Free Vibration of Composite Rotating Beams - An Integral Method Based on Green's Functions

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Abstract

An approximative method using Green's functions has been developed to analyse the free vibration characteristics of clamped-free composite rotating or non-rotating beams with structural bending-torsion coupling. In this formulation the coupling terms are considered as external applied distributed loads. Numerical examples concerning natural frequency calculation of several uniform composite beams are also presented and compared with the results of other studies.

Notation

$EI_y(K_{55})$ = flap stiffness
$EI_z(K_{66})$ = lag stiffness
e_1 = root blade offset
$GJ(K_{44})$ = torsional stiffness
K, K_{45} = bending-torsion (pitch-flap) coupling stiffness
K_{46} = bending-torsion(pitch-lag) coupling stiffness
k_A = effective polar radius of gyration of blade cross section
k_m = polar radius of gyration of cross sectional mass about elastic axis
$k_m^2 = k_{m1}^2 + k_{m2}^2$
L = beam length
m = blade mass per unit length
v, w = elastic displacements in y, z directions, respectively
x, ξ = coordinates along the blade
ϕ = elastic twist of cross section of blade about shear center
Ω = angular velocity of rotation, rad/s
ω = frequency of vibration, rad/s
$(.)'$ = differentiation w.r.t. x
$(.)''$ = double differentiation w.r.t. x
$(.)'''$ = triple differentiation w.r.t. x
$(.)^T$ = transpose of a matrix

Introduction

The study of static and dynamic characteristics of composite beams has attracted considerable interest in recent years, especially because of their increasing applications in aerospace. Many papers deal with this subject, references [1]-[10] being some selected examples. Several different methods may be used to solve the free vibration problem: the influence coefficient method [5], a mixed formulation developed by Hodges [8], a mixed finite element method [7], a Galerkin method [9], the dynamic stiffness matrix method [10]. The approach presented here provides a simple matrix method for obtaining the natural frequencies and mode shapes of a clamped-free uniform composite beam with structural bending-torsion coupling. The static stiffness properties of the beam are considered knowns using a formulation as in [3]..[6]. This approach is valid for small linear deflections. The

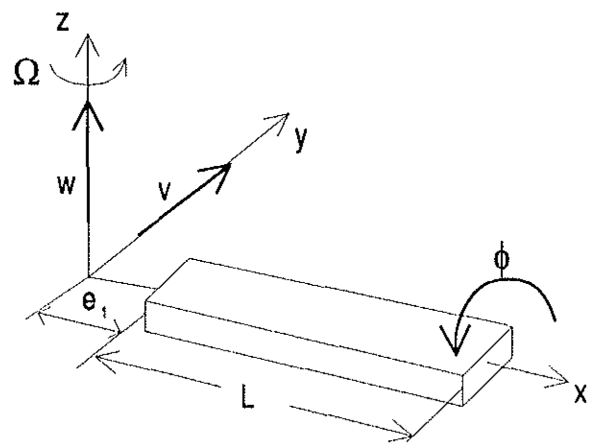

Figure 1: Configuration of the rotating beam

effects such as shear, rotatory inertia or warping are also neglected. The numerical examples presented concern natural frequencies and mode shapes calculation of non-rotating and rotating beams. All these calculations are performed in the MATLAB environment.

Problem Formulation

As many results of the present study are compared with those of [9] the notation used in this reference will also be employed here. The analysed configuration is shown in Fig. 1. We start with the linearized governing equations of motion for flap-lag-torsion vibration analysis of a uniform composite rotating beam in the case $e_1 = 0$, [9] written in the form:

$$K_{55}\frac{\partial^4 w}{\partial x^4} = m\omega^2 w - K_{45}\frac{\partial^3 \phi}{\partial x^3} + \frac{1}{2}m\Omega^2(L^2 - x^2)w'' - m\Omega^2 x w' \quad (1)$$

