DEVELOPMENT OF AN INTERMEDIATE-LEVEL ANALYTICAL FRAMEWORK FOR FLUID-STRUCTURE INTERACTION IN HELICOPTERS

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Abstract. : Helicopter rotor exhibits an environment different from that of a fixed-wing aircraft due to its complex aerodynamic condition as well as structural behavior. To understand the helicopter's characteristics and alleviate problems caused by these phenomena, it is strongly required to develop an accurate aeroelastic analytical tool.
To describe the behavior of helicopter rotor exactly, geometrical exactness must be considered. The generalized Greenberg-type unsteady aerodynamics is incorporated in the torsional-flap-lag aeroelastic problem.
This paper is focused on the development of an intermediate-level analytical framework for fluid-structure interaction of the helicopter rotor. The present framework is established for a pretwisted, off-hinged and preconed blade model in hover by combining advanced structural model and a relatively simple aerodynamic model. Transient time response in time domain and eigenanalysis for stability boundary in frequency domain are attempted in this paper.

1 INTRODUCTION

As helicopter is quite different from that of the fixed-wing airplanes, rotary-wing aeroelasticity is related with several complicated phenomena, and that can cause severe problems. Excessive noise and vibration upon helicopter fuselage induce fatigue problem in the helicopter components. Noise imposes severe limits upon operation in urban areas and restrictions on the pilot and passenger comfort. To correctly understand and alleviate these problems, an accurate analytical framework is required for the interaction between structure and aerodynamics. To describe the motion of the rotor blade precisely which exhibits large deflection and moderate rotation, geometrical nonlinearity should be considered. In this paper, geometrical exactness is completely formulated to include the coupling between bending and torsion deformation. An exact intrinsic equation is utilized in the present approach, and it also requires cross-sectional properties. Geometrically nonlinear formulation of three-dimensional elasticity for a moving beam is separated into a nonlinear one-dimensional problem and a linear two-dimensional problem. The two-dimensional cross-sectional analysis adopts a variational asymptotic approach, taking into account of curvature, twist and warping effects, and is consistent with three-dimensional elasticity theory to obtain the cross-section stiffness constants for an anisotropic beam. In the one-dimensional analysis, mixed variational formulation based on exact intrinsic equations for dynamics of moving beams is used to compute the global deformation or response of the blade.

Aerodynamic environment around a helicopter rotor is quite different from that of the fixed wing. Airfoil motion which is usually represented by a non-circulatory term and the wake effect as a circulatory term are the two main ingredients of the fixed-wing unsteady aerodynamics, and the unsteady inflow effect is added in the rotary-wing aerodynamics. This paper investigates effects of the blade motion and the wake in the rotary wing, but the unsteady inflow effect is neglected at this stage. The two-dimensional arbitrary motion airfoil theory in time domain is applied to a helicopter rotor blade undergoing large flap, lead-lag, and torsional motion in hover.

There have been a number of investigations which attempt to combine structural and aerodynamic analysis, such as Cesnik *et al.*[1]. These investigations turned out to be quite useful for improving accuracy of analysis, however they required considerable computational resources as well as computation time and model complicacy. To trade off among these problems while maintaining accuracy of prediction, the present study adopts a relatively simple aerodynamic model and a sophisticated blade structural dynamics model.

In this paper, two ways of analyses will be described. Time domain analysis obtains the transient responses, while frequency domain analysis predicts the stability boundaries.

2 THEORETICAL MODELING

2.1 Global Frame and Frame Transformation

To apply the mixed variational formulation for dynamics of moving beams, it is necessary to introduce a global frame which is shown in Figure 1. The global frame named *a*, with its axes labeled a_1, a_2 and a_3 is rotating with the rotor at the predefined angular velocity. The blade's undeformed reference frame is named *b*, with its axes labeled by b_1, b_2 and b_3. Its deformed reference frame is named *B*, with its axes expressed by B_1, B_2 and B_3.

Using the transformation matrices, transformation among the frames is enabled without destroying its compactness. For example, an arbitrary vector Z can be expressed by its components in frames of *a*, *b* or *B*, with their relations being

$$Z_a = C^{ba} Z_a\,, \qquad Z_B = C^{Ba} Z_a \tag{1}$$

where C^{ba} is the transformation matrix from *a* to *b*, and C^{Ba} is that from *a* to *B*.

