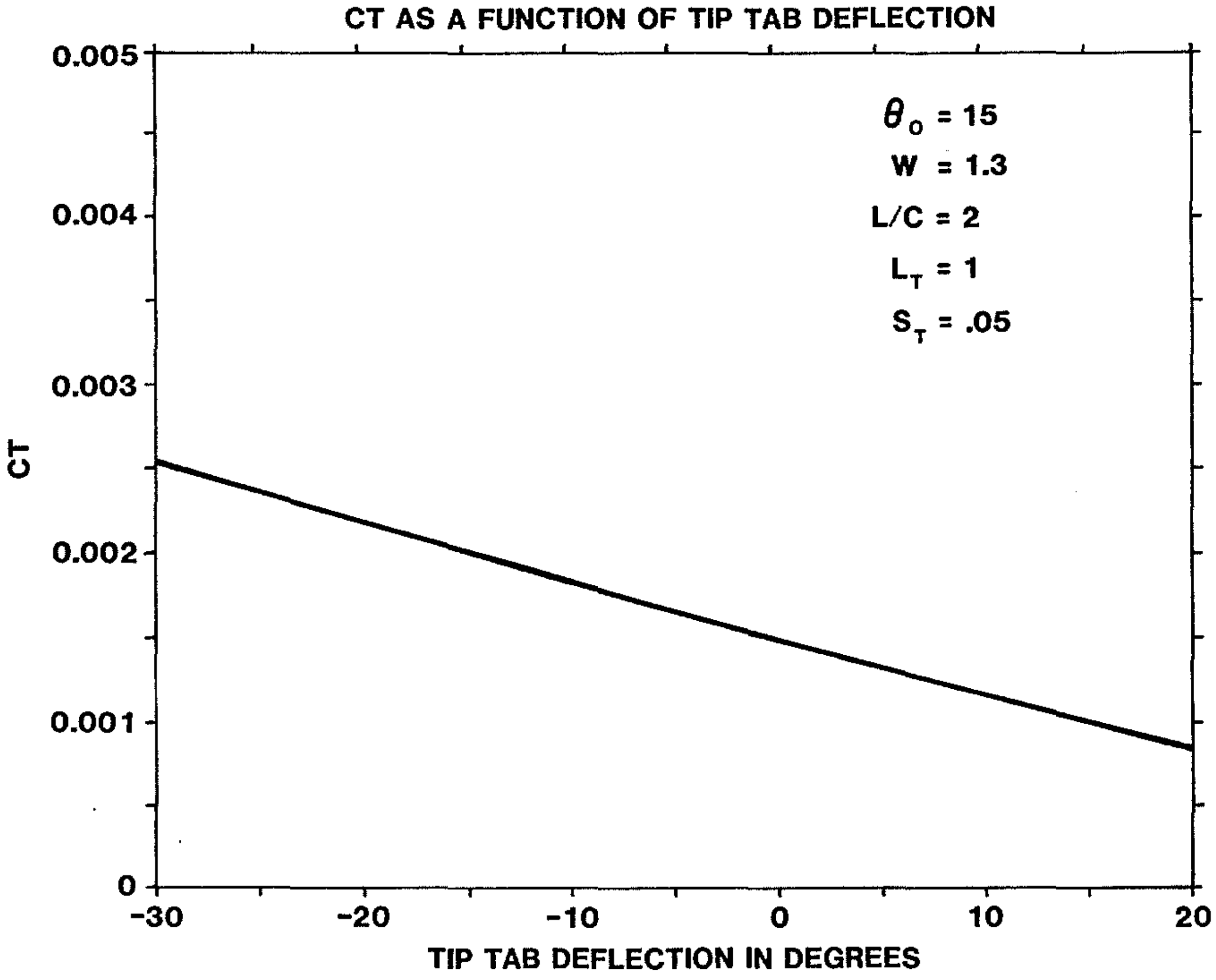

CT AS A FUNCTION OF TIP TAB DEFLECTION
θ_o = 15
W = 1.3
L/C = 2
L_T = 1
S_T = .05
CT
0.005
0.004
0.003
0.002
0.001
0
-30
-20
-10
0
10
20
TIP TAB DEFLECTION IN DEGREES