$$\begin{aligned} K_{66}\frac{\partial^4 v}{\partial x^4} &= m\omega^2 v + m\Omega^2 v - K_{46}\frac{\partial^3 \phi}{\partial x^3} \\ &+ \frac{1}{2}m\Omega^2(L^2 - x^2)v'' \\ &- m\Omega^2 x v' - m\Omega^2 K_{66}L_{22}v'' \end{aligned} \quad (2)$$

$$\begin{aligned} & K_{44}\frac{\partial^2 \phi}{\partial x^2} + K_{45}\frac{\partial^3 w}{\partial x^3} + K_{46}\frac{\partial^3 v}{\partial x^3} \\ + & mk_m^2\omega^2\phi - m\Omega^2(k_{m2}^2 - k_{m1}^2)\phi \\ + & m\Omega^2 K_{46}L_{22}v' + \frac{1}{2}m\Omega^2 k_A^2(L^2 - x^2)\phi'' \\ - & m\Omega^2 k_A^2 x\phi' = 0 \end{aligned} \quad (3)$$

If we consider only the flap-torsion equations of free vibration of a non-rotating uniform beam, these relations take the form used in [10]:

$$EI_y\frac{\partial^4 w}{\partial x^4} = m\omega^2 w - K\frac{\partial^3 \phi}{\partial x^3} \quad (4)$$

$$GJ\frac{\partial^2 \phi}{\partial x^2} + K\frac{\partial^3 w}{\partial x^3} + mk_m^2\omega^2\phi = 0 \quad (5)$$

According to [11], [12] a differential equation of the type:

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 u}{\partial x^2}] = p(x) \quad (6)$$

can be written in the following integral form:

$$\begin{aligned} u(x) &= \int_0^x G_u(x,\xi)p(\xi)d\xi \\ &+ \int_x^L G_u(x,\xi)p(\xi)d\xi \end{aligned} \quad (7)$$

where $G_u(x,\xi)$ is the Green's function, which has a discontinuity at $x = \xi$. Also according to [11]..[13], the equation:

$$\frac{\partial}{\partial x}[GJ(x)\frac{\partial \phi}{\partial x}] + m_x(x) = 0 \quad (8)$$

can be written in the following integral form:

$$\begin{aligned} \phi(x) &= \int_0^x G_t(x,\xi)m_x(\xi)d\xi \\ &+ \int_x^L G_t(x,\xi)m_x(\xi)d\xi \end{aligned} \quad (9)$$

G_u and G_t are the appropriate Green's functions which include boundary conditions. The idea is to consider equations (1), (2), (3) or (4), (5) of the form (6), (8) including the coupling terms as external applied loading. For performing the integrals required by the forms (7), (9), n collocation points are choosen in order to evaluate the G_w, G_v, G_t matrices, while a weighting matrix $[P]$ is also used. This corresponds to the Simpson integration scheme. $[D_1]$ and $[D_2]$ are matrices for differentiation, and are used to obtain the vectors $\{w'\}, \{w''\}, \{v'\}, \{v''\}, \{\phi'\}, \{\phi''\}$. They correspond to a differentiation layout with central differences. For handling the coupling bending torsion terms which require the vectors $\{w'''\}, \{v'''\}, \{\phi'''\}$ an integration by parts is first performed. Triple differentiation is thus avoided, while the $[G'_w]$, $[G'_v]$, $[G'_t]$ (differentiation w.r.t. x of $[G_w]$, $[G_v]$, $[G_t]$) are also required. Equations (1), (2), (3) take the forms:

$$\begin{aligned}\{w\} &= \omega^2 m[G_w][P]\{w\} \\ &+ K_{45}[G'_w][P][D_2]\{\phi\} \\ &+ [G_w][P][M_{in}][D_2]\{w\} \\ &+ [G_w][P][M_x][D_1]\{w\} \end{aligned} \quad (10)$$