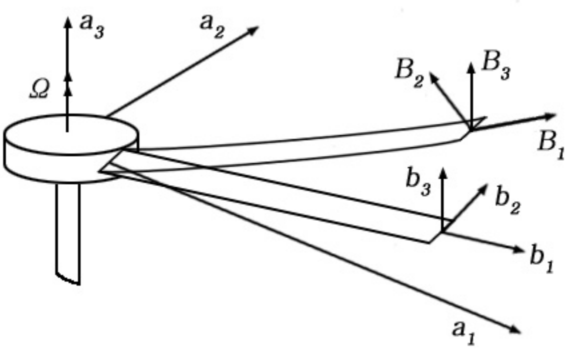

Figure 1. Frame sets used in the formulation for dynamics of moving beams

2.2 Structure Model

For the blade structure modeling, mixed form of the variational equation is used. The formulation is derived using Hamilton's principle and it can be written as

$$\int_{t_1}^{t_2}\int_0^l [\delta(K-U)+\overline{\delta W}]dx_1 dt = \overline{\delta A} \tag{2}$$

where t_1 and t_2 are arbitrary fixed times, l is the length of the beam, K and U are the kinetic and potential energy densities per unit length, respectively. $\overline{\delta A}$ is the virtual action at the ends of the beam and at the ends of the time interval, and $\overline{\delta W}$ is the virtual work of applied loads per unit length.

The variation of kinetic energy and potential terms is with respect to the column vectors of $\mathrm{V_B}, \Omega_B, \gamma$ and κ, which are linear and angular velocity, force and momentum strain, respectively. The internal force and moment vectors F_B and M_B, and linear and angular momentum vectors P_B and H_B are defined as

$$F_B = \left(\frac{\partial U}{\partial \gamma}\right)^T, \qquad M_B = \left(\frac{\partial U}{\partial \kappa}\right)^T$$
$$P_B = \left(\frac{\partial K}{\partial V_B}\right)^T, \qquad H_B = \left(\frac{\partial K}{\partial \Omega_B}\right)^T \tag{3}$$

The generalized strain and force measure, and velocity and momentum measures are related through the constitutive laws in the following form:

$$\begin{Bmatrix} F_B \\ M_B \end{Bmatrix} = [S]\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}, \qquad \begin{Bmatrix} P_B \\ H_B \end{Bmatrix} = \begin{bmatrix} m\Delta & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} V_B \\ \Omega_B \end{Bmatrix} \tag{4}$$

where $[S]$ is 6×6 stiffness matrix, m is the mass density per unit length, and I is the matrix of moment of inertia.

With the above equations, equation (2) can be written as

$$\int_{t_1}^{t_2}\int_0^l \left[\delta V_B^{*T} P_B + \delta\Omega_B^{*T} H_B - \delta\gamma^{*T} F_B - \delta\kappa^{*T} M_B\right] + \int_{t_1}^{t_2}\int_0^l \overline{\delta W} dt = \overline{\delta A} \tag{5}$$

where the superscript * means that the specific terms must satisfy the geometrically exact equations in *a* frame. The geometrically exact equations are

$$\gamma^* = C^{Ba}\left(C^{ab}e_1 + U_a'\right) - e_1\ , \qquad \kappa^* = C^{ba}\left(\frac{\Delta - \dfrac{\tilde{\theta}}{2}}{1 + \dfrac{\theta^T\theta}{4}}\right)\theta'$$

$$V_B^* = C^{Ba}\left(v_a + \dot{u}_a + \tilde{\omega}_a u_a\right)\ , \qquad \Omega_B^* = C^{ba}\left(\frac{\Delta - \dfrac{\tilde{\theta}}{2}}{1 + \dfrac{\theta^T\theta}{4}}\right)\dot{\theta} + C^{Ba}\omega_a \tag{6}$$

where u_a is the displacement vector measured in the *a* frame, θ is the rotation vector expressed in terms of Rodrigues parameters. The Rodrigues parameters are defined in terms of a rotation of magnitude about a unit vector, $e = e_i b_i$, as $\theta_i = 2e_i \tan(\alpha/2)$. Δ is the 3×3 identity matrix, v_a and w_a are the initial velocity and initial angular velocity of a generic point on the *a* frame, and e_1 is the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$. The rotation matrix, $C = C^{ab}C^{Ba}$ is expressed in terms of the rotation parameter, θ, as follows

$$C = \frac{\left(1 - \dfrac{\theta^T\theta}{4}\right)\Delta - \tilde{\theta} + \dfrac{\theta\theta^T}{2}}{1 + \dfrac{\theta^T\theta}{4}} \tag{7}$$

where $\tilde{\theta}$ operator converts θ to its dual matrix.
The Lagrange multipliers are used to enforce the satisfaction of the kinematical equations. The *a* frame version of the complete variational formulation, based on an exact intrinsic equation for dynamics of moving beams is described in detail in Shang[2].