Figure 9

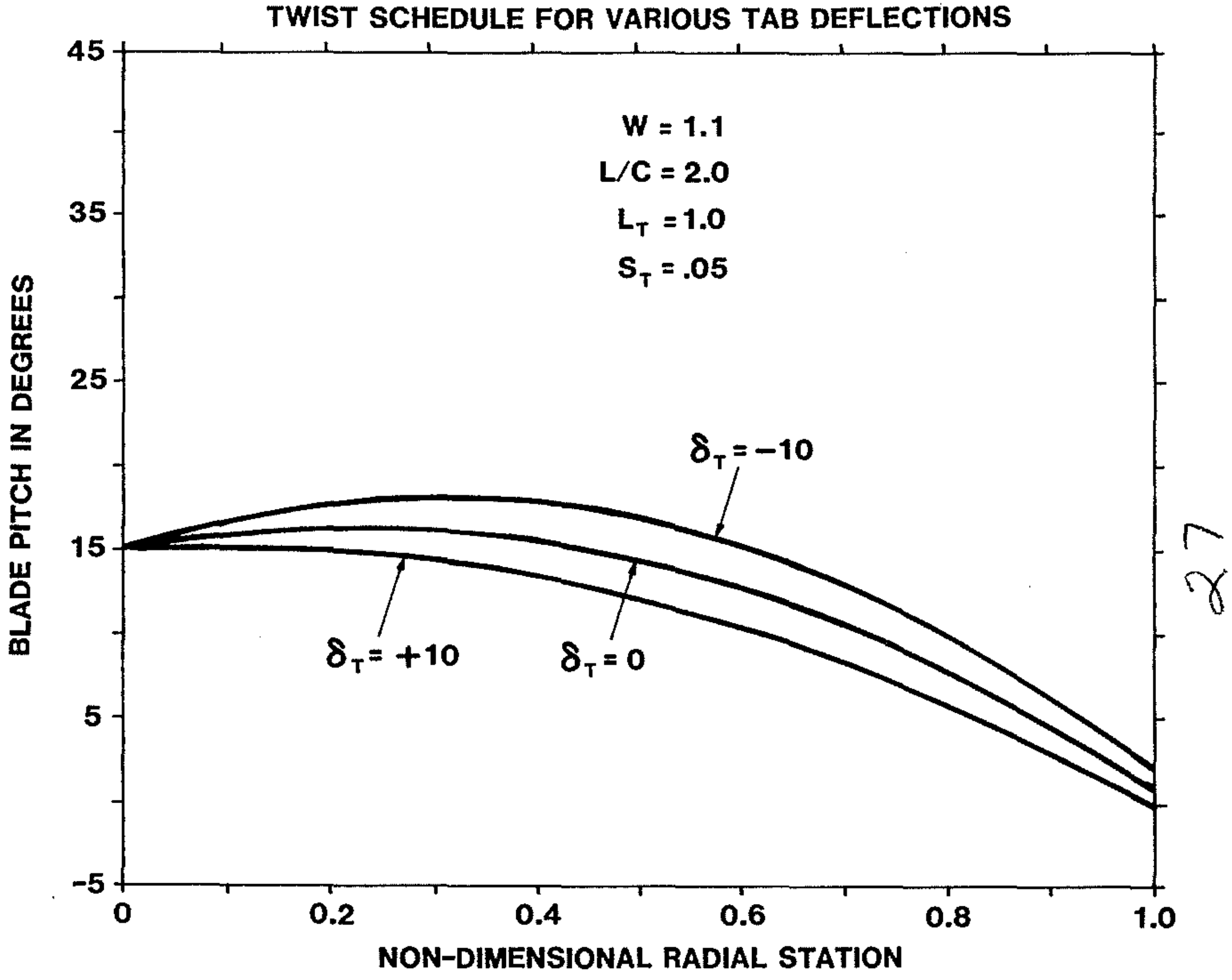

TWIST SCHEDULE FOR VARIOUS TAB DEFLECTIONS
W = 1.1
L/C = 2.0
L_T = 1.0
S_T = .05
$\delta_T = -10$
$\delta_T = +10$
$\delta_T = 0$
BLADE PITCH IN DEGREES
45
35
25
15
5
-5
0
0.2
0.4
0.6
0.8
1.0
NON-DIMENSIONAL RADIAL STATION

