EFFICIENT SOLUTION OF GOLDSTEIN'S EQUATIONS FOR PROPELLERS WITH APPLICATION TO ROTOR INDUCED-POWER EFFICIENCY

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Abstract

Betz and Prandtl (1919) presented the optimum velocity distribution for a rotor in axial flow having an infinite number of blades. Goldstein (1929) derived an expression for the circulation that would give the ideal inflow of Betz-Prandtl. Goldstein offered an elegant, numerical solution to this equation in order to find the optimum circulation to give Betz induced flow. He presented solutions for two blades at a number of inflow ratios and for four blades at one particular inflow ratio. The objective of this work is to develop a more computationally accurate and robust method of finding the optimum circulation for the ideal propeller. We look for a solution that would be taken to any desired accuracy and applied for any number of blades and any tip-speed ratio. With such a solution, one can have benchmarks against which to compare other methodologies. In addition, an accurate solution will allow computation of induced power efficiency for the Goldstein optimum such that other blade designs can be measured against it.

Nomenclature

a_k	constant part of h'_k
$[A]$	Goldstein derivative matrix
b_k	part of h'_k due to h_k
$\{B\}$	forcing factor of particular solution
$\{c\}$	coefficients of particular solution
$\{d\}$	coefficients of homogenious solution
$\{D\}$	focing function of homogenius solution
e	Prandtl tip-correction function
$[E]$	Galerkin stiffness matrix
$f_k(\mu)$	velocity potential expansions
$g_k(\mu)$	homogenius part of h_k
$[G]$	matrix of boundary conditions
$h_k(\mu)$	correction functions
i	summation index
I	modified Bessel function
IPE	induced power efficiency
j	summation index
k	harmonic number, $k = Q/2, 3Q/2, 5Q/2, \dots.$
$K(y)$	modified Bessel function o f y

$K'(y)$ derivative of K with respect to y (dK/dy)

l harmonic number, $l = Q, 2Q, 3Q, \ldots$

$\{M\}$ forcing function for boundary

n number of k terms

N number of the terms in Galerkin function

$P_j(x)$ Legendre polynomials

Q number of blades

r radial coordinate, m

R blade radius, m

$trig()$ either sin( ) or cos( )

$v(r)$ induced velocity at disk normal to vortex sheet, m/sec

V climb rate, m/sec

x mapping coordinate, $2(\mu/\mu_0) - 1$, $-1 < x < 1$

z axial coordinate, m

$\gamma(\mu)$ nondimensional normalized circulation, $\Gamma\Omega/V^2$

γ_b nondimensional circulation per blade, $\Gamma_b\Omega/V^2$

$\bar{\gamma}$ corrected circulation

$\hat{\gamma}$ Galerkin optimum circulation

$\Gamma(r)$ normalized nominal circulation, uncorrected, m^2/sec

$\Gamma_b(r)$ total circulation per blade, m^2/sec

κ correction factor

ζ nondimensional screw coordinate, $\theta - \Omega z/V$

θ angle of screw surface, rad

$\lambda(\mu)$ nondimensional induced velocity downstream, $2\nu(\mu)$

μ radial coordinate, $\Omega r/V = \cot(\phi)$

μ_0 value of μ at blade tip, $\Omega R/V$

$\nu(\mu)$ nondimensional induced velocity at disk, $v(r)/V$

ϕ inflow angle, $\arctan(1/\mu)$

$\Phi(\mu, \zeta)$ velocity potential, normalized on V^2/Ω

Φ_j admissible functions of either μ or x

Ω rotor speed, rad/sec

Introduction

Betz and Prandtl, Ref. [1], found the optimum velocity distribution (i.e., for minimum power) for a rotor in axial flow. Although they were unable to find an exact solution for the circulation distribution that would result in such a velocity distribution, they were able to find this optimum circulation for a rotor with an infinite number of blades and offered an approximate tip correction that would account for the effect of blade number. Although the Prandtl correction factor is based on a two-dimensional inflow model, it is quite accurate and is used extensively in rotorcraft analysis to account for blade number.

It fell to Goldstein, Ref. [2], to find the exact solution for the optimal circulation on a propeller with a finite number of blades. He treated both two-bladed and four-bladed rotors at various inflow angles. The results agree nicely with computations based on Prandtl's equation, as shown in Fig. (1)—taken from Ref. [2]—where the condition chosen is $\mu_0 = 5.0$, which is a fairly high climb rate.

For four blades, the two solutions are quite close, although there is a small discrepancy that occurs near the root of the blade. For the two-bladed rotor, this discrepancy is more pronounced. The behavior of the Prandtl approximation at small μ is nearly identical for all Q, but the Goldstein solution increasingly differs from the Prandtl solution (at small μ) as Q decreases.

The objective of this work is to develop a more computationally accurate and robust method of finding the optimum circulation for the ideal propeller. We look for a solution that would be taken to any desired accuracy and applied for any number of blades and any tip-speed ratio.

For formulating the problem, we will follow the general outline of Ref. [2] but proceed along what we believe is a more direct and compact approach. To begin, note that the pressure and velocity around a propeller in axial flow are governed by the following velocity potential:

(1) $$\Phi(\mu,\zeta) = \left[\frac{\pi}{Q} - \zeta\right]\gamma(\mu) + \sum f_k(\mu)\, trig(k\zeta), \quad 0 < \zeta < \frac{2\pi}{Q}$$

where $\gamma(\mu)$ is some nominal circulation, μ is the nondimensional radial co-ordinate, and Q is the number of blades. (Note that μ is also the cotangent of the inflow angle ϕ.)

The first term in Eq. (1) is the nominal velocity potential that is chosen to give a nondimensional velocity distribution $\lambda(\mu) = \gamma(\mu)/\cos(\phi)$. The second part of Eq. (1), involving $f_k(\mu)$, is a correction term. (The summation is taken over appropriate k's as will be defined later.)

The total velocity potential (nominal plus correction) must satisfy Laplace's equation in helical coordinates.