2.3 Aerodynamics Model

Aerodynamic analysis is based on the derivation of Dinyavari[3]. This is a generalized version of Greenberg's theory[4] of the incompressible, finite-time, arbitrary motion airfoil theories, suitable for coupled flap-lag-torsional aeroelastic analysis of the rotary blade. The generalized aerodynamic loads consist of two parts: 1) circulatory loads that involve the augmented states as well as the blade dynamics, and 2) noncirculatory loads that involve only the blade dynamics.
The non-dimensionalized circulatory and noncirculatory airloads in terms of local blade velocities are:

$$\begin{aligned}\bar{L}_{NC} &= \left[\frac{L_{NC}}{a_i \rho_A (bR)(\Omega l)^2}\right] \\ &= \left(\frac{b\bar{R}}{2}\right)\{\bar{U}_y''(\overset{*}{\theta}_G + \overset{*}{\phi}) - \bar{U}_z'' - b\bar{R}(\bar{x}_A - 1/2)(\overset{**}{\theta}_G + \overset{**}{\phi}) + \overset{*}{\bar{U}}{}_y''(\theta_G + \phi)\}\end{aligned} \tag{8a}$$

$$\bar{L}_C = \left[\frac{L_c}{\rho_A a_i (bR)(\Omega l)^2}\right] = \left[1/2 + H(\psi)\right]\bar{U}_y''\bar{Q} \tag{8b}$$

$$\bar{M}_{NC}=\left[\frac{M_{NC}}{a_i\rho_A(bR)^2(\Omega l)^2}\right]$$
$$=(\frac{b\bar{R}}{2})\{(\bar{x}_A-1)\bar{U}_y''(\overset{*}{\theta}_G+\overset{*}{\phi})-(\bar{x}_A-1/2)\overset{*}{\bar{U}}_z''$$
$$+(\bar{x}_A-1/2)\overset{*}{\bar{U}}_y''(\theta_G+\phi)-b\bar{R}(3/8+\bar{x}_A^2-\bar{x}_A)(\overset{**}{\theta}_G+\overset{**}{\phi})\} \tag{8c}$$

$$\bar{M}_C=\left[\frac{M_C}{\rho_A a_i(bR)^2(\Omega l)^2}\right]$$
$$=\bar{x}_A\bar{L}_C=\bar{x}_A[1/2+H(\psi)]\bar{U}_y''\bar{Q} \tag{8d}$$

where x_0 is the location of typical section in spanwise direction of the deformed coordinate, e is the hinge offset, μ is the advance ratio, ψ is an azimuth angle, ζ is lag angle, β is flapping angle, β_p is precone angle, θ_G is total geometric pitch angle from feathering angle, ϕ is torsional elastic deformation of the blade, bR is half chord of the blade, x_A is the aero-dynamic center offsets from the elastic center, $\overline{(\ )}$ means non-dimensionalized value, and $\overset{*}{(\ )}$ is $\partial/\partial\psi$. U'' is the velocity at the blade elastic center, and can be expressed as

$$\bar{U}_y''=[(\bar{x}_0+\bar{e})+\mu\bar{R}\sin\psi+\bar{x}_0\overset{*}{\zeta}+\mu\bar{R}\sin\psi\zeta]$$
$$\bar{U}_z''=[\bar{R}\lambda+\bar{x}_0\overset{*}{\beta}+(\mu\bar{R}\cos\psi+\bar{x}_0\zeta)(\beta_p+\beta)] \tag{9}$$
$$\bar{Q}=[\bar{U}_y''(\theta_G+\phi)-\bar{U}_z''+b\bar{R}(1-\bar{x}_A)(\overset{*}{\theta}_G+\overset{*}{\phi})]$$

$H(\psi)$ is the lift deficiency function which is a function of the augmented states. The augmented states are used to convey information regarding the unsteady wake. 3/4-span location is selected as the typical section for the augmented states because which represents good average for the phase and amplitude of the lift deficiency function for the entire blade. The expression for the lift deficiency function in terms of the augmented states is

$$H(\bar{X}_{TS1},\bar{X}_{TS2},\bar{Q}_{TS}:\psi)$$
$$=\frac{\left[0.006825\left(\frac{\bar{U}_{TS0}}{b\bar{R}}\right)^2\right]\bar{X}_{TS1}+\left[0.10805\left(\frac{\bar{U}_{TS0}}{b\bar{R}}\right)\right]\bar{X}_{TS2}}{\bar{Q}_{TS}} \tag{10}$$