Figure 10

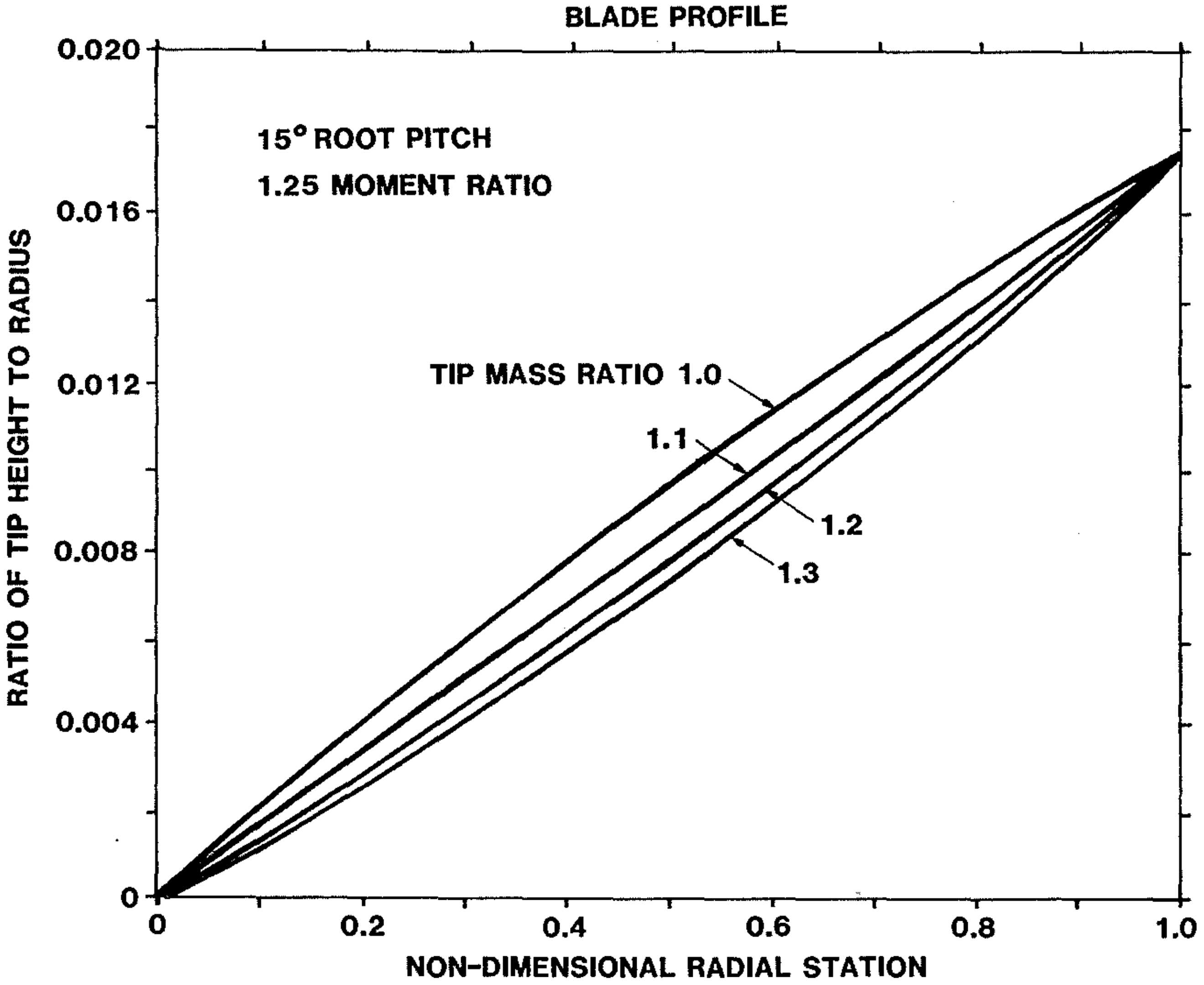

BLADE PROFILE
15° ROOT PITCH
1.25 MOMENT RATIO
TIP MASS RATIO 1.0
1.1
1.2
1.3
RATIO OF TIP HEIGHT TO RADIUS
0.020
0.016
0.012
0.008
0.004
0
0
0.2
0.4
0.6
0.8
1.0
NON-DIMENSIONAL RADIAL STATION

