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INTERACTION OF TORSION AND TENSION IN BEAM THEORY

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Abstract

Beam theories are approximations in order to present stresses, strains, and section forces as unknowns of the beam length only. Therefore some basic hypotheses have to be observed (for instance Bernoulli's hypothesis).

If the implications of these hypotheses are disregarded, the obtained equations present often a lot of terms pretending a higher accuracy. Whereas indeed the basic conditions are violated. - As a matter of fact the beam theory is limited by its own preconditions to a certain degree of accuracy.

The aim is to show the consequences on the interaction of torsion and tension. The limitations of beam theory are outlined. A solution which agrees with the preconditions is given with the conclusion that interaction between torsion and tension occurs only in beams with warping sections. I.e. there is no stiffening effect from the centrifugal force on the torsional stiffness in quasi warp-free helicopter rotor blades.

1. Introduction

All beam theories are approximations in order to prevent a three dimensional stress and strain analysis. The aim is to present stresses, strains, and section forces as functions of the length of the beam only.

The results obtained in deriving the differential equations depend on the philosophy applied to the problem under consideration. But there are some basic hypotheses and assumptions to be observed [7].

- Bernoulli's hypothesis of plane sections under deformation.
- The contour of the section is not changed under deformation, i.e. shear strains perpendicular to the beam axis in planes parallel to it must vanish.
- Normal stresses penpendicular to the beam axis are assumed to be zero.
- Shear strains due to lateral forces or a change in the magnitude of a bimoment are negligibly small, but strains due to St. Venant's torsion are taken into account.

For short beams the last assumption needs a correcting approximation. But for long beams it is fulfilled up to a certain degree of accuracy. Now, the desired degree of accuracy and the approximations made in beam theory become contradictory to each other if higher order theories are deduced neglecting the limitations given by the previous assumptions.

Equations obtained in this way present often a lot of terms which pretend a higher accuracy but indeed violate the basic conditions for a beam theory [2, 3, 4]. The object in deriving a beam theory is, to estimate the order of magnitude of terms and to stop the development of additional terms before the contradiction with the starting conditions is reached. This was done in [5].

The aim of this paper is to show the consequences of these criticisms on the interaction of torsion and tension. The main conclusion will be, that there is only an interaction in the case of warping sections. If warp-free or nearly (quasi) warp-free sections are considered there is no coupling between torsion and tension in beam theory.

2. Very Slender Pretwisted Beam Without Prewarping

If warping effects under torsion are taken into account, Bernoulli's hypothesis of plane sections is modified. Parts of the section differ from the plane section which is considered as a theoretical average one. If the deviations are infinitesimally small, the consequences are not so important. For closed section beams this is the case. Open section beams however exhibit considerable warping. But the contour of the section remains plane within itself, even though the plane performs no longer a right angle with the beam axis. In both cases the additions from the warping torsion theory are valid [6, 7].

Starting the presentations with a very slender beam means that the length of it is two orders of magnitude longer than the thickness. Thus, we have the following hierarchy of orders.

$$
\begin{aligned}
&O(\varepsilon):\ \theta',\ \theta'',\ \nu\\
&O(\varepsilon^2):\ \eta,\ \xi,\ \frac{\partial\varphi_s}{\partial\eta},\ \frac{\partial\varphi}{\partial\xi},\ u'\\
&O(\varepsilon^3):\ \frac{\partial f_2}{\partial\eta},\frac{\partial f_2}{\partial\xi},\frac{\partial f_3}{\partial\eta},\frac{\partial f_3}{\partial\xi}\\
&O(\varepsilon^4):\ \varphi_s\\
&O(\varepsilon^5):\ f_2,\ f_2',\ f_3,\ f_3',\frac{\partial f_1}{\partial\eta},\frac{\partial f_1}{\partial\xi}
\end{aligned}
\tag{2.1}
$$

For simplicity the problem is reduced to torsion and tension without any bending. The beam is manufactured with a pretwist, but in the unstressed state no warping occurs. Warping is only obtained in the elastically strained state. An axis point in the unstressed but pretwisted state is given by

$$r_s = x\,\bar{e}_1 \quad , \tag{2.2}$$

and any section point may be described as follows.

$$r = x\bar{e}_1 + \eta\bar{e}_2 + \xi\bar{e}_3 \tag{2.3}$$

The tripod $\bar{e}_i$ is the accompanying one and is related to the reference system by

$$\begin{Bmatrix}\bar{e}_1\\ \bar{e}_2\\ \bar{e}_3\end{Bmatrix} \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_0 & \sin\theta_0\\ 0 & -\sin\theta_0 & \cos\theta_0\end{pmatrix} \begin{Bmatrix}e_1\\ e_2\\ e_3\end{Bmatrix} . \tag{2.4}$$