where,

$$\bar{Q}_{TS}=\bar{Q}\ \ \textit{evaluated at}\ \ \bar{x}_0=\bar{x}_{TS}$$
$$\bar{U}_{TS0}=\bar{x}_{TS}+\bar{e} \tag{11}$$
$$\begin{Bmatrix}\overset{*}{\bar{X}}_1(t)\\ \overset{*}{\bar{X}}_2(t)\end{Bmatrix}=\begin{bmatrix}0 & 1\\ -0.01365(\bar{U}_{T0}/b\bar{R})^2 & -0.3455(\bar{U}_{T0}/b\bar{R})\end{bmatrix}\begin{Bmatrix}\bar{X}_1(t)\\ \bar{X}_2(t)\end{Bmatrix}+\begin{Bmatrix}0\\ \bar{Q}_{TS}\end{Bmatrix}$$

Now consider a hovering flight assuming a constant inflow. Then advance ratio becomes $\mu = 0$, cyclic pitch angle change also $\overset{*}{\theta}_G = 0$, and thus the inflow ratio change will be $\overset{*}{\lambda} = 0$. When there exists a linear pretwist in the blade, the total geometrical pitch angle becomes a function of x_0, $\theta_G(\bar{x}_0) = \theta_0 + \kappa \bar{x}_0$.
It is finally required to transform the airloads equations in the undeformed coordinate to the global *a* coordinate system, f_a, m_a. Simple transformation matrix can be used for this as follows.

$$\begin{bmatrix} 1 & -\zeta & -\beta \\ \beta\phi + \zeta & 1 & \phi \\ \beta - \zeta\phi & -\zeta\beta - \phi & 1 \end{bmatrix} \tag{12}$$

2.4 Finite Element Discretization and the Governing Equations

By discretizing the blade into *N* elements, the structural and aerodynamic equations can be rewritten in the following form:

$$\int_{t_1}^{t_2} \sum_i \delta\Pi_i dt = 0 \tag{13}$$

where index *i* indicates an *i*th element with length Δl_i, and $\delta\Pi_i$ is the corresponding spatial integration over the *i*th element. The simplest shape functions can be used in this form, and the effective nodal airload vectors can be obtained using the following relations.

$$\begin{aligned} \bar{f}_i &= \int_{l_i} (1-\xi) f_a dx_1 \ , \quad \bar{f}_{i+1} = \int_{l_i} \xi f_a dx_1 \\ \bar{m}_i &= \int_{l_i} (1-\xi) m_a dx_1 \ , \quad \bar{m}_{i+1} = \int_{l_i} \xi m_a dx_1 \end{aligned} \tag{14}$$

Once discretization is accomplished on the structural and aerodynamic part, the resulting equations can be expressed in a simple form as follows.

$$F_S(X, \dot{X}) - F_L(X, \dot{X}) = 0 \tag{15}$$

where F_S is the structural operator, F_L is the airload operator, and X is the unknown vector consisting of structural variables in the following form:

$$X = [\hat{F}_1^T \ \hat{M}_1^T \ u_1^T \ \theta_1^T \ F_1^T \ M_1^T \ P_1^T \ H_1^T \cdots u_N^T \ \theta_N^T \ F_N^T \ M_N^T \ P_N^T \ H_N^T \ \hat{u}_N^T \ \hat{\theta}_N^T]^T \tag{16}$$

Structural variables in (15) can be further expressed in the following form:

$$X = \bar{X} + \breve{X}(t) \tag{17}$$

where $\bar{X}$ is the steady component of the solution, which is independent of time, and $\breve{X}(t)$ is the transient component of the solution, or, the perturbed motion, which has time dependency.

2.5 Steady Response

In the steady state, $\breve{X}(t)=0$, therefore the governing equations can be simply written as:

$$F_S(\bar{X})-F_L(\bar{X})=0 \tag{18}$$

Since the pertinent operators have explicit expressions, the Jacobian matrix can then be derived by differentiation:

$$[J]=\left[\frac{\partial F_S}{\partial \bar{X}}-\frac{\partial F_L}{\partial \bar{X}}\right] \tag{19}$$

whose detailed expressions for the structural part are listed in [2], and those for the aerodynamics are provided in Appendix.

Equation (18) can be calculated by Newton-Raphson method, and this solution will be used successively through next two sections.