$$\begin{aligned}\{v\} &= \omega^2 m[G_v][P]\{v\} + \Omega^2 m[G_v][P]\{v\} \\ &+ K_{46}[G'_v][P][D_2]\{\phi\} + [G_v][P][[M_{in}] \\ &+ [M_1]][D_2]\{v\} \\ &+ [G_v][P][M_x][D_1]\{v\} \end{aligned} \quad (11)$$

$$\begin{aligned}\{\phi\} &= \omega^2 m k_m^2 [G_t][P]\{\phi\} \\ &- K_{45}[G'_t][P][D_2]\{w\} \\ &- K_{46}[G'_t][P][D_2]\{v\} \\ &- m\Omega^2(k_{m2}^2 - k_{m1}^2)[G_t][P]\{\phi\} \\ &+ m\Omega^2 K_{46} L_{22}[G_t][P][D_1]\{v\} \\ &+ k_A^2[G_t][P][[M_{in}][D_2] \\ &+ [M_x][D_1]]\{\phi\} \end{aligned} \quad (12)$$

where the (n,n) matrices $[M_{in}]$, $[M_x]$, $[M_1]$ have $\frac{1}{2}m\Omega^2(L^2 - x^2)$, $-m\Omega^2 x$, $-m\Omega^2 K_{66} L_{22}$ respectively on the main diagonal. Equations (10), (11), (12) may be written in either the form:

$$\{z\} = \omega^2[A_1]\{z\} + [B_1]\{z\} \quad (13)$$

or the form:

$$[[A] - \omega^2[I]]\{z\} = \{0\} \quad (14)$$

where:

$$[A] = [A_1]^{-1}[[I] - [B_1]] \quad (15)$$

is a (3n,3n) matrix and the vector $\{z\} = [[w][v][\phi]]^T$ has dimension $3n$. This is an eigenvalues and eigenvectors problem which gives the natural frequencies of the free vibrations and the corresponding mode shapes.

Numerical Examples

The first example is to find the natural frequencies of a flap bending-torsion coupled uniform non-rotating composite cantilever beam. The properties of this beam are given in [10] and are taken from [7]: $EI_y = 0.5317Nm^2$, $GJ = 0.3586Nm^2$, $K = 0.099Nm^2$, $m = 0.07383Kg/m$, $mk_m^2 = 5.56210^{-6}kgm$. Table 1 compares the results of the present integral formulation (I.F.) with the finite element calculated natural frequencies of reference [7] and with the natural frequencies obtained using the exact dynamic stiffness matrix method given in [10].

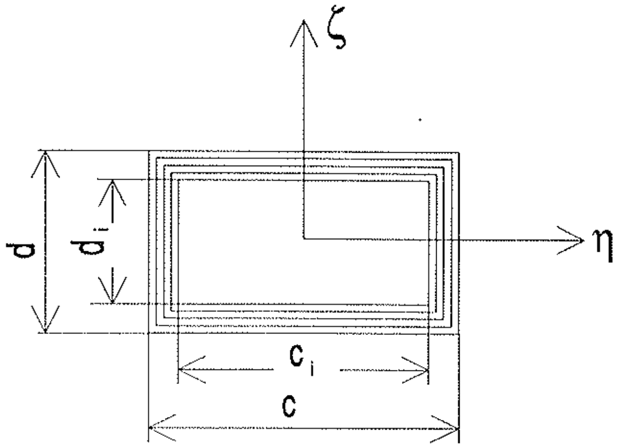

Figure 2: Composite rectangular box beam

Table 1 Natural frequencies of a coupled bending-torsion cantilever composite non-rotating beam

Mode	Natural frequencies (Hz)		
	$F.E.M$ [7]	[10]	$I.F., n = 20$
1	4.66	4.66	4.68
2	29.60	29.17	29.28
3	84.89	81.63	82.37
4	113.43	113.28	113.29

The next examples deal with the free vibration characteristics analysis of a rotating composite box beam(Fig. 2). All comparisons are made with the results published by Chandra and Chopra [9], who used Galerkin's method to solve of the same coupled equations. They also obtained experimental results. Four symmetric graphite-epoxy, kevlar-epoxy and glass-epoxy box-beam configurations with bending-torsion coupling, presented in [9] were considered. Beam length is $L = 33.25in$, while the inner dimensions of the cross-section are $0.893in$ and $0.477in$ (Fig. 2). Other geometrical details of the beams are shown in Table 2.