In the stressed state we get:

$$R_s = (x+u)\,\hat{e}_1 \ , \tag{2.5}$$

$$R = R_s + (f_1 + u_w)\,\hat{e}_1 + (\eta + f_2)\,\hat{e}_2 + (\xi + f_3)\,\hat{e}_3 \ , \tag{2.6}$$

wit the warping effect

$$u_w = \frac{\theta'}{1+u'}\ \varphi_s \ . \tag{2.7}$$

The tripod $\hat{e}_i$ has the following relation to the reference system:

$$(2.8)\qquad \begin{Bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Theta_0+\Theta) & \sin(\Theta_0+\Theta) \\ 0 & -\sin(\Theta_0+\Theta) & \cos(\Theta_0+\Theta) \end{pmatrix}$$

The unknown functions f_i in eq. (2.6) are introduced in order to fit the preconditions of the beam theory. It is to be shown that they are of higher order of magnitude than to be included in a beam theory. Nevertheless, their derivatives with respect to the section coordinates, which present lower order in (2.1), have to be regarded.

Let us perform a slender beam theory up to fifth order due to the above hierarchy. The basic vectors are obtained as follows.

$$(2.9)\qquad \begin{aligned} G_1 &= \frac{\partial R}{\partial x} = (1+u'+\Theta''\varphi_s)\,\hat{e}_1 + \left[-\xi(\Theta_0'+\Theta') - f_3\Theta' + f_2'\right]\hat{e}_2 + \\ &\qquad + \left[\eta(\Theta_0'+\Theta') + f_2\Theta' + f_3'\right]\hat{e}_3 \\ G_2 &= \frac{\partial R}{\partial \eta} = \left(\frac{\Theta'}{1+u'}\cdot\frac{\partial\varphi_s}{\partial\eta} + \frac{\partial f_1}{\partial\eta}\right)\hat{e}_1 + \left(1+\frac{\partial f_2}{\partial\eta}\right)\hat{e}_2 + \frac{\partial f_3}{\partial\eta}\,\hat{e}_3 \\ G_3 &= \frac{\partial R}{\partial \xi} = \left(\frac{\Theta'}{1+u'}\cdot\frac{\partial\varphi_s}{\partial\xi} + \frac{\partial f_1}{\partial\xi}\right)\hat{e}_1 + \frac{\partial f_2}{\partial\xi}\,\hat{e}_2 + \left(1+\frac{\partial f_3}{\partial\xi}\right)\hat{e}_3 \end{aligned}$$

The basic vectors for the unstressed state are:

$$(2.10)\qquad \begin{aligned} g_1 &= \bar{e}_1 - \xi\,\Theta_0'\,\bar{e}_2 + \eta\,\Theta_0'\,\bar{e}_3 \\ g_2 &= \bar{e}_2 \\ g_3 &= \bar{e}_3 \end{aligned}$$

The metric coefficients are performed by scalar products.

$$(2.11)\qquad \begin{aligned} G_{11} &= 1 + 2(u'+\Theta''\varphi_s) + u'^2 + (\eta^2+\xi^2)(\Theta_0'^2 + 2\Theta_0'\Theta') \\ G_{22} &= 1 + 2\frac{\partial f_2}{\partial\eta} \\ G_{33} &= 1 + 2\frac{\partial f_3}{\partial\xi} \\ G_{12} &= \Theta'\frac{\partial\varphi_s}{\partial\eta} + \frac{\partial f_1}{\partial\eta} + f_2' - \xi(\Theta_0'+\Theta') + \Theta_0'\left(-\xi\frac{\partial f_2}{\partial\eta} + \eta\frac{\partial f_3}{\partial\eta} - f_3\right) \\ G_{13} &= \Theta'\frac{\partial\varphi_s}{\partial\xi} + \frac{\partial f_1}{\partial\xi} + f_3' + \eta(\Theta_0'+\Theta') + \Theta_0'\left(-\xi\frac{\partial f_2}{\partial\xi} + \eta\frac{\partial f_3}{\partial\xi} + f_2\right) \\ G_{23} &= \frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta} \end{aligned}$$

$$
(2.12) \quad \begin{aligned}
g_{11} &= 1 + (\eta^2 + \xi^2)\,\Theta_0'^2 \\
g_{22} &= 1 \\
g_{33} &= 1 \\
g_{12} &= -\xi\,\Theta_0' \\
g_{13} &= \eta\,\Theta_0' \\
g_{23} &= 0
\end{aligned}
$$

The strain tensor components are given by the difference of the metric coefficients, $\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij})$.