2.6 Unsteady Response

To investigate the stability of the rotor blade at a steady state, the system needs to be perturbed by performing differentiation about $\bar{X}$. The perturbed system is obtained as follows.

$$\left(\frac{\partial F_S}{\partial \bar{X}}-\frac{\partial F_L}{\partial \bar{X}}\right)\breve{X}+\left(\frac{\partial F_S}{\partial \dot{X}}-\frac{\partial F_L}{\partial \dot{X}}\right)\dot{\breve{X}}=0 \tag{20}$$

A complex Lanczos algorithm for a general eigenvalue problem with sparse, unsymmetric matrices is designed to solve the present eigenvalue problems. The eigenvalue solution will provide the stability characteristics of the coupled structural and aerodynamic system at the specified steady state.

2.7 Time Integration

To deal with time-derivative terms, a second-order backward Euler method is adopted. The following finite difference discretization scheme is applied at each time step, *n*:

$$\dot{X}_i^n=\frac{3X_i^n-4X_i^{n-1}+X_i^{n-2}}{2\Delta t} \tag{21}$$

where Δt is the time-step size. Superscripts indicate the time step and subscripts are the node index.

Now Equation (15) can be solved using Newton's method. The Jacobian matrix can be derived explicitly by differentiation:

$$[J]=\left[\frac{\partial F_S}{\partial X}-\frac{\partial F_L}{\partial X}\right] \tag{22}$$

whose detailed expressions for the structural part are listed in [5], and those for the aerodynamics can be developed in same way as that of steady case with unsteady aerodynamic terms.

2.8 Solution Procedure

Figure 2 shows the structure of the present framework. In the first block of the program, data for the geometry of the blade, rotating speed, finite element mesh, material properties and some control variables are provided. Depending upon the desired output, starting condition must be chosen carefully. Steady response analysis must always be processed before the other analysis is conducted. It expedites convergence of the solution process. The frequency domain solver iterates until the solution converges within a given tolerance, while the time domain solver integrates each time step. Once integration of the time solver is completed, next time step will be undertaken.

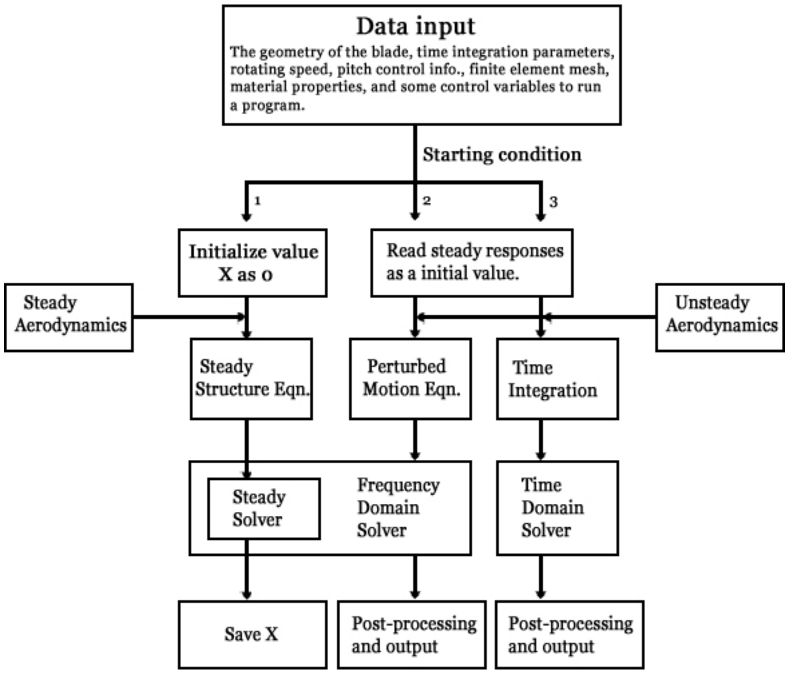

Figure 2: Block-Diagram for for the present framework

3 RESULTS AND DISCUSSION

Time domain analysis is first performed in this paper, prior to frequency domain analysis. The offset-hinged, pretwisted, articulated, small-scaled experimental blade is examined in hover. Uniform inflow is assumed in the analysis, and it is evaluated from the following relation.

$$\lambda = (\sigma a_i / 16)[\sqrt{1 + (24\theta_G / \sigma a_i)} - 1] \tag{23}$$

The resulting lift force when the rotating speed is linearily increases for the first 1 second is illustrated in Figure 3. Comparing it with the result from CAMRAD II, it is slightly larger by approximately 20%. This is due to the single blade aerodynamics adopted in this paper. The effects of the returning wake among the multiple blades will act as damping, and this will decrease overall forces and deformations. Also the root rotation is illustrated in Figure 4 at the same hover condition. By applying 15.5° of the collective pitch, it is shown a quite amount of the flap, lead-lag, torsional rotations occurring. The results about stability boundary are also possible, but not contained in this paper.