Material characteristics are presented in Table 3. In this table the mass density is given in $10^{-3} lbs^2/in^4$.

Table 2 Details of the thin-walled composite box-beams

Config., Layers	Ply thick. < in >	Material	Flanges	Webs
S1, 6	0.005	graph. epoxy	$[30]_6$	$[\pm 30]_3$
S2, 6	0.005	graph. epoxy	$[45]_6$	$[\pm 45]_3$
S3, 2	0.01	kevlar epoxy	$[45]_2$	± 45
S4, 2	0.016	glass epoxy	$[45]_2$	± 45

Table 3 Material characteristics

Material	Density	E_L $10^6 psi$	E_T $10^6 psi$	G_{LT} $10^6 psi$	μ_{LT}
graph. epoxy	0.135	20.59	1.42	0.89	0.42
kevlar epoxy	0.104	11	0.8	0.34	0.34
glass epoxy	0.167	7	2.1	0.8	0.26

All coefficients K_{ij} were calculated using the model developed in [6]. Figure 3 shows the influence of rotation on the first three natural frequencies for the $[30]_6$ symmetric graphite-epoxy beam with bending-torsion coupling (configuration S1). The theoretical and experimental results of [9] are also presented for comparison. Figures 4, 5 and 6 show results for configurations S2, S3, S4. Correlation with the Galerkin method is very good, while correlation between the theoretical and experimental results of [9] is within 10%.

Conclusions

Theoretical values for the free vibration characteristics of coupled bending-torsion composite clamped-free uniform beams were obtained starting with an integral formulation of the corresponding equations of motion. The results show good agreement with those of the classic Galerkin method. In this formulation, the boundary conditions are included by using appropriate structural influence (Green's) functions, while the bending-torsion coupling