$$
(2.13) \quad \begin{aligned}
\gamma_{11} &= u' + \frac{1}{2}u'^2 + \Theta''\varphi_s + (\eta^2 + \xi^2)\,\Theta_0'\,\Theta' \\
\gamma_{22} &= \frac{\partial f_2}{\partial \eta} \\
\gamma_{33} &= \frac{\partial f_3}{\partial \xi} \\
\gamma_{12} &= \frac{1}{2}\left[\Theta'\left(\frac{\partial \varphi_s}{\partial \eta} - \xi\right) + \frac{\partial f_1}{\partial \eta} + f_2' - \Theta_0'\left(f_3 + \xi\frac{\partial f_2}{\partial \eta} - \eta\frac{\partial f_3}{\partial \eta}\right)\right] \\
\gamma_{13} &= \frac{1}{2}\left[\Theta'\left(\frac{\partial \varphi_s}{\partial \xi} + \eta\right) + \frac{\partial f_1}{\partial \xi} + f_3' + \Theta_0'\left(f_2 - \xi\frac{\partial f_2}{\partial \xi} + \eta\frac{\partial f_3}{\partial \xi}\right)\right] \\
\gamma_{23} &= \frac{1}{2}\left(\frac{\partial f_2}{\partial \xi} + \frac{\partial f_3}{\partial \eta}\right)
\end{aligned}
$$

The γ_{ij} are the Green strain components related to the undeformed state. Applying a higher order theory we need of course the Almansi strain components related to the deformed state [1].

In order to get the transformation coefficients we have to calculate the contravariant base G^i to the covariant one in eqs. (2.9). Because the lowest order in γ_{ij} is ε^2. It is sufficient to use for the geometrical relations a maximum order of ε^3.

$$
(2.14) \quad \begin{aligned}
G^1 &= \frac{G_2 \times G_3}{\sqrt{G}} = (1 - u')\,\hat{e}_1 - \Theta'\frac{\partial \varphi_s}{\partial \eta}\,\hat{e}_2 - \Theta'\frac{\partial \varphi_s}{\partial \xi}\,\hat{e}_3 \\
G^2 &= \frac{G_3 \times G_1}{\sqrt{G}} = \xi(\Theta_0' + \Theta')\,\hat{e}_1 + \left(1 - \frac{\partial f_2}{\partial \eta}\right)\hat{e}_2 - \frac{\partial f_2}{\partial \xi}\,\hat{e}_3 \\
G^3 &= \frac{G_1 \times G_2}{\sqrt{G}} = -\eta(\Theta_0' + \Theta')\,\hat{e}_1 - \frac{\partial f_3}{\partial \eta}\,\hat{e}_2 + \left(1 - \frac{\partial f_3}{\partial \xi}\right)\hat{e}_3
\end{aligned}
$$

with the scalar triple product

$$
(2.15) \quad \sqrt{G} = G_1 \cdot (G_2 \times G_3) .
$$

From eqs. (2.14) we have the coefficients

(2.16) $$G^i = b^i_j \; \hat{e}_j \; .$$

The transformations from the Green to the Almansi strain components are

(2.17) $$\hat{\gamma}_{kl} = b_k^i \, b_l^j \, \gamma_{ij} \quad ,$$

from which the following expressions result up to the fifth order.