Figure 11

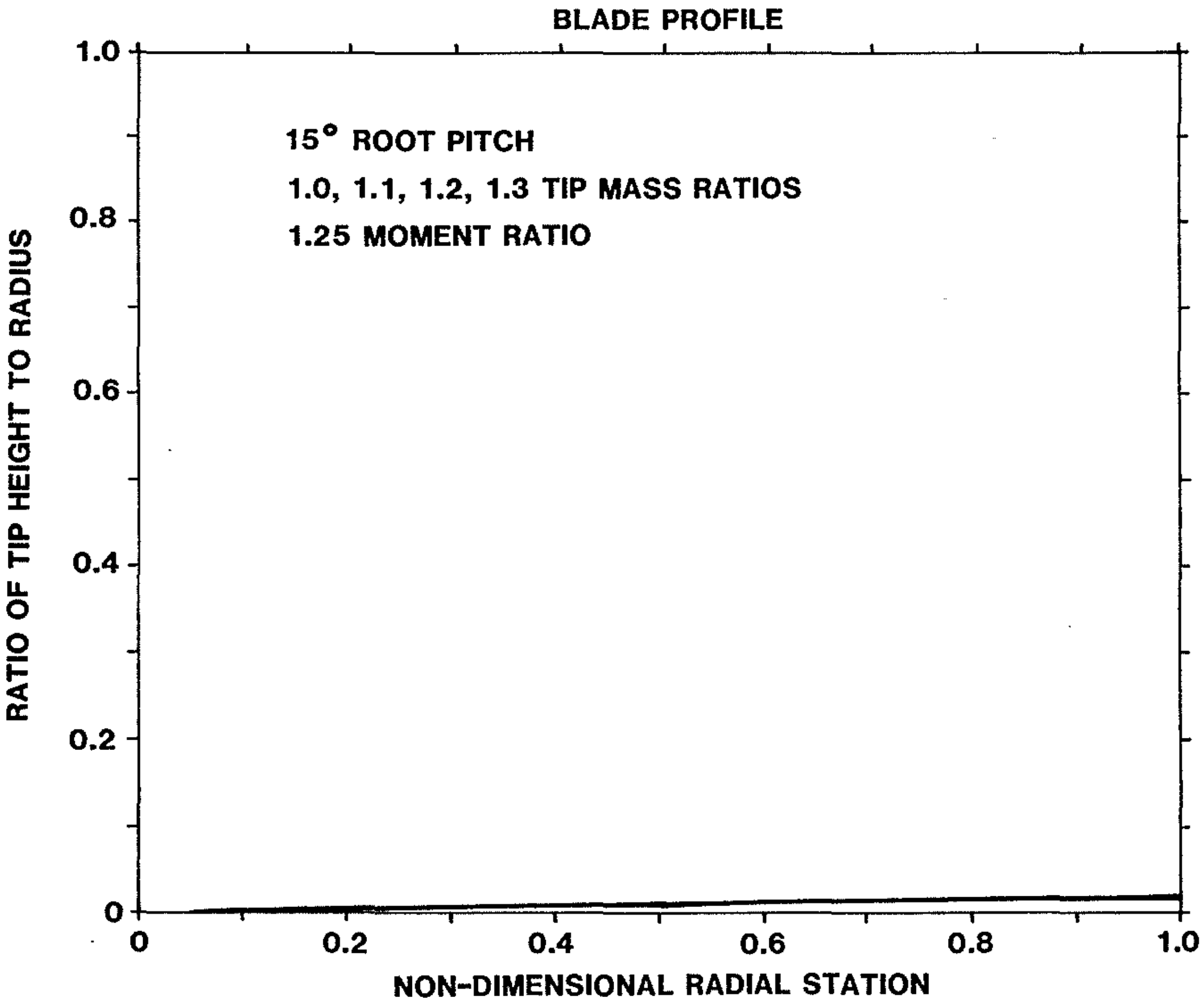

BLADE PROFILE
15° ROOT PITCH
1.0, 1.1, 1.2, 1.3 TIP MASS RATIOS
1.25 MOMENT RATIO
RATIO OF TIP HEIGHT TO RADIUS
0
0.2
0.4
0.6
0.8
1.0
NON-DIMENSIONAL RADIAL STATION
0
0.2
0.4
0.6
0.8
1.0