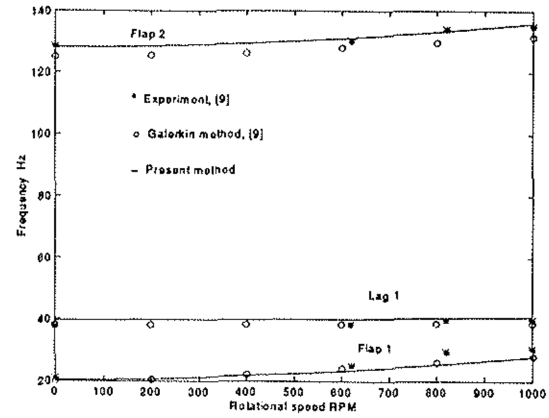

Figure 3: Natural frequencies for configuration S1

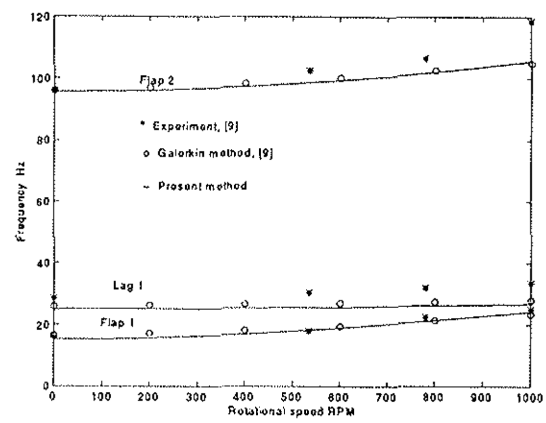

Figure 4: Natural frequencies for configuration S2

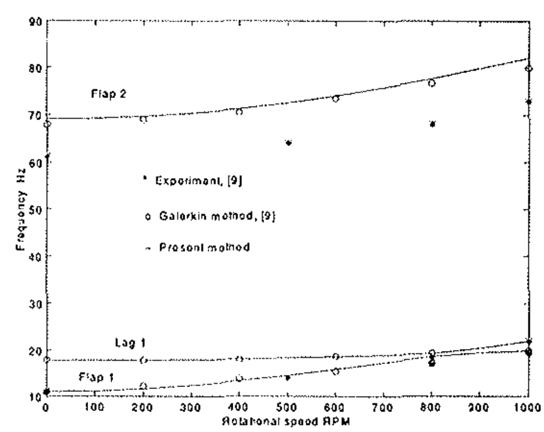

Figure 5: Natural frequencies for configuration S3

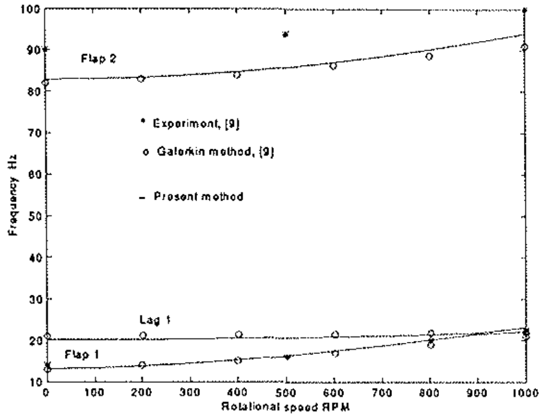

Figure 6: Natural frequencies for configuration S4

terms are considered as external applied distributed loads. Integration along the length of the beam is performed by using a weighting matrix corresponding to the Simpson method. The differentiation matrices $[D_1]$, $[D_2]$ correspond to a differentiation layout with central differences. Another more general integral method, which also use integrating and differentiating matrices is the Integrating Matrix Method [14], [15]. The present method has been successfully used to determine the flap-lag-torsion dynamic characteristics of a non-uniform isotropic pretwisted rotating blade [16]. It may be concluded that this formulation based on Green's functions is another useful way of performing such an analysis.

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Appendix

The following Green's functions are used:

$$G_w(x,\xi) = \begin{cases} \frac{\xi^2(3x-\xi)}{6K_{55}} & \xi < x \\ \frac{x^2(3\xi-x)}{6K_{55}} & \xi \geq x \end{cases}$$

$$G_v(x,\xi) = \begin{cases} \frac{\xi^2(3x-\xi)}{6K_{66}} & \xi < x \\ \frac{x^2(3\xi-x)}{6K_{66}} & \xi \geq x \end{cases}$$

$$G_t(x,\xi) = \begin{cases} \frac{\xi}{K_{44}} & \xi < x \\ \frac{x}{K_{44}} & \xi \geq x \end{cases}$$

These functions are differentiated as follows:

$$G_w'(x,\xi) = \begin{cases} \frac{3\xi(2x-\xi)}{6K_{55}} & \xi < x \\ \frac{3x^2}{6K_{55}} & \xi \geq x \end{cases}$$

$$G_v'(x,\xi) = \begin{cases} \frac{3\xi(2x-\xi)}{6K_{66}} & \xi < x \\ \frac{3x^2}{6K_{66}} & \xi \geq x \end{cases}$$

$$G_t'(x,\xi) = \begin{cases} \frac{1}{K_{44}} & \xi < x \\ 0 & \xi \geq x \end{cases}$$

The following relations are also used:

$$\int_0^L G_w \phi''' d\xi = -\int_0^L G_w' \phi'' d\xi$$

$$\int_0^L G_v \phi''' d\xi = -\int_0^L G_v' \phi'' d\xi$$

$$\int_0^L G_t w''' d\xi = -\int_0^L G_t' w'' d\xi$$

$$\int_0^L G_t v''' d\xi = -\int_0^L G_t' v'' d\xi$$