(2.18)
$$\begin{aligned}
\hat{\gamma}_{11} &= \gamma_{11}(1-2u') + 2\gamma_{12}(\theta_0'+\theta')\xi - 2\gamma_{13}(\theta_0'+\theta')\eta = \\
&= u' - \frac{3}{2}u'^2 + \theta''\varphi_s + \theta'\theta_0'(\xi\frac{\partial\varphi_s}{\partial\eta} - \eta\frac{\partial\varphi_s}{\partial\xi}) \\
\hat{\gamma}_{12} &= -\gamma_{11}\theta'\frac{\partial\varphi_s}{\partial\eta} + \gamma_{12}(1-u'-\frac{\partial f_2}{\partial\eta}) + \gamma_{13}(-\frac{\partial f_3}{\partial\eta}) + \gamma_{22}(\theta_0'+\theta')\xi - \gamma_{23}(\theta_0'+\theta)\eta = \\
&= \frac{1}{2}\left[\theta'(\frac{\partial\varphi_s}{\partial\eta} - \xi)(1-u') - 2u'\theta'\frac{\partial\varphi_s}{\partial\eta} + \frac{\partial f_1}{\partial\eta} + f_2' + \theta_0'(-f_3 + \xi\frac{\partial f_2}{\partial\eta} - \eta\frac{\partial f_2}{\partial\xi})\right] \\
\hat{\gamma}_{13} &= -\gamma_{11}\theta'\frac{\partial\varphi_s}{\partial\xi} + \gamma_{12}(-\frac{\partial f_2}{\partial\xi}) + \gamma_{13}(1-u'-\frac{\partial f_3}{\partial\xi}) + \gamma_{23}(\theta_0'+\theta')\xi - \gamma_{33}(\theta_0'+\theta')\eta = \\
&= \frac{1}{2}\left[\theta'(\frac{\partial\varphi_s}{\partial\xi} + \eta)(1-u') - 2u'\theta'\frac{\partial\varphi_s}{\partial\xi} + \frac{\partial f_1}{\partial\xi} + f_3' + \theta_0'(f_2 - \eta\frac{\partial f_3}{\partial\xi} + \xi\frac{\partial f_3}{\partial\eta})\right] \\
\hat{\gamma}_{22} &= -2\gamma_{12}\theta'\frac{\partial\varphi_s}{\partial\eta} + \gamma_{22}(1-2\frac{\partial f_2}{\partial\eta}) - 2\gamma_{23}\frac{\partial f_3}{\partial\eta} = \frac{\partial f_2}{\partial\eta} \\
\hat{\gamma}_{33} &= -2\gamma_{13}\theta'\frac{\partial\varphi_s}{\partial\xi} - 2\gamma_{23}\frac{\partial f_2}{\partial\xi} + \gamma_{33}(1-2\frac{\partial f_3}{\partial\xi}) = \frac{\partial f_3}{\partial\xi} \\
\hat{\gamma}_{23} &= -\gamma_{13}\theta'\frac{\partial\varphi_s}{\partial\eta} - \gamma_{21}\theta'\frac{\partial\varphi_s}{\partial\xi} - \gamma_{22}\frac{\partial f_2}{\partial\xi} + \gamma_{23}(1-\frac{\partial f_2}{\partial\eta} - \frac{\partial f_3}{\partial\xi}) = \\
&= \frac{1}{2}(\frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta})
\end{aligned}$$

Applying Hooke's extendend law of elasticity to the Almansi strain components we get the Euler stress components. In order to verify the beam theory we have to observe the following conditions.

a) $f_1 \approx 0 , \quad$ i.e. $\; f_1 < 0(\varepsilon^5)$ | Bernoulli's hypothesis

b) $\hat{\tau}^{22} = \hat{\tau}^{33} = 0 \; , \;$ i.e. $\; \hat{\gamma}_{22} = \hat{\gamma}_{33} = -\nu\hat{\gamma}_{11} = \frac{\partial f_2}{\partial\eta} = \frac{\partial f_3}{\partial\xi}$

(2.19)

c) $\gamma_{23} = \frac{1}{2}(\frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta}) = 0$ — conservation of section contour

(2.19) $\dfrac{\partial f_1}{\partial \eta} + f_2{}' \approx 0$

d) $\dfrac{\partial f_1}{\partial \xi} + f_3{}' \approx 0$

i.e. negligible shear strains due to lateral forces or bimoment variation

The condition (b) is met by the solution

$$f_2 = -\nu\left[(u'-\tfrac{3}{2}u'^2)\eta + \theta''\int \varphi_s \, d\eta + \theta'\theta_0{}'(\xi\varphi_s - \int \eta \frac{\partial \varphi_s}{\partial \xi} d\eta)\right] + c_2(\eta) \; ,$$

$$f_3 = -\nu\left[(u'-\tfrac{3}{2}u'^2)\xi + \theta''\int \varphi_s \, d\xi + \theta'\theta_0{}'(\int \xi \frac{\partial \varphi_s}{\partial \eta} d\xi - \eta\varphi_s)\right] + c_2(\eta) \; , \qquad (2.20)$$

and

$$c_2(\xi) = \frac{\nu}{6}\theta''\xi^3 + c_2^{\circ} \; ,$$

$$c_3(\eta) = -\frac{\nu}{6}\theta''\eta^3 + c_3^{\circ} \; . \qquad (2.21)$$

These functions fulfill also the condition (c) which is outlined in appendix 1. Thus, the strongest condition that no change in the shape of the cross section may occur is satisfied. If terms in beam theory are taken into account which will produce shear strains γ_{23} the stiffnesses in bending and torsion will vary with the deformation due to the change of the section contour. This is not allowed within the beam theory. In order to regard these effects shell theory or continuum mechanics have to be taken for further analysis.