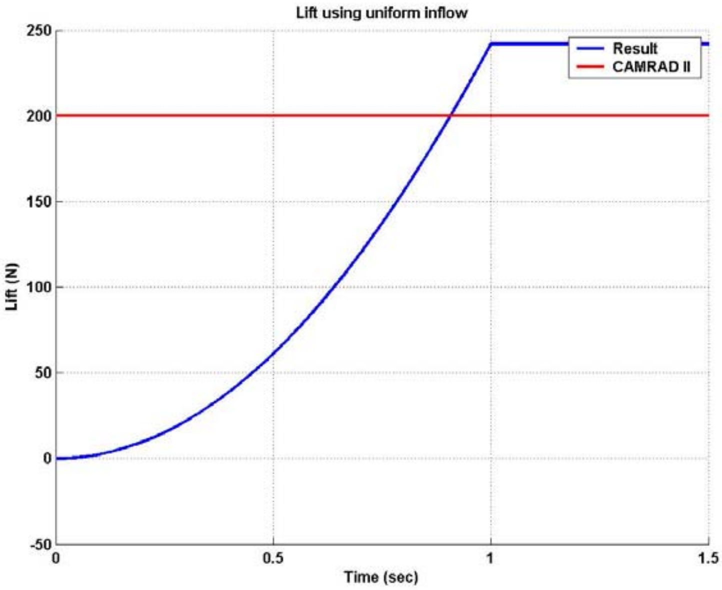

Figure 3: Lift force on the experimental blade in hover

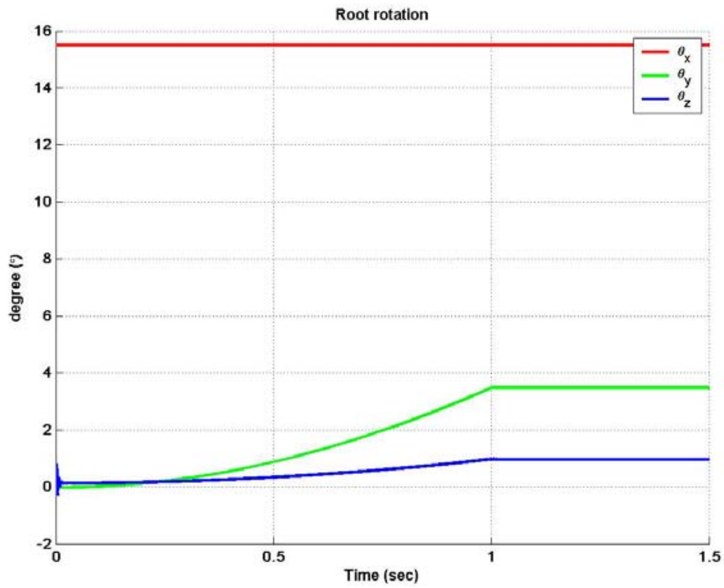

Figure 4: Root rotation of the experimental blade

4 ACKNOWLEDGEMENT

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6 APPENDIX

6.1 Jacobian matrix for steady state aerodynamic equations

$$\frac{\partial F_L}{\partial \overline{X}} = \sum_i \left(\frac{\partial F_L}{\partial X} \right)_i$$

where the expression for the submatrix of *i*th element is:

$$\frac{\partial F_L}{\partial X_i} = \begin{bmatrix} 0 & \frac{\partial \overline{f}_i}{\partial \theta} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \overline{m}_i}{\partial \theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \overline{f}_j}{\partial \theta} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial \overline{m}_i}{\partial \theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

At first, steady state aerodynamic equations are

$$f_{xAe} = \left[\rho_A a_i (bR)(\Omega l)^2\right] \times (1/2 + H_e)\left[-(\bar{x}_0 + \bar{e})^2(\theta_0 + \phi)\beta \right.$$
$$\left. -\kappa(\bar{x}_0 + \bar{e})^2 \bar{x}_0 \beta + (\bar{x}_0 + \bar{e})(\bar{R}\lambda\beta + \bar{x}_0 \overset{*}{\beta}\beta)\right]$$