(2) $$\left(\frac{\mu\partial}{\partial\mu}\right)^2 \Phi + (1+\mu^2)\frac{\partial^2\Phi}{\partial\zeta^2} = 0$$

This implies that the correction functions $f_k(\mu)$ are related to a set of basic functions $h_k(\mu)$—that are governed by a differential equation that follows from Eqs. (1) and (2)—namely:

(3) $$\left(\frac{\mu d}{d\mu}\right)^2 [h_k(\mu)] - k^2(1+\mu^2)h_k(\mu) = -\left(\frac{\mu d}{d\mu}\right)^2 [\gamma(\mu)]$$

(we will demonstrate the relationship between f_k and h_k later.)

The resultant circulation per blade $\gamma_b(\mu)$ and the resultant velocity distribution in the wake $\lambda(\mu)$ can be given in terms of the total velocity potential:

(4) $$\gamma_b(\mu) = \Phi(\mu,0) - \Phi(\mu,2\pi/Q)$$
$$\lambda(\mu) = -\sqrt{\frac{1+\mu^2}{\mu^2}}\frac{\partial\Phi}{\partial\zeta}\bigg|_{\zeta=0}$$

Therefore, once $h_k(\mu)$ and $f_k(\mu)$ are determined, the circulation and velocity can be found; and that is the focus of what is to follow.

Now, consider the case in which we are given the velocity distribution $\lambda(\mu)$, and want to find the applied circulation $\gamma_b(\mu)$ that would produce it. In order to preserve the desired velocity $\lambda(\mu)$ from Eq. (4), we take $trig(k\zeta) = \cos(k\zeta)$ in the summation of Eq. (1) with $k = Q/2, 3Q/2, 5Q/2, etc.$ This ensures that the derivative of Φ will be zero at the boundary. It is then convenient to expand $[\pi/Q - \zeta]$ in cosine terms summed over these same k's.

(5) $$[\pi/Q - \zeta] = \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\dots} \frac{2Q}{\pi k^2}\cos(k\zeta)$$

The nominal circulation to obtain the desired velocity is:

(6) $$\gamma(\mu) = \frac{\mu}{\sqrt{1+\mu^2}}\lambda(\mu)$$

and this becomes the forcing function for the correction terms in Eq.(3). When Eq. (1) and Eq. (5) are placed into Eq. (2), it is clear that one must define the relation $f_k(\mu) \equiv 2Qh_k(\mu)/(\pi k^2)$ in order to obtain the standard form of Eq. (3).

It follows that the total circulation distribution per blade is given from Eq. (4) as:

(7) $$\gamma_b(\mu) = \frac{2\pi}{Q}\left[\gamma(\mu) + \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\dots} \frac{2Q^2}{\pi^2 k^2} h_k(\mu)\right]$$

For the above summation over k in Eq. (7), $\pi k/Q = \pi/2, 3\pi/2, 5\pi/2,$ etc. Notice that the circulation is increased due to positive $h_k(\mu)$.

Numerical Computation

Now we need to formulate a numerical solution to the correction function. The general solution to Eq. (3) would be the sum of the particular solution and the homogenous solution.

To find the particular solution with boundary conditions $h_k(0) = h_k(\mu_0) = 0$, we take Eq. (3) and expand the unknown $h_k(\mu)$ in a Galerkin series of admissible functions $c_j\Phi_j(\mu)$.

(8) $$h_k(\mu) = \sum_{i=1,2,3,\dots}^{N} c_j^k \Phi_j(\mu)$$

We then use a change of variable to map the domain onto $-1 < x < 1$

(9) $$x = 2\frac{\mu}{\mu_0} - 1 \qquad -1 < x < 1$$

This change of variable allows admissible and comparison functions to be chosen on a more convenient interval, $-1 < x < 1$.

For test functions, we chose the combination of Legendre polynomials that have been applied to the p-version finite element method, Ref. [5],

$$(10) \qquad \Phi_j = \frac{P_{j+1}(x) - P_{j-1}(x)}{\sqrt{2(2j-1)}}$$

Substituting Eq.(8) into Eq.(3), multiplying by the comparison functions $\Phi_i(\mu)/\mu$ and integrating from zero to μ_0, one obtains:

$$(11) \quad \left[\int_0^{\mu_0} \frac{1}{\mu}\Phi_i\left(\mu\frac{d}{d\mu}\right)\left(\mu\frac{d}{d\mu}\right)\Phi_j d\mu - k^2\int_0^{\mu_0}\frac{1}{\mu}\Phi_i\Phi_j(1+\mu^2)d\mu\right]\{c_j^k\} = -\int_0^{\mu_0}\frac{1}{\mu}\Phi_i\left(\mu\frac{d}{d\mu}\right)^2\gamma(\mu)d\mu$$

The first integral in Eq. (11) can be written in the form:

$$(12) \quad \int_0^{\mu_0}\frac{1}{\mu}\Phi_i\left(\mu\frac{d}{d\mu}\right)\left(\mu\frac{d}{d\mu}\right)\Phi_j d\mu = \int_0^{\mu_0}\Phi_i\frac{d}{d\mu}\left(\mu\frac{d\Phi_j}{d\mu}\right)d\mu = \int_0^{\mu_0}\Phi_i\left[\frac{d\Phi_j}{d\mu}+\mu\frac{d^2\Phi_j}{d\mu^2}\right]d\mu = -\int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\left(\frac{d\Phi_j}{d\mu}\right)d\mu + \underbrace{\Phi_i\left(\mu\frac{d\Phi_j}{d\mu}\right)\Bigg|_0^{\mu_0}}_{0} = -\int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\left(\frac{d\Phi_j}{d\mu}\right)d\mu$$

Similarly the integral on the right hand side of Eq. (11), can be written in the form:

$$(13) \quad \int_0^{\mu_0}\frac{1}{\mu}\Phi_i\left(\mu\frac{d}{d\mu}\right)^2\gamma(\mu)d\mu = -\int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\frac{d\gamma}{d\mu}d\mu$$

As a result, Eq. (11) can be rewritten in the form:

$$(14) \quad \left[\int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\frac{d\Phi_j}{d\mu}d\mu + k^2\int_0^{\mu_0}\frac{1+\mu^2}{\mu}\Phi_i\Phi_j d\mu\right]\{c_j^k\} = -\int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\frac{d\gamma}{d\mu}d\mu$$