It can be seen from eqs. (2.20) the lowest order of magnitude in f_2 and f_3 is ε^5. Therefore, if the condition of negligible shear strains is still valid, we have to admit a deviation from Bernoulli's hypothesis of order ε^7, which can be neglected.

$$f_1 = \frac{1}{2}\nu(\eta^2+\xi^2)u'' = O(\varepsilon^7) \qquad (2.22)$$

Thus, the Almansi strain components are:

$$\hat{\gamma}_{11} = (u'-\tfrac{1}{2}u'^2)(1-u') + \theta''\varphi_s + \theta'\theta_0{}'(\xi\frac{\partial \varphi_s}{\partial \eta} - \eta\frac{\partial \varphi_s}{\partial \xi})$$

$$\hat{\gamma}_{12} = \frac{1}{2}\theta'(\frac{\partial \varphi_s}{\partial \eta} - \xi)(1-u') - u'\theta'\frac{\partial \varphi_s}{\partial \eta}$$

$$\hat{\gamma}_{13} = \frac{1}{2}\theta'(\frac{\partial \varphi_s}{\partial \xi} + \eta)(1-u) - u'\theta'\frac{\partial \varphi_s}{\partial \xi}$$

$$\hat{\gamma}_{22} = \hat{\gamma}_{33} = -\nu\hat{\gamma}_{11} \qquad (2.23)$$

$$\hat{\gamma}_{23} = 0$$

In eqs. (2.23) $*(1 - u')$ means differentation with respect to the elongated length.

$$\frac{\partial *}{\partial s} = \frac{\partial *}{(1+u')\partial x} \approx (1-u')\frac{\partial *}{\partial s} \qquad (2.24)$$

From eqs. (2.23) it is seen that there is an effect of tension marked by the elongation u' to the torsional shear strain. This effect will vanish, if the section does not warp, i.e. $\varphi_s = 0$ with $\partial\varphi_s/\partial\eta = 0$ and $\partial\varphi_s/\partial\xi = 0$.

3. Fourth Order Normal Beam Theory

Let us make the attempt to improve the beam theory for normally long beams instead for very slender ones. The length now is only one order of magnitude longer than the thickness.

$$\begin{aligned} O(\varepsilon) &: \ \eta\,,\ \xi\,,\ \frac{\partial\varphi_s}{\partial\eta}\,,\ \frac{\partial\varphi_s}{\partial\xi}\,,\ \theta'\,,\ \theta''\,,\ v \\ O(\varepsilon^2) &: \ \varphi_s\,,\ u' \\ O(\varepsilon^3) &: \ \frac{\partial f_2}{\partial\eta}\,,\ \frac{\partial f_2}{\partial\xi}\,,\ \frac{\partial f_3}{\partial\eta}\,,\ \frac{\partial f_3}{\partial\xi} \\ O(\varepsilon^4) &: \ f_2\,,\ f_3 \end{aligned} \qquad (3.1)$$

As it can be seen from eqs. (2.9), (2.11), and (2.13) under these conditions (3.1) the Green component γ_{11} will get a further term.

$$\gamma_{11} : \ \frac{1}{2}\theta'^2(\eta^2+\xi^2) \qquad (3.2)$$

In connection with the transformations done in eqs. (2.17) the Almansi component will contain this term in negative form.

$$\hat{\gamma}_{11} : \ -\frac{1}{2}\theta'^2(\eta^2+\xi^2) \qquad (3.3)$$

This will lead also to further terms in f_2 and f_3. From

$$\frac{\partial f_2}{\partial\eta} = \frac{\partial f_3}{\partial\xi} = -v\hat{\gamma}_{11} : \ \frac{1}{2}v\theta'^2(\eta^2+\xi^2) \qquad (3.4)$$

we get

$$\begin{aligned} f_2 &: \ \frac{1}{2}v\theta'^2\left(\frac{1}{3}\eta^3+\eta\xi^2\right) + c_2(\xi) \\ f_3 &: \ \frac{1}{2}v\theta'^2\left(\eta^2\xi+\frac{1}{3}\xi^3\right) + c_3(\eta) \end{aligned} \qquad (3.5)$$

This result will change the contour of the section, because γ_{23} does not vanish.

(3.6) $$\frac{\partial f_2}{\partial \xi} + \frac{\partial f_3}{\partial \eta} = 0 \neq \frac{1}{2} \nu \theta'^2 \left[2\eta\xi + 2\eta\xi \right] + \frac{\partial c_2(\xi)}{\partial \xi} + \frac{\partial c_3(\eta)}{\partial \eta}$$

The order of magnitude is due to the above convention (3.1) of fifth order. But for a theory of fourth order the solution up to the fifth order should be free of contradictions, because the fourth order terms result by differentiation with respect to the section coordinates from the fifth order terms. On the other hand the conservation of the section contour is just the strongest condition in beam theory. If the shape will change under deformation all the preconditions necessary for establishing a beam theory will be violated. Thus, the beam theory limits itself to an allowable degreee of accuracy.