Figure 12

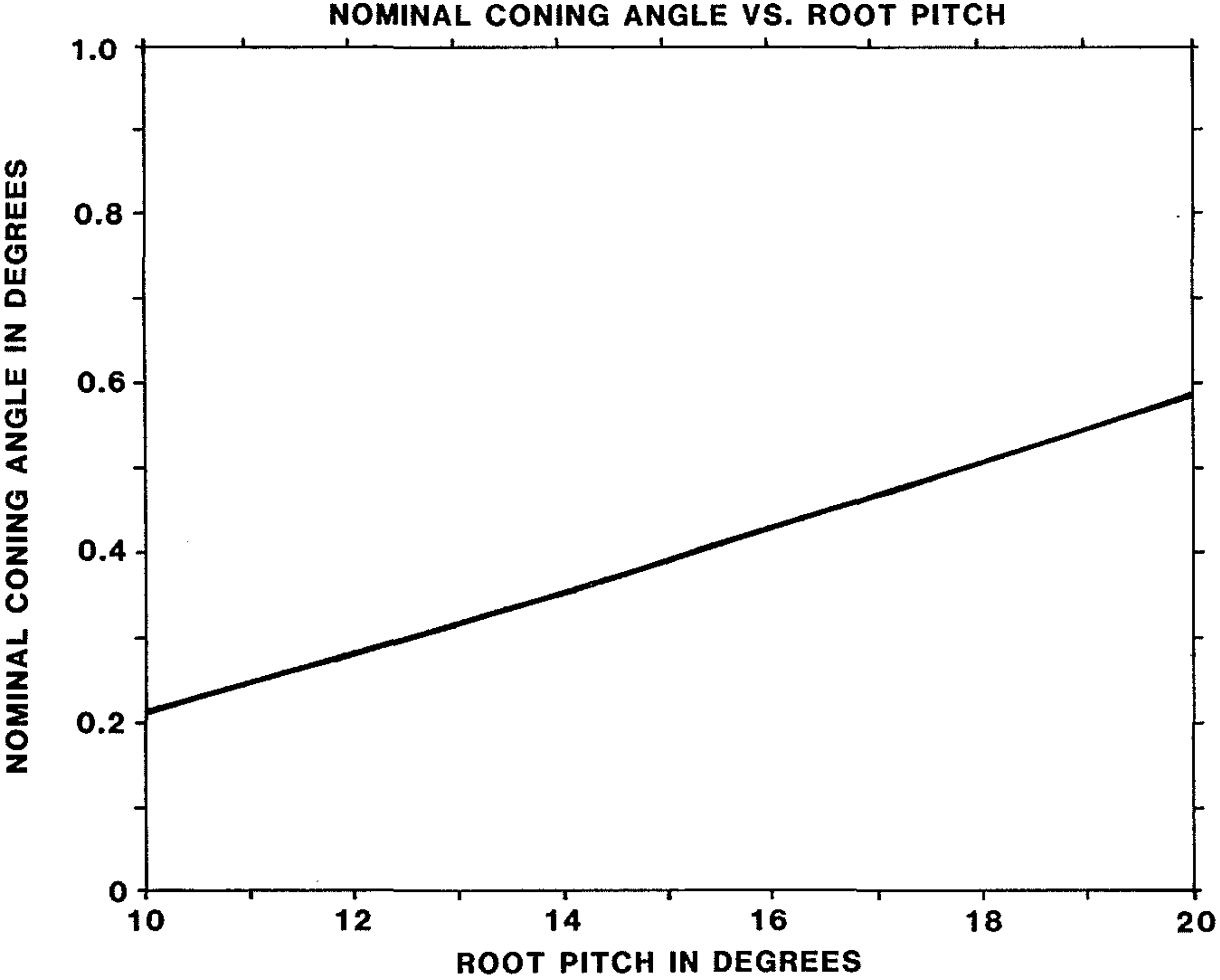

NOMINAL CONING ANGLE VS. ROOT PITCH
NOMINAL CONING ANGLE IN DEGREES
1.0
0.8
0.6
0.4
0.2
0
10
12
14
16
18
20
ROOT PITCH IN DEGREES

Figure 13

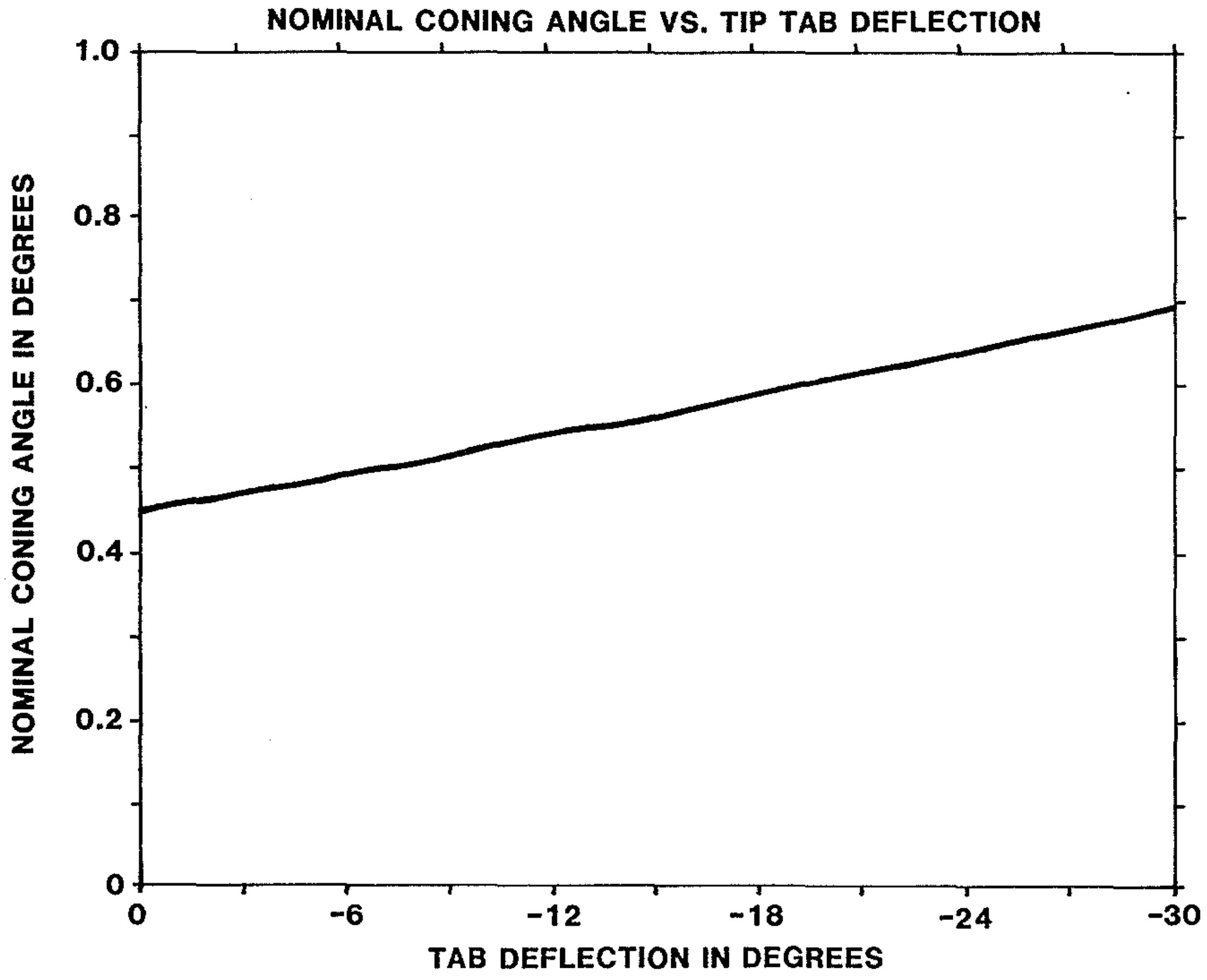

NOMINAL CONING ANGLE VS. TIP TAB DEFLECTION
NOMINAL CONING ANGLE IN DEGREES
1.0
0.8
0.6
0.4
0.2
0
0
-6
-12
-18
-24
-30
TAB DEFLECTION IN DEGREES