Taking:

$$(15) \quad [A_{ij}] = \int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\frac{d\Phi_j}{d\mu}d\mu, \quad [E_{ij}] = \int_0^{\mu_0}\frac{1+\mu^2}{\mu}\Phi_i\Phi_j d\mu, \quad \{B_i\} = -\int_0^{\mu_0}\mu\frac{d\Phi_i}{d\mu}\frac{d\gamma}{d\mu}d\mu$$

one can write Eq. (14) in the form:

$$(16) \quad [A + k^2E]\{c_j^k\} = \{B\}$$

and:

$$(17) \quad \{c_j^k\} = [A + k^2E]^{-1}\{B\}$$

The procedure for finding the homogeneous solution is similar to that used for finding the particular solution. For the homogenous case, the boundary conditions are $h_k(0) = 0\ and\ \ h_k(\mu_0) \neq 0$, and the differential equation that follows from Eq. (3) is:

$$(18) \quad \left(\frac{\mu d}{d\mu}\right)^2[h_k(\mu)] - k^2(1+\mu^2)h_k(\mu) = 0$$

For the homogenous boundary conditions we use the change of variable:

$$(19) \quad h_k(\mu) = \mu + g_k(\mu), \qquad g_k(0) = 0,\ \ g_k(\mu_0) = 0 \rightarrow\ h_k(\mu_0) = \mu_0$$

So, Eq. (18), can be written in the form:

$$(20) \quad \left(\mu\frac{d}{d\mu}\right)^2 g_k(\mu) - k^2(1+\mu^2)g_k(\mu) = [k^2(1+\mu^2)-1]\mu$$

We take Eq. (20) and expand $g_k(\mu)$ in a Galerkin series of admissible functions $d_i\Phi_i(\mu)$.

$$g_k(\mu) = \sum_{j=1,2,3,\ldots}^{N} d_i^k \Phi_i(\mu) \tag{21}$$

Once again, we map the domain onto $-1 < x < 1$, and choose a combination of Legendre polynomials for our test functions:

$$\Phi_i = \frac{P_{i+1}(x) - P_{i-1}(xf)}{\sqrt{2(2i-1)}} \tag{22}$$

Substituting Eq.(21) into Eq.(20), multiplying by the comparison functions $\Phi_j(\mu)/\mu$, and integrating from zero to μ_0, yields:

$$[A + k^2 E]\{d_i^k\} = -\int_0^{\mu_0} \Phi_j \{[k^2(1+\mu^2) - 1]\} d\mu \tag{23}$$

Taking:

$$D_j = -\int_0^{\mu_0} \Phi_j \{[k^2(1+\mu^2) - 1]\} d\mu \tag{24}$$

Eq. (23) becomes:

$$[A + k^2 E]\{d_i^k\} = \{D\} \tag{25}$$

and:

$$\{d_i^k\} = [A + k^2 E]^{-1}\{D\} \tag{26}$$

The total solution for h is then the sum of the particular solution and the homogenous solution.

$$h_k = \sum_{j=1}^{N} c_j^k \Phi_j + \left[\sum_{i=1}^{N} d_i^k \Phi_i + \mu\right] \frac{h_k(\mu_0)}{\mu_0} \tag{27}$$

With the above, we can find the solution to the potential problem. It is not difficult to see that the conditions of the problem are such that Φ is an odd function of ζ (or $\frac{1}{2}\pi - \zeta$). Furthermore, Φ is a single-valued function of position, continuous for $r \geq R$ ($\mu \geq \mu_0$). Therefore, it can be expanded, for $r \geq R$, in a series of even multiples of ζ. Taking this expansion, differentiating term by term, and then substituting in Eq. (2), we find that the coefficients of $\sin(l\zeta)$ must be a linear functions of $I_{l/2}(l\mu)$ and $K_l(l\mu)$, where $I_{l/2}$ and $K_{l/2}$ are the modified Bessel functions.

But $I_l(l\mu)$ cannot occur, since grad Φ must vanish when *r*, or μ, is infinite. Hence we may assume:

$$\Phi_0(\zeta, \mu) = \sum_{l=Q,2Q,3Q,\ldots} c_l^0 \frac{K_l(l\mu)}{K_l(l\mu_0)} \sin(l\zeta) \tag{28}$$

For $0 \leq r \leq R$ ($0 \leq \mu \leq \mu_0$), the velocity potential can be obtained from Eq. (1). Since the velocity potential is a continuous function at $r = R$ ($\mu = \mu_0$), it should satisfy the continuity conditions:

$$\Phi_0(\mu_0) = \Phi(\mu_0), \quad \Phi'_0(\mu_0) = \Phi'(\mu_0) \tag{29}$$

where the ($'$) sign implies the derivative with respect to μ.

According to first continuity condition, Φ from Eq. (28) equals Φ from Eq. (1) at $\mu = \mu_0$

$$\sum_{l=Q,2Q,3Q,\ldots} c_l^0 \frac{K_l(l\mu_0)}{K_l(l\mu_0)} \sin(l\zeta) = \left[\frac{\pi}{Q} - \zeta\right]\gamma(\mu_0) + \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} f_k \cos(k\zeta) \tag{30}$$

Equation (5) and $f_k(\mu) \equiv 2Q h_k(\mu)/(\pi k^2)$ can be substituted into Eq. (30). Expanding $\cos(k\zeta)$ in sine terms:

$$\cos(k\zeta) = \frac{4}{\pi} \sum_{l=Q,2Q,3Q,\ldots} \frac{l}{l^2 - k^2} \sin(l\zeta) \tag{31}$$

one can rewrite Eq. (30) in the form:

$$\sum_{l=Q,2Q,3Q,\ldots} c_l^0 \frac{K_l(l\mu_0)}{K_l(l\mu_0)} \sin(l\zeta) = \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{2Q}{\pi k^2} [\gamma(\mu_0) + h_k(\mu_0)] \left(\frac{4}{\pi}\right)\left(\frac{l}{l^2 - k^2}\right) \sin(l\zeta) \tag{32}$$

and, as a result:

$$c_l^0 = \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{2Q}{\pi k^2} [\gamma(\mu_0) + h_k(\mu_0)] \left(\frac{4}{\pi}\right)\left(\frac{l}{l^2 - k^2}\right) \tag{33}$$