4. Iterative Linearized Solution

The previous statements were made by differential geometry calculus, which is indeed a mathematical way of derivation. Much more convenient to engineers is the way of an iterative procedure.

Let us consider the case of linear standard beam theory where no distinction has to be made whether the components of stress or strain are related to the deformed or undeformed state. Than the longitudinal stress will give a first approximate relation between elongation and tensile force.

(4.1) $$U' = \frac{N_0}{EA}$$

Corresponding to eqs. (2.13) we obtain the linearized Green components of strain.

(4.2)
$$\begin{aligned}
\gamma_{11} &= \frac{N_0}{EA} + \theta''\varphi_S + (\eta^2 + \xi^2)\,\theta_0'\theta' \\
\gamma_{12} &= \frac{1}{2}\theta'\left(\frac{\partial \varphi_S}{\partial \eta} - \xi\right) \\
\gamma_{13} & \frac{1}{2}\theta'\left(\frac{\partial \varphi_S}{\partial \xi} + \eta\right) \\
\gamma_{22} &= \gamma_{33} = -\nu\frac{N_0}{EA} \quad \hat{=}\; O(\varepsilon^3) \\
\gamma_{23} &= 0
\end{aligned}$$

Within the standard beam theory no further use is made of the lateral contraction occurring in γ_{22} and γ_{33}. If it is taken into account the auxiliary functions f_i have to be introduced. In the previous sections however it was shown that the consequent use of these additional functions will give no effect as long as the limits of beam theory are respected.

For transformation linearized coefficients b^i_j , see eqs. (2.14) and (2.16), are used.

$$(4.3)\qquad b_j^{\ i} = \begin{pmatrix} 1 & -\theta'\frac{\partial\varphi_s}{\partial\eta} & -\theta'\frac{\partial\varphi_s}{\partial\xi} \\ \xi(\theta_0'+\theta') & 1 & 0 \\ -\eta(\theta'+\theta') & 0 & 1 \end{pmatrix}$$

The linearized Almansi components are given by the following equations.

$$(4.4)\qquad \begin{aligned} \hat{\gamma}_{11} &= \gamma_{11} + 2\gamma_{12}\,\theta_0'\xi - 2\gamma_{13}\,\theta_0'\eta &&= \frac{N_0}{EA} + \theta''\varphi_s + \theta'\theta_0'\left(\xi\frac{\partial\varphi_s}{\partial\eta} - \eta\frac{\partial\varphi_s}{\partial\xi}\right) \\ \hat{\gamma}_{12} &= -\gamma_{11}\,\theta'\frac{\partial\varphi_s}{\partial\eta} + \gamma_{12} &&= \frac{1}{2}\theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right) - \theta'\frac{N_0}{EA}\frac{\partial\varphi_s}{\partial\eta} \\ \hat{\gamma}_{13} &= -\gamma_{11}\,\theta'\frac{\partial\varphi_s}{\partial\xi} + \gamma_{13} &&= \frac{1}{2}\theta'\left(\frac{\partial\varphi}{\partial\xi} + \eta\right) - \theta'\frac{N_0}{EA}\frac{\partial\varphi_s}{\partial\xi} \end{aligned}$$

The Euler stress components are given by the extended Hooke's law of elasticity. Using only linearized terms, where the longitudinal force is assumed to be a known function, the Lagrange components of the cross section stress vector do not differ from the Euler components. Langrange components are related to the undeformed state but acting in the deformed direction. Therefore these components are suitable for simple integration up to section forces.

$$(4.5)\qquad \begin{aligned} \hat{\tau}^{11} &= \frac{N_0}{A} + E\left[\theta''\varphi_s + \theta'\theta_0'\left(\xi\frac{\partial\varphi_s}{\partial\eta} - \eta\frac{\partial\varphi_s}{\partial\xi}\right)\right] &&= \hat{T}^{11} \\ \hat{\tau}^{12} &= \mu\,\theta'\left[\frac{\partial\varphi_s}{\partial\eta} - \xi - 2\frac{N_0}{EA}\frac{\partial\varphi_s}{\partial\eta}\right] &&= \hat{T}^{12} \\ \tau^{13} &= \mu\,\theta'\left[\frac{\partial\varphi}{\partial\xi} + \eta - 2\frac{N_0}{EA}\frac{\partial\varphi}{\partial\xi}\right] &&= \hat{T}^{13} \end{aligned}$$

with

$$(4.6)\qquad \hat{T}^{1k} = \sqrt{G}\; b^1_{\ j}\,\hat{\tau}^{jk}$$

5. Section Forces and Conclusions

From eqs. (4.5) the section forces are obtained. The moments of inertia are defined as follows.