Figure 14

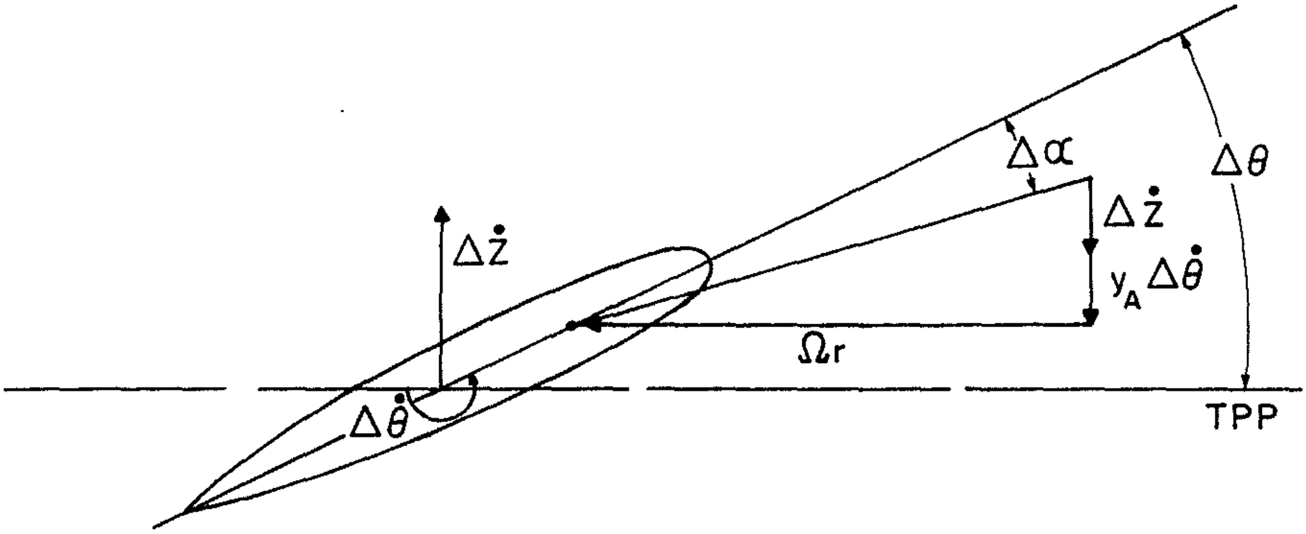
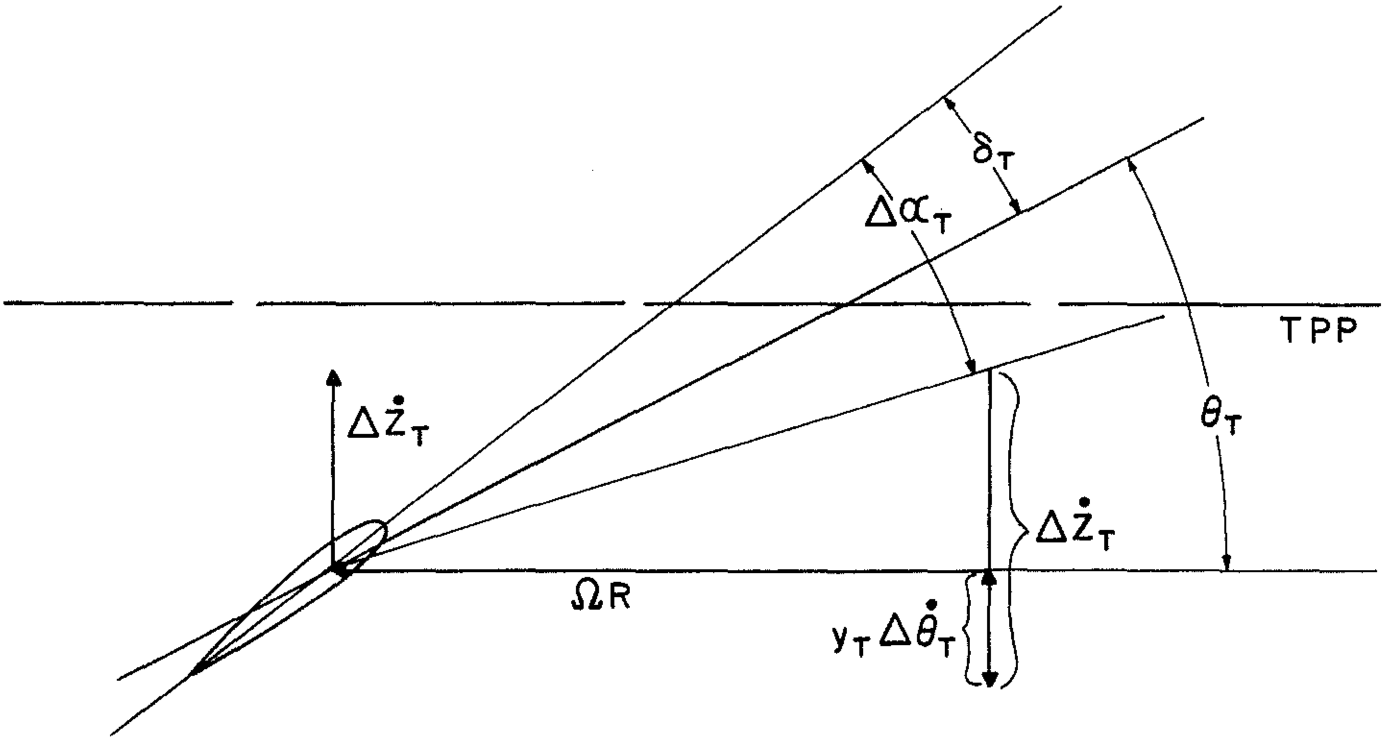