The second continuity condition implies that:

$$l c_l^0 \frac{K'_l(l\mu_0)}{K_l(l\mu_0)} = \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{2Q}{\pi k^2} [\gamma'(\mu_0) + h'_k(\mu_0)] \left(\frac{4}{\pi}\right)\left(\frac{l}{l^2 - k^2}\right) \tag{34}$$

Substituting Eq. (33) in Eq. (34) and simplifying, we obtain:

$$(35)\quad \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{1}{k^2(l^2-k^2)}\{lK'_l(l\mu_0)[\gamma(\mu_0) + h_k(\mu_0)] - K_l(l\mu_0)[\gamma'(\mu_0) + h'_k(\mu_0)]\} = 0$$

On the other hand, from Eq. (27), one obtains an expression for h'_k:

$$(36)\quad h'_k(\mu) = \sum_{j=1}^{N} c_j^k \frac{2}{\mu_0}\frac{d\Phi_j}{dx} + \left[1 + \sum_{i=1}^{N} d_i^k \frac{2}{\mu_0}\frac{d\Phi_i}{dx}\right]\left(\frac{h_k(\mu_0)}{\mu_0}\right)$$

and at $\mu = \mu_0$:

$$(37)\quad h'_k(\mu_0) = \sum_{j=1}^{N} c_j^k \frac{2}{\mu_0}\frac{d\Phi_j}{dx}\bigg|_{\mu_0} + \left[1 + \sum_{i=1}^{N} d_i^k \frac{2}{\mu_0}\frac{d\Phi_i}{dx}\bigg|_{\mu_0}\right]\left(\frac{h_k(\mu_0)}{\mu_0}\right)$$

Taking:

$$(38)\quad a_k = \sum_{j=1}^{N} c_j^k \frac{2}{\mu_0}\frac{d\Phi_j}{dx}\bigg|_{\mu_0}$$

$$(39)\quad b_k = \left[1 + \sum_{i=1}^{N} d_i^k \frac{2}{\mu_0}\frac{d\Phi_i}{dx}\bigg|_{\mu_0}\right]\left(\frac{1}{\mu_0}\right)$$

one can rewrite Eq. (37) in the form:

$$(40)\quad h'_k(\mu_0) = a_k + b_k h_k(\mu_0)$$

Substituting Eq. (40) in Eq. (35), and dividing the whole equation by K (to make the matrices better conditioned) one obtains:

$$(41)\quad \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{1}{k^2(l^2-k^2)}\{lK'_l(l\mu_0)[\gamma(\mu_0)\; h(\mu_0)] - K_l(l\mu_0)[\gamma'(\mu_0) + a_k + b_k h_k(\mu_0)]\} = 0$$

Taking

$$(42)\quad [G_{lk}] = \frac{lK'_l(l\mu_0) - b_k}{k^2 K(l^2-k^2)}$$

$$(43)\quad \{M_l\} = \left[\sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{1}{k^2(l^2-k^2)}\right][\gamma'(\mu_0) - \frac{lK'_l(l\mu_0)}{K(l\mu_0)}\gamma(\mu_0)] + \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{a_k}{k^2(l^2-k^2)}$$

one can write Eq. (41) in the form:

$$(44)\quad [G_{lk}]\{h_k(\mu_0)\} = \{M_l\}$$

and:

$$(45)\quad \{h_k(\mu_0)\} = [G_{lk}]^{-1}\{M_l\}$$

Now, that we have $\{c\}$ (from Eq. (17)), $\{d\}$ (from Eq. (26)), and $h_k(\mu_0)$ (from Eq. (45)), we can calculate the value of $h_k(\mu)$ from Eq. (27). One may also calculate the Galerkin optimum circulation, $\hat{\gamma}$, which is actually the expression inside the square brackets of Eq. (7).

$$(46)\quad \hat{\gamma}(\mu) = \gamma(\mu) + \sum_{k=\frac{Q}{2},\frac{3Q}{2},\frac{5Q}{2},\ldots} \frac{2Q^2}{\pi^2 k^2} h_k(\mu)$$

$\gamma(\mu)$ is the non-dimensional circulation for a case with an infinite number of blades, and can be calculated from Eq. (6); but since the optimum Betz velocity distribution is $\lambda(\mu) = \mu/\sqrt{1+\mu^2}$, then $\gamma(\mu)$ would be:

$$(47)\quad \gamma(\mu) = \frac{\mu^2}{1+\mu^2}$$

With the Galerkin optimum circulation, one obtains the corrected circulation, $\bar{\gamma}$, from Eq. (48):

$$(48)\quad \bar{\gamma}(\mu) = \hat{\gamma}(\mu) - \hat{\gamma}(\mu_0)[1-e]$$

Where e is the Prandtl correction factor Ref. [1] used to make the tip correction, and h_k are the solutions from the Galerkin method. In order to maximize convergence, the Prandtl factor is used—but designed only to eliminate the residual—not correct the entire function. We add a acceleration factor, κ, to account for the fact that the residual dies out more quickly as more terms are added. The factor is chosen so as to minimize the number of terms required for convergence in the matrix formulation. The modified Prandtl function e is therefore of the form:

$$(49)\quad e = \frac{2}{\pi}\arccos[\exp(-f)]$$

with:

$$f = \frac{\kappa Q(1 - \frac{\mu}{\mu_0})\sqrt{1+\mu_0^2}}{2} \quad (50)$$

The correction factor for optimized convergence has been expressed in the following form:

$$\begin{aligned}\kappa = 1 - 5.6429\, ln\left(\frac{n}{n+25}\right)\left[\frac{15625n}{(n+25)^4}\right] \\ -3.2449\left[\frac{25n}{(n+25)^2}\right] \\ +13.3817\left[\frac{625n}{(n+25)^3}\right] \\ -3.7991\left[\frac{15625n}{(n+25)^4}\right]\end{aligned} \quad (51)$$