$$f_{yAe} = \left[\rho_A a_i (bR)(\Omega l)^2\right] \times$$
$$\left\langle -(1/2 + H_e)\left[\left\{(\bar{x}_0 + \bar{e})\bar{R}\lambda + (\bar{x}_0 + \bar{e})\bar{x}_0 \zeta(\beta_p + \beta)\right\}(\theta_0 + \phi)\right.\right.$$
$$+ \kappa\left\{\bar{R}\lambda(\bar{x}_0 + \bar{e})\bar{x}_0 + (\bar{x}_0 + \bar{e})\bar{x}_0{}^2 \zeta(\beta_p + \beta)\right\} - (\bar{R}\lambda)^2 - 2\bar{x}_0 \bar{R}\lambda\zeta(\beta_p + \beta)$$
$$\left. -\bar{x}_0^2 \zeta^2 (\beta_p + \beta)^2\right]$$
$$-(\frac{b\bar{R}}{2})\left[\left\{-\bar{x}_0 \zeta(\beta_p + \beta) - \bar{R}\lambda\right\}(\theta_0 + \phi) - \kappa\left\{\bar{x}_0^2 \zeta(\beta_p + \beta) + \bar{x}_0 \bar{R}\lambda\right\}\right]$$
$$\left. -(\frac{c_{d0}}{a_i})(\bar{x}_0 + \bar{e})^2 \right\rangle$$

$$f_{zAe} = \left[\rho_A a_i (bR)(\Omega l)^2\right]\left\langle (1/2 + H_e)\left[(\bar{x}_0 + \bar{e})^2(\theta_0 + \phi) + \kappa(\bar{x}_0 + \bar{e})^2 \bar{x}_0\right.\right.$$
$$\left. -\left\{\bar{R}\lambda(\bar{x}_0 + \bar{e}) + \zeta(\beta_p + \beta)(\bar{x}_0 + \bar{e})\bar{x}_0\right\}\right]$$
$$\left. + \left(\frac{b\bar{R}}{2}\right)\left\{-\bar{R}\lambda - \zeta(\beta_p + \beta)\bar{x}_0\right\} \right\rangle$$

$$m_{xAe} = [a_i \rho_A (bR)\Omega^2 l^4] \times$$
$$\left\langle b\bar{R}\bar{x}_A (1/2 + H_e)\left[(\bar{x}_0 + \bar{e})^2(\theta_0 + \phi) + \kappa\bar{x}_0(\bar{x}_0 + \bar{e})^2\right.\right.$$
$$\left. -(\bar{x}_0 + \bar{e})\{\bar{R}\lambda + \bar{x}_0 \zeta(\beta_p + \beta)\}\right]$$
$$+ (1/2 + H_e)\left[(\bar{x}_0 + \bar{e})^2 \bar{x}_0 \zeta(\theta_0 + \phi) + \kappa(\bar{x}_0 + \bar{e})^2 \bar{x}_0^2 \zeta\right.$$
$$\left. -\bar{R}\lambda\zeta(\bar{x}_0 + \bar{e})\bar{x}_0 + \zeta^2(\beta_p + \beta)(\bar{x}_0 + \bar{e})\bar{x}_0^2\right]$$
$$-\left(\frac{b\bar{R}}{2}\right)\left\{\bar{R}\lambda\zeta\bar{x}_0 + \zeta^2(\beta_p + \beta)\bar{x}_0^2\right\}$$
$$+ (1/2 + H_e)\left[\left\{(\bar{x}_0 + \bar{e})\bar{x}_0 \beta\bar{R}\lambda + (\bar{x}_0 + \bar{e})\bar{x}_0^2 \beta\zeta(\beta_p + \beta)\right\}(\theta_0 + \phi)\right.$$
$$+ \kappa\left\{\bar{R}\lambda(\bar{x}_0 + \bar{e})\bar{x}_0^2 \beta + (\bar{x}_0 + \bar{e})\bar{x}_0^3 \beta\zeta(\beta_p + \beta)\right\} - (\bar{R}\lambda)^2 \beta\bar{x}_0$$
$$\left. -2\bar{x}_0^2 \bar{R}\lambda\beta\zeta(\beta_p + \beta)\right]$$
$$+(\frac{b\bar{R}}{2})\left[\left\{-\bar{x}_0^2 \beta\zeta(\beta_p + \beta) - \beta\bar{R}\lambda\bar{x}_0\right\}(\theta_0 + \phi) - \kappa\left\{\bar{x}_0^3 \beta\zeta(\beta_p + \beta) + \bar{x}_0^2 \beta\bar{R}\lambda\right\}\right]$$
$$\left. +(\frac{c_{d0}}{a_i})\beta(\bar{x}_0 + \bar{e})^2 \bar{x}_0 \right\rangle$$