$$J_T = \int_A \left[\eta\frac{\partial\varphi_s}{\partial\xi} - \xi\frac{\partial\varphi_s}{\partial\eta} + \eta^2 + \xi^2\right] dA$$ St. Venant's torsional moment of inertia

$$(5.1)\qquad \bar{J}_p = \int_A (\eta^2 + \xi^2)\, dA$$ polar moment of inertia related to shear center

$$\bar{J}_p - J_T = \int_A \left(\xi \frac{\partial \varphi_s}{\partial \eta} - \eta \frac{\partial \varphi_s}{\partial \xi} \right) dA$$

(5.1)

$$C_s = \int_A \varphi_s^2 \, dA$$

$$C_T = \int_A \varphi_s \left(-\eta \frac{\partial \varphi_s}{\partial \xi} + \xi \frac{\partial \varphi_s}{\partial \eta} \right) dA$$

warping moment of inertia (sixth order of cross section dimensions)

All other combinations have a zero result.

The bimoment is given by

(5.2) $$\hat{B} = \int_A \varphi_s \hat{T}^{11} dA = E C_s \theta'' + E C_T \theta_0' \theta' \; .$$

The torsional moment is given by

(5.3)
$$\hat{M}_T = \int_A (\eta \hat{T}^{13} - \xi \hat{T}^{12}) \, dA =$$
$$= \mu \theta' \left[J_T + 2 \frac{N_0}{EA} (\bar{J}_p - J_T) \right] \; .$$

As it can be seen from eq. (5.3) there are the following consequences for the coupling of torsion and tension:

(a) Open section beams with considerable warping:

$$\bar{J}_p \gg J_T$$

(b) Closed section beams,

warp-free (e.g. circular tube): $\bar{J}_p = J_T$

quasi warp-free: $\bar{J}_p \approx J_T$

With the approximation $2\mu = \frac{E}{1+\nu} \approx E$ we get for the torsional equilibrium [7]:

(5.4)
$$-\hat{B}'' + \hat{M}_T' + \hat{m}_1 = 0$$
$$E C_s \theta^{IV} + E C_T (\theta_0' \theta')'' - \mu J_T \theta'' - (\bar{J}_p - J_T)(\frac{N_0}{A} \theta')' - \hat{m}_1 = 0$$

In the case of a warping, open section with constant longitudinal force the terms with θ'' are nearly

(5.5) $$-\left[\mu J_T + \frac{N_0}{A} \bar{J}_p\right] \theta'' .$$

That means the effective St. Venant's torsional stiffness μJ_T is strengthened by $\frac{N_0}{A} \bar{J}_p$, if a tensile force is acting. In the case of a compressive force the stiffness is reduced and stability failure may occur.

If warp-free sections or nearly (quasi) warp-free sections are considered the effect of the tensile force on torsion is cancelled. That is the obvious result of a physically evident conclusion. If only St. Venant's torsion occurs, there is no longitudinal displacement of any section point. Thus, an interaction with a longitudinal force is not given. The shear and the tensile effects are perpendicular to each other, and any coupling of them is excluded.

6. References

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Appendix 1: Proof of Conservation of Section Shape

The condition to met was formulated in eqs. (2.19) as part (c).

(A 1) $$\frac{\partial f_2}{\partial \xi} + \frac{\partial f_3}{\partial \eta} = 0$$

From eqs. (2.20) and (2.21) we get

(A 2) $$\frac{\partial f_2}{\partial \xi} = -\nu\left[\theta''\int\frac{\partial \varphi_s}{\partial \xi}\,d\eta + \theta'\theta_0'\left(\varphi_s + \xi\frac{\partial \varphi_s}{\partial \xi} - \int\eta\,\frac{\partial^2 \varphi_s}{\partial \xi^2}\,d\eta\right)\right] + \frac{\nu}{2}\theta''\xi^2$$
$$\frac{\partial f_3}{\partial \eta} = -\nu\left[\theta''\int\frac{\partial \varphi_s}{\partial \eta}\,d\xi + \theta'\theta_0'\left(\int\xi\,\frac{\partial^2 \varphi_s}{\partial \eta^2}\,d\xi - \varphi_s - \eta\,\frac{\partial \varphi_s}{\partial \eta}\right)\right] - \frac{\nu}{2}\theta''\eta^2$$

Let us first consider the terms of θ''.

(A 3) $$-\int\left(\frac{\partial \varphi_s}{\partial \xi}\,d\eta + \frac{\partial \varphi_s}{\partial \eta}\,d\xi\right) + \frac{1}{2}(\xi^2 - \eta^2) = 0$$

We introduce the torsional stress function ϕ.