Once the solution for the corrected circulation is found, one can define the induced-power efficiency (*IPE*) as the ratio of the Goldstein optimum power (for a given number of blades) to the Glauert ideal power for an actuator disk:

$$IPE = \frac{2}{\mu_0^2}\int_0^{\mu_0} \bar{\gamma}\, \mu d\mu \quad (52)$$

Results

The present methodology is first used to compute cases already found in Goldstein as a verification of the convergence and accuracy of the method. Next, results, not found in earlier work are computed. Figures 2-4 compare the corrected circulation (circulation with the tip correction), and the Galerkin optimum circulation (circulation without the tip correction) for 2, 4 and 6 bladed rotors. The results are for $n = 10$ and $N = 15$ which we found was sufficient for convergence in all cases. Figure 2 shows our results in red for μ_0 =5 and $Q = 2$, for which we have a known solution from Goldstein. One can see that convergence is slow near the tip in that the zero boundary condition has not converged. Goldstein noted the same effect with his solution; and he mentions in his paper that he adds a correction to bring the tip to zero

From the singularity at the edges the convergence may be very slow. The corresponding point in the graph may be displaced this amount if the curve can thereby be smoothed. Goldstein

We similarly smooth the curves at the tip with our accelerated Prandtl tip-correction function, and that is shown in the blue curve which is virtually identical to the Goldstein solution. Figure 3 is for $\mu_0 = 5$ and $Q = 4$, another case for which Goldstein gives a solution. Similarly good convergence is seen. Figure 4 is for a six-bladed rotor, a result which has not heretofore been published.

We next compute the induced-power efficiency (IPE) for these cases. These are plotted versus the Glauert tip-speed ratio μ_0 (the ratio of tip speed to free-stream velocity) and also versus its reciprocal $\lambda = 1/\mu_0$ in Fig. 5 for rotors with 2, 4 and 6 blades.

The IPE decreases with increasing λ because the local blade lift is perpendicular to the vortex sheet and thus tilts—implying energy is lost in wake swirl. Figure 5 also shows how a decrease in blade number reduces efficiency because there are tip losses associated with upwash at the tip.

Figures 6, 7 and 8 compare the IPE under various induced-flow assumptions. The curve labeled "Betz Approximation" is the IPE for the infinite-blade case, which includes only the effect of lift tilt. The curve noted as "Prandtl Approx-imation" is the result of the Prandtl blade-number correction applied to the Glauert actuator-disk model. It includes only tip effects. The curve, "Betz-Prandtl Approximation" is the methodology suggested by Betz and Prandtl (and implemented by Goldstein in Fig. 1) in which the Prandtl correction is applied to the Betz solution. The final curve, labeled "Goldstein Exact Solution" is the result of our analysis which gives the complete solution including root losses as well as tip losses. One can see the relative effects of the various physical processes on the induced power efficiency.

Figures 6-8 reveal the magnitude of the various contributions: of lift tilt (the Glauert curves), of tip losses (the Prandtl curves), of combined tilt and tip losses 9the Betz-Prandtl curves), and of root corrections (the exact curves). It is clear that the Prandtl approximation gives an almost exact result for the IPE when $\lambda > 2.0$, and a very good approximation even for $\lambda < 2.0$. This is because the Goldstein correction, clearly seen as a large effect in Fig. 1, has both positive and negative corrections to the Betz-Prandtl circulation. Thus, the net effect on efficiency is small. With the new, numerical method for finding the true Goldstein circulation, it has been possible for the first time to verify the effect of the Betz-Prandtl approximation on induced power efficiency (IPE).

Summary and Conclusions

With the use of a Galerkin procedure, we have obtained an efficient and accurate method for solving the Goldstein optimum circulation distribution for propellers with arbitrary blade number and tip-speed ratio. The numerical procedure is verified against results given by Goldstein for two specific cases, and it is then used to compute results not given by Goldstein. The results are used in order to find induced power efficiency of propellers. These results show that the effect of Goldstein's root corrections on IPE are quite small such that the Prandtl-Betz approximation is generally adequate. However, the optimum circulation is significantly affected by Goldstein's root effect for large wake spacing (i.e., for small blade number and small inflow ratio μ.