$$
\begin{aligned}
m_{yAe} = {} & [a_i \rho_A (bR)\Omega^2 l^4]\times \\
& \Big\langle (-1/2 - H_e)\Big[(\bar{x}_0 + \bar{e})^2 \bar{x}_0 (1 - 1/2\beta^2 - 1/2\zeta^2)(\theta_0 + \phi) \\
& + (\bar{x}_0 + \bar{e})^2 \bar{x}_0{}^2 \kappa (1 - 1/2\beta^2 - 1/2\zeta^2) - (\bar{x}_0 + \bar{e})\bar{x}_0 \bar{R}\lambda \\
& - (\bar{x}_0 + \bar{e})\bar{x}_0{}^2 (\beta_p + \beta)\zeta - (\bar{x}_0 + \bar{e})^2 b\bar{R}\bar{x}_A (\theta_0 + \phi)\zeta \\
& - (\bar{x}_0 + \bar{e})^2 \bar{x}_0 \kappa b\bar{R}\bar{x}_A \zeta + (\bar{x}_0 + \bar{e})^2 \bar{x}_0 \beta^2 (\theta_0 + \phi) + (\bar{x}_0 + \bar{e})^2 \bar{x}_0^2 \kappa \beta^2 \Big] \\
& + (\frac{b\bar{R}}{2})\bar{x}_0 \bar{R}\lambda \Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
m_{zAe} = {} & [a_i \rho_A (bR)\Omega^2 l^4]\times \\
& \Big\langle -(1/2 + H_e)\Big[(\bar{x}_0 + \bar{e})\bar{x}_0 \bar{R}\lambda(\theta_0 + \phi) + (\bar{x}_0 + \bar{e})\bar{x}_0^2 \kappa \bar{R}\lambda \\
& + (\bar{x}_0 + \bar{e})\bar{x}_0^2 \zeta (\theta_0 + \phi)(\beta_p + \beta) + (\bar{x}_0 + \bar{e})\bar{x}_0^3 \kappa \zeta (\beta_p + \beta) \\
& - \bar{x}_0 (\bar{R}\lambda)^2 - 2\bar{x}_0^2 \zeta \bar{R}\lambda (\beta_p + \beta)\Big] \\
& + (1/2 + H_e)(b\bar{R})\bar{x}_A \Big[(\bar{x}_0 + \bar{e})^2 \beta (\theta_0 + \phi) + (\bar{x}_0 + \bar{e})^2 \bar{x}_0 \kappa \beta \\
& - (\bar{x}_0 + \bar{e})\bar{R}\lambda\beta\Big] + (1/2 + H_e)\Big[(\bar{x}_0 + \bar{e})^2 \bar{x}_0 \beta\zeta(\theta_0 + \phi) \\
& + (\bar{x}_0 + \bar{e})^2 \bar{x}_0^2 \kappa\beta\zeta - (\bar{x}_0 + \bar{e})\bar{x}_0 \bar{R}\lambda\beta\zeta\Big] - (\frac{c_{d0}}{a_i})(\bar{x}_0 + \bar{e})^2 \bar{x}_0 \Big\rangle
\end{aligned}
$$

The submatrices can be obtained using these relations:

$$
\frac{\partial \bar{f}_i}{\partial \theta} = \int_{l_i} (1-\xi) \begin{bmatrix} f_{xAe,\phi} & f_{xAe,\beta} & f_{xAe,\zeta} \\ f_{yAe,\phi} & f_{yAe,\beta} & f_{yAe,\zeta} \\ f_{zAe,\phi} & f_{zAe,\beta} & f_{zAe,\zeta} \end{bmatrix} dl
$$

$$
\frac{\partial \bar{f}_j}{\partial \theta} = \int_{l_i} \xi \begin{bmatrix} f_{xAe,\phi} & f_{xAe,\beta} & f_{xAe,\zeta} \\ f_{yAe,\phi} & f_{yAe,\beta} & f_{yAe,\zeta} \\ f_{zAe,\phi} & f_{zAe,\beta} & f_{zAe,\zeta} \end{bmatrix} dl
$$

$$
\frac{\partial \bar{m}_{\boldsymbol{i}}}{\partial \theta} = \int_{l_i} (1-\xi) \begin{bmatrix} m_{xAe,\phi} & m_{xAe,\beta} & m_{xAe,\zeta} \\ m_{yAe,\phi} & m_{yAe,\beta} & m_{yAe,\zeta} \\ m_{zAe,\phi} & m_{zAe,\beta} & m_{zAe,\zeta} \end{bmatrix} dl
$$

$$
\frac{\partial \bar{m}_j}{\partial \theta} = \int_{l_i} \xi \begin{bmatrix} m_{xAe,\phi} & m_{xAe,\beta} & m_{xAe,\zeta} \\ m_{yAe,\phi} & m_{yAe,\beta} & m_{yAe,\zeta} \\ m_{zAe,\phi} & m_{zAe,\beta} & m_{zAe,\zeta} \end{bmatrix} dl
$$