$\Delta\alpha$
$\Delta\theta$
$\Delta\dot{z}$
$\Delta\dot{z}$
$y_A\Delta\dot{\theta}$
Ωr
$\Delta\dot{\theta}$
TPP

Figure 15



δ_T
$\Delta\alpha_T$
TPP
θ_T
$\Delta\dot{z}_T$
$\Delta\dot{z}_T$
ΩR
$y_T\Delta\dot{\theta}_T$

Figure 16

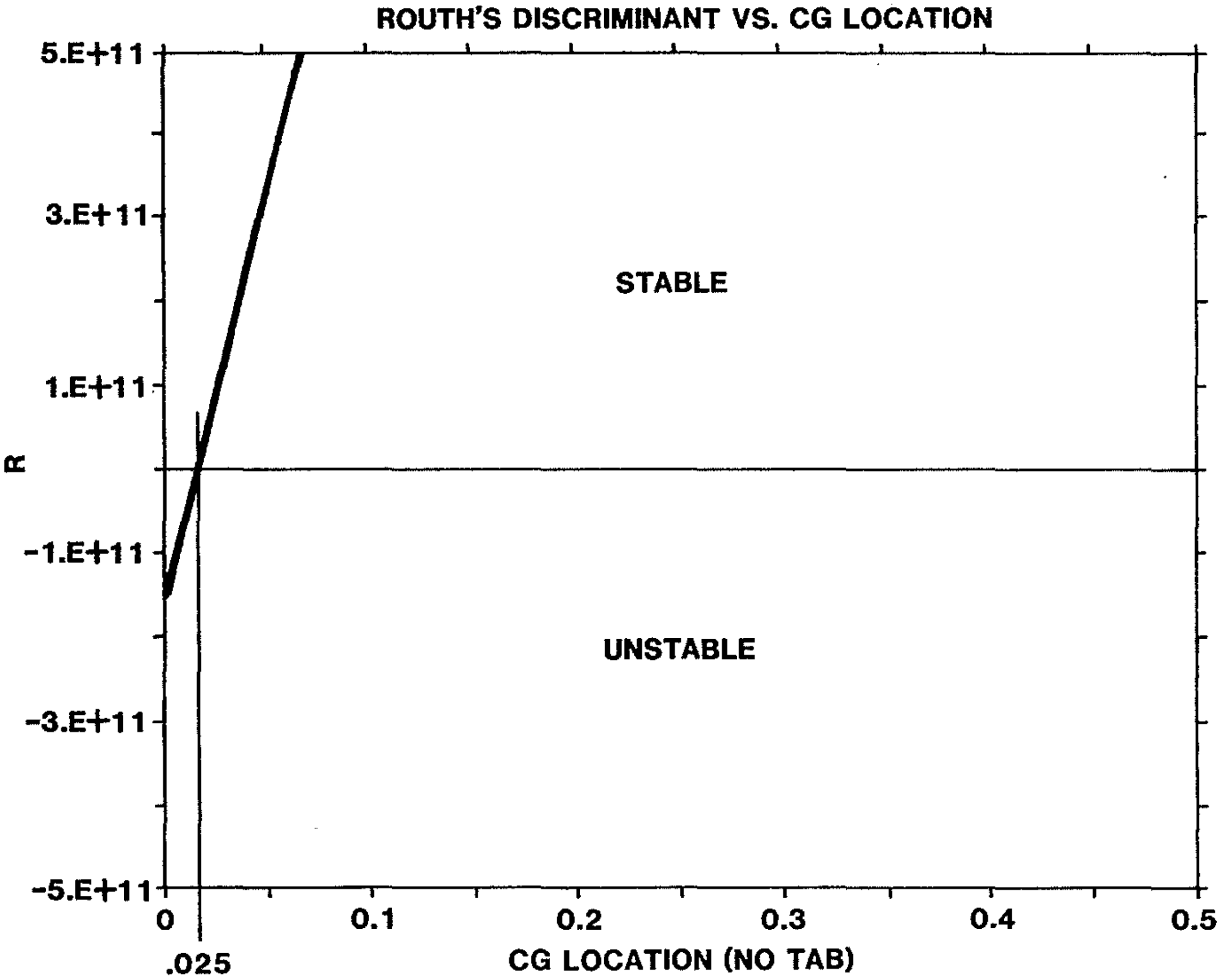

ROUTH'S DISCRIMINANT VS. CG LOCATION
5.E+11
3.E+11