Acknowledgement

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<u>**Figures**</u>

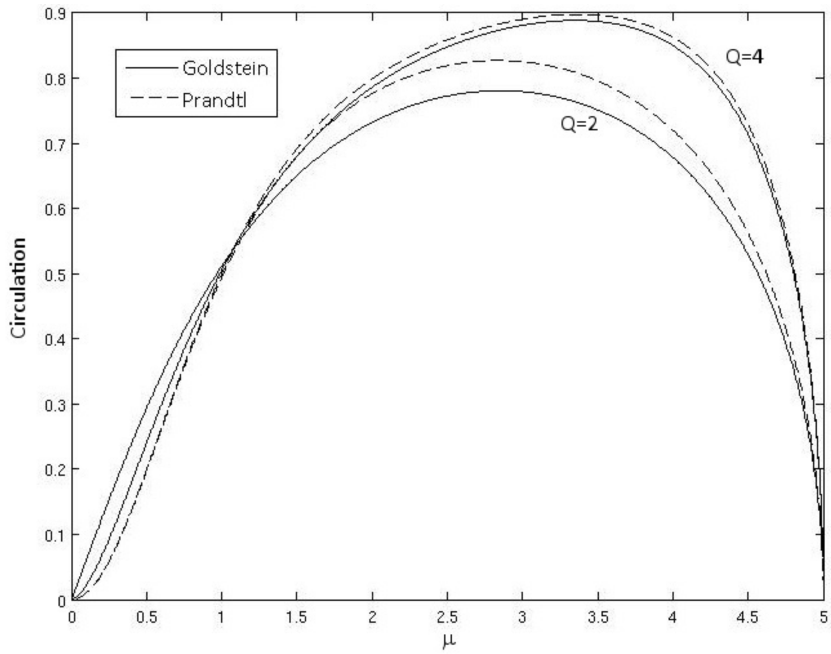

Figure 1. Optimum Circulation for 2-Blades and 4-Blades Rotors

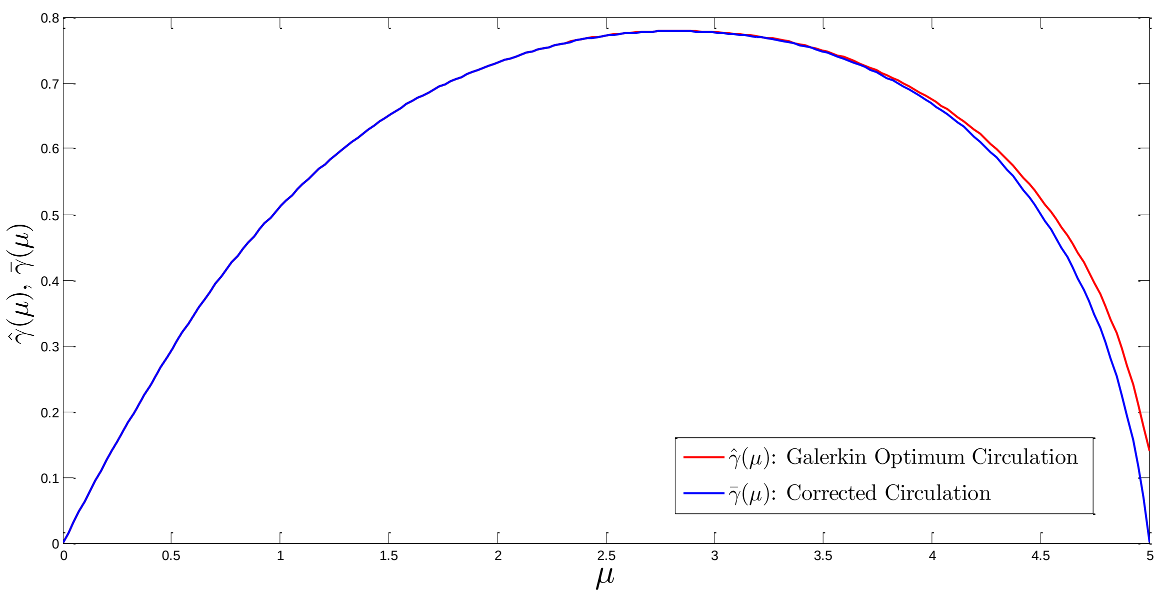

Figure 2. Corrected circulation vs. Galerkin optimum circulation for 2-bladed rotor

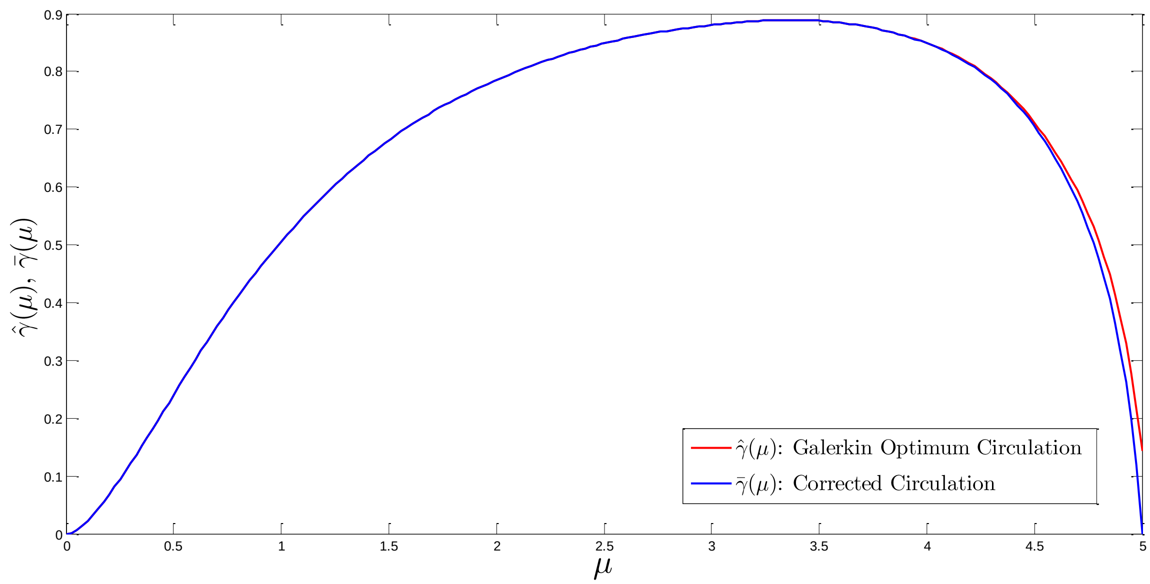

Figure 3. Corrected circulation vs. Galerkin optimum circulation for 4-bladed rotor

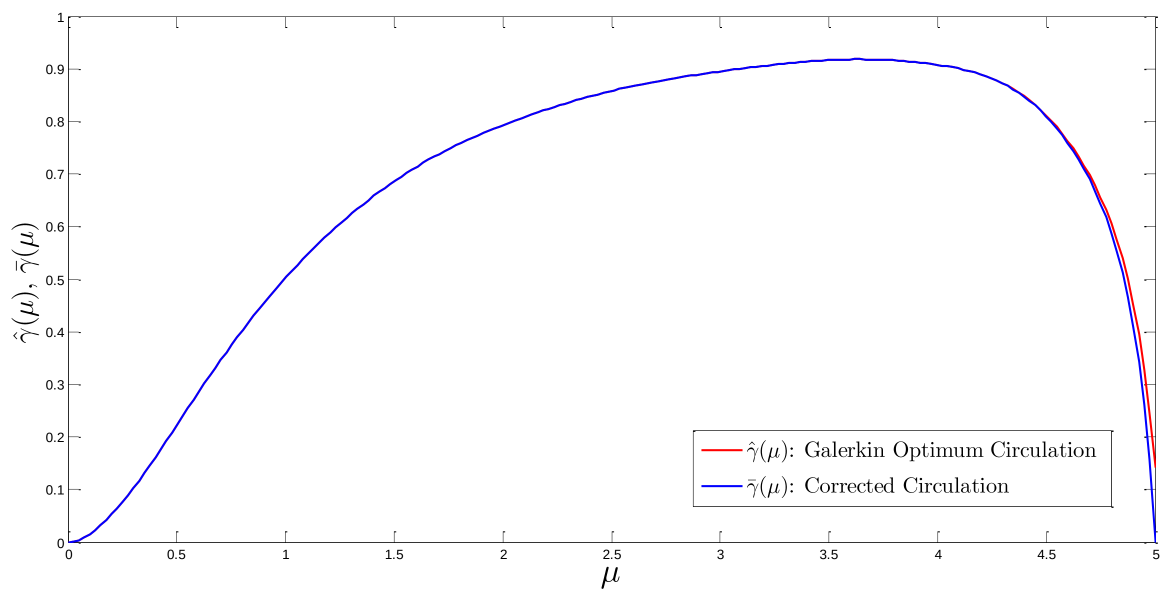

Figure 4. Corrected circulation vs. Galerkin optimum circulation for 6-bladed rotor