(A 4) $$\frac{\partial \phi}{\partial \xi} = \frac{\partial \varphi}{\partial \eta} - \xi \quad \text{and} \quad \frac{\partial \phi}{\partial \eta} = \frac{\partial \varphi}{\partial \xi} - \eta$$

With eqs. (A 4) in eq. (A 3) we have:

(A 5) $$\int\left[\left(\frac{\partial \phi}{\partial \eta} + \eta\right)d\eta - \left(\frac{\partial \phi}{\partial \xi} + \xi\right)d\xi\right] + \frac{1}{2}(\xi^2 - \eta^2) = 0 \ .$$

Performing the integrations in eq. (A 5) the zero result is obtained.

Now, the terms of $\theta_0'\theta$ are taken.

(A 6) $$-\xi\,\frac{\partial \varphi_s}{\partial \xi} + \eta\,\frac{\partial \varphi_s}{\partial \eta} + \int\left(\eta\,\frac{\partial^2 \varphi_s}{\partial \xi^2}\,d\eta - \xi\,\frac{\partial^2 \varphi_s}{\partial \eta^2}\,d\xi\right) = 0$$

We make again use of eq. (A 4) for insertion in eq. (A 6).

(A 7)' $$-\xi\,\frac{\partial \varphi_s}{\partial \xi} + \eta\,\frac{\partial \varphi_s}{\partial \eta} + \int\left[-\eta\,\frac{\partial^2 \phi}{\partial \eta \partial \xi}\,d\eta - \xi\,\frac{\partial^2 \phi}{\partial \eta \partial \xi}\,d\xi\right] = 0$$

Integration by parts produces

(A 8) $$-\xi\,\frac{\partial \varphi_s}{\partial \xi} + \eta\,\frac{\partial \varphi_s}{\partial \eta} - \eta\,\frac{\partial \phi}{\partial \xi} - \xi\,\frac{\partial \phi}{\partial \eta} + \int\left(\frac{\partial \phi}{\partial \xi}\,d\eta + \frac{\partial \phi}{\partial \eta}\,d\xi\right) = 0$$

Now, the stress function ϕ is replaced by the warping function φ_s .

$$- \xi \frac{\partial \varphi_s}{\partial \xi} + \eta \frac{\partial \varphi_s}{\partial \eta} - \eta \left(\frac{\partial \varphi_s}{\partial \eta} - \xi \right) - \xi \left(- \frac{\partial \varphi_s}{\partial \xi} - \eta \right) +$$

(A 9)

$$+ \int \left[\left(\frac{\partial \varphi_s}{\partial \eta} - \xi \right) d\eta - \left(\frac{\partial \varphi_s}{\partial \xi} + \xi \right) d\xi \right] = 0$$

Performing the integrations all terms cancel themselves.

Appendix 2: List of Symbols

A	section area
$\hat{B}$	bimoment from warping in $\hat{e}_i$ system
C_s	warping moment of inertia
C_T	special moment of inertia, eq. (5.1)
E	Young's modulus
G_i	covariant basic vectors of strained state
G^j	contravariant basis vectors of strained state
G_{ij}	metric coefficients of strained state
$\bar{J}_p$	polar moment of inertia related to shear center
J_T	St. Venant's torsional moment of inertia
$\hat{M}_T$	torsional moment in $\hat{e}_i$ system
N_0	normal force from standard linear beam theory
R	vector of section point in the elastically strained state
R_s	vector of shear center axis point in the elastically strained state
$\hat{T}^{ik}$	Lagrange stress components
b^i_j	transformation coefficients, eq. (2.16)
c_2 , c_3	integration functions
c_2° , c_3°	integration constants
e_i	tripod of reference coordinate system

$\bar{e}_i$	tripod of pretwisted beam
$\hat{e}_i$	tripod of elastically strained beam
f_1, f_2, f_3	auxiliary functions to fit beam theory conditions
g_i	covariant basis vectors of unstressed state
g_{ij}	metric coefficients of unstressed state
$\hat{m}_1$	torsional loading in $\hat{e}_i$ system
r	vector of section point in the unstressed state
r_s	vector of shear center axis point in the unstressed state
s	elongated beam length coordinate
u	motion in x-direction
u_w	motion in x-direction due to warping
x	beam length coordinate of unstressed state
θ	angle of elastic torsion
θ_0	angle of built-in-torsion (pretwisted state)
ϕ	torsional stress function
γ_{ij}	Green strain components
$\hat{\gamma}_{kl}$	Almansi strain components
η, ξ	section coordinates of main axes related to shear center
μ	shear modulus of elasticity
ν	Poisson's ratio
$\hat{\tau}^{kl}$	Euler stress components
φ_s	warping function of section