1.E+11
-1.E+11
-3.E+11
-5.E+11
R
STABLE
UNSTABLE
0
0.1
0.2
0.3
0.4
0.5
.025
CG LOCATION (NO TAB)

Figure 17

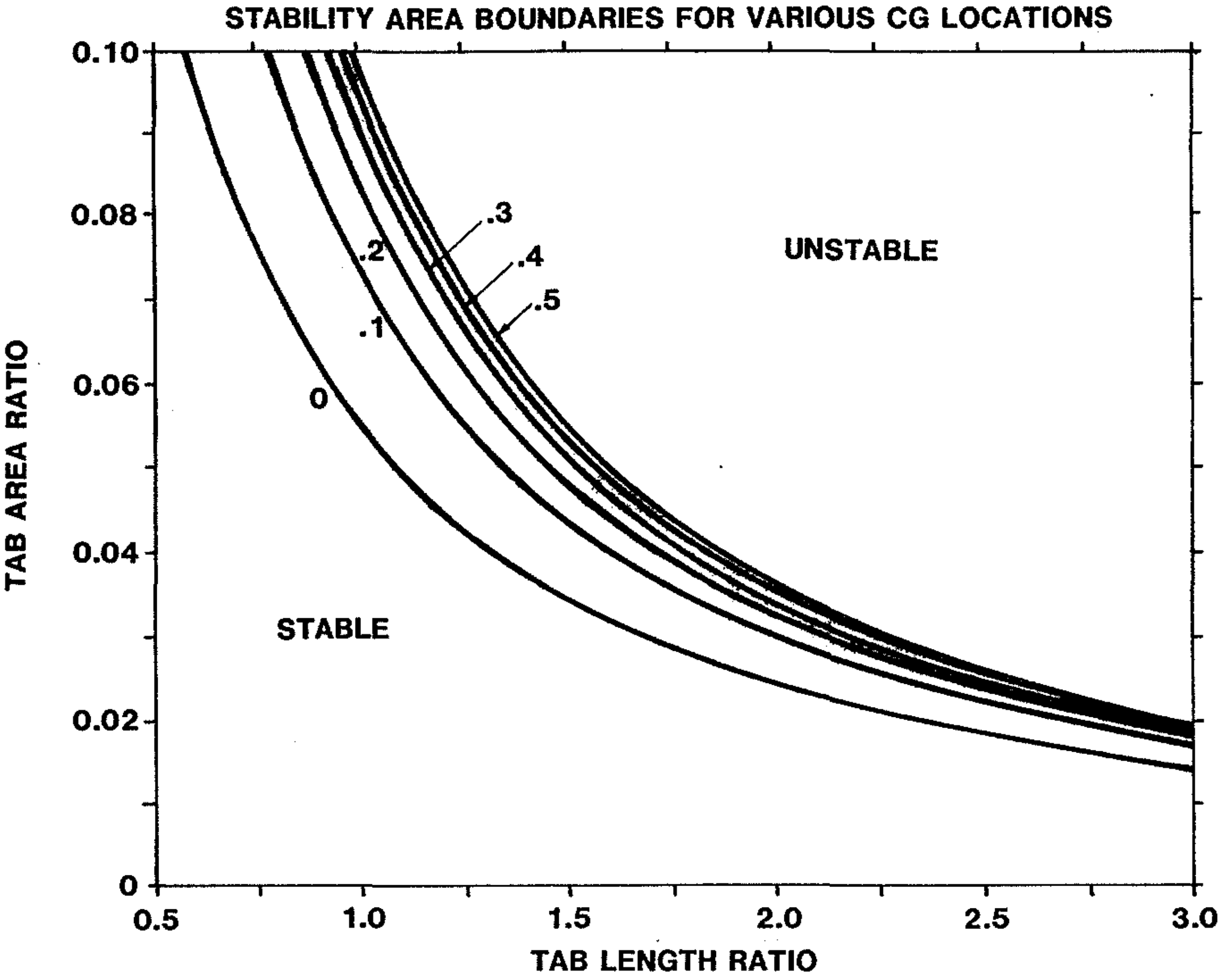

STABILITY AREA BOUNDARIES FOR VARIOUS CG LOCATIONS
TAB AREA RATIO
0.10
0.08
0.06
0.04
0.02
0
TAB LENGTH RATIO
0.5
1.0
1.5
2.0
2.5
3.0
.3
.2
.4
.5
.1
0
UNSTABLE
STABLE

Figure 18