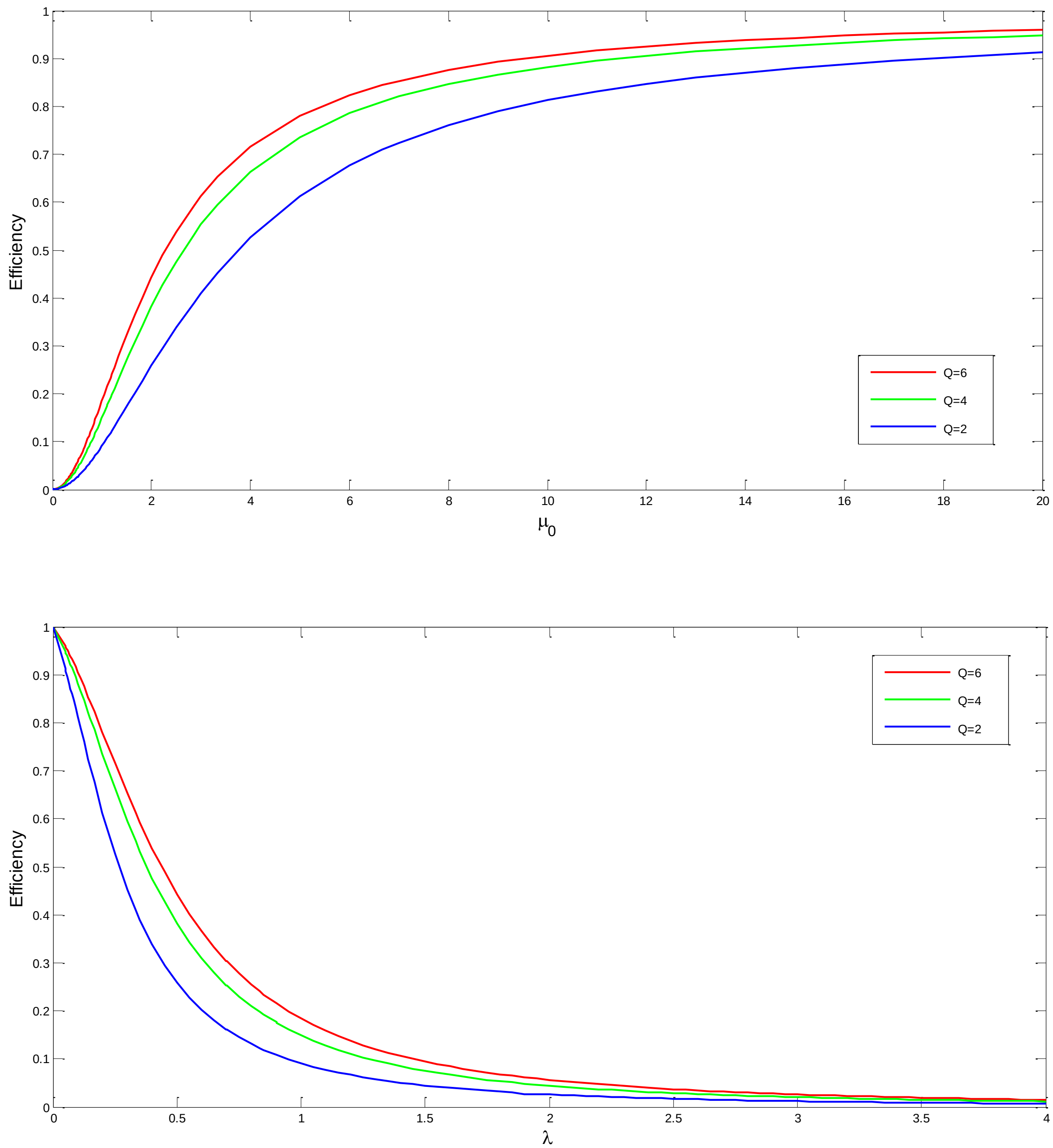

Figure 5. Induced power efficiency for 2, 4 and 6 blades rotor by Goldstein's Solution

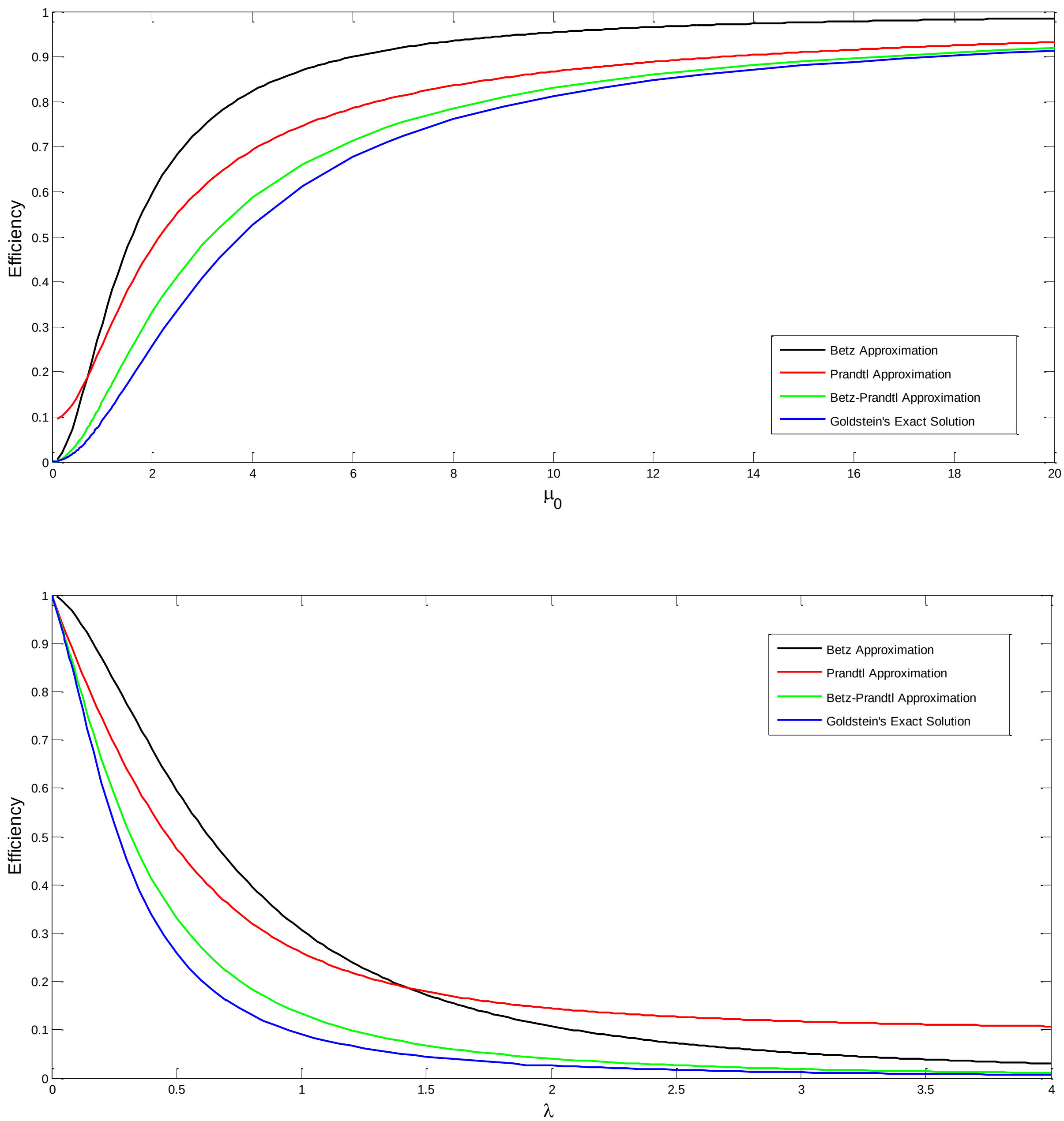

Figure 6. Induced power efficiency, Goldstein's Exact Solution vs. Other approximations, **2** bladed rotor

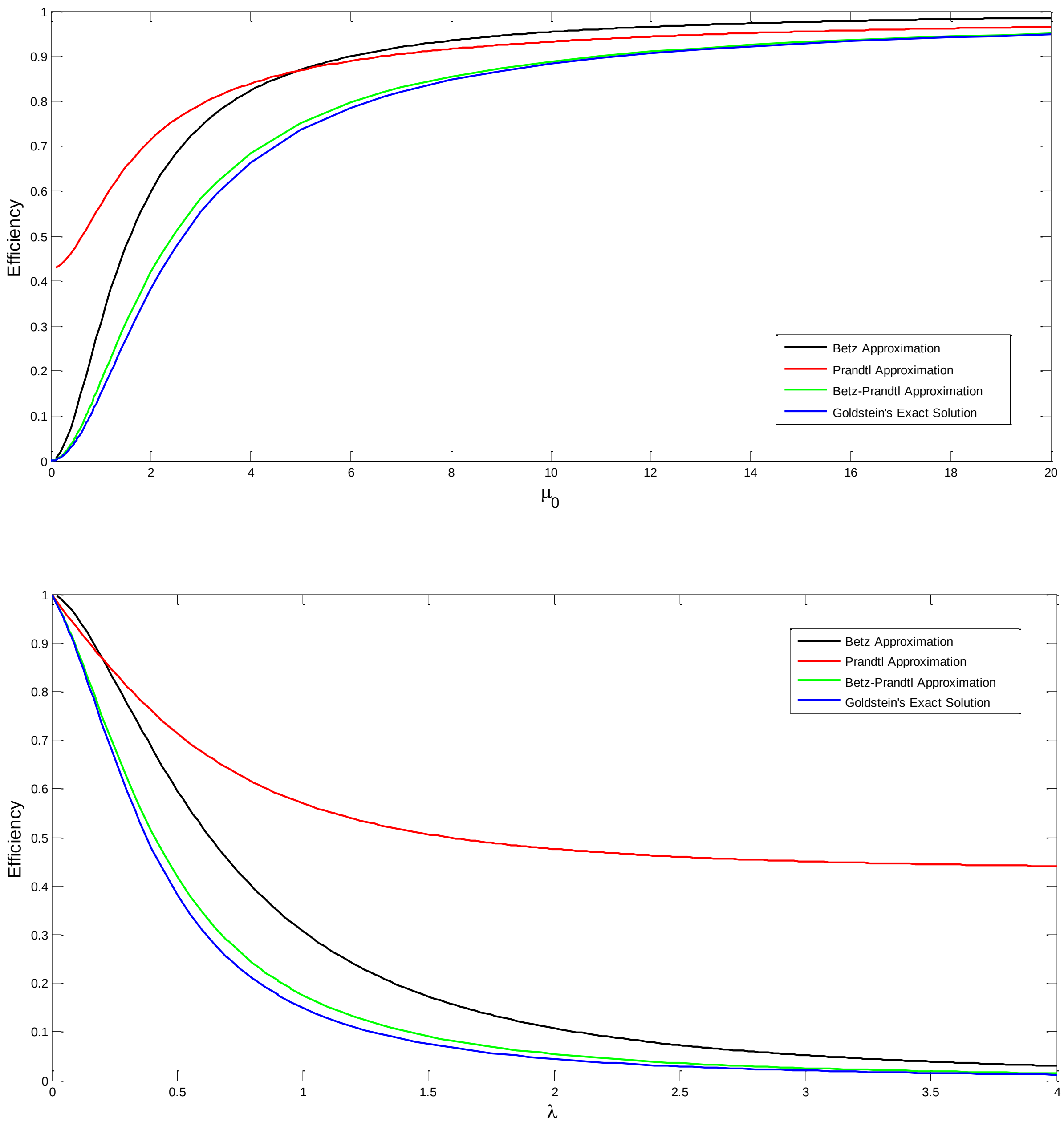

Figure 7. Induced power efficiency, Goldstein's Exact Solution vs. Other approximations, **4** bladed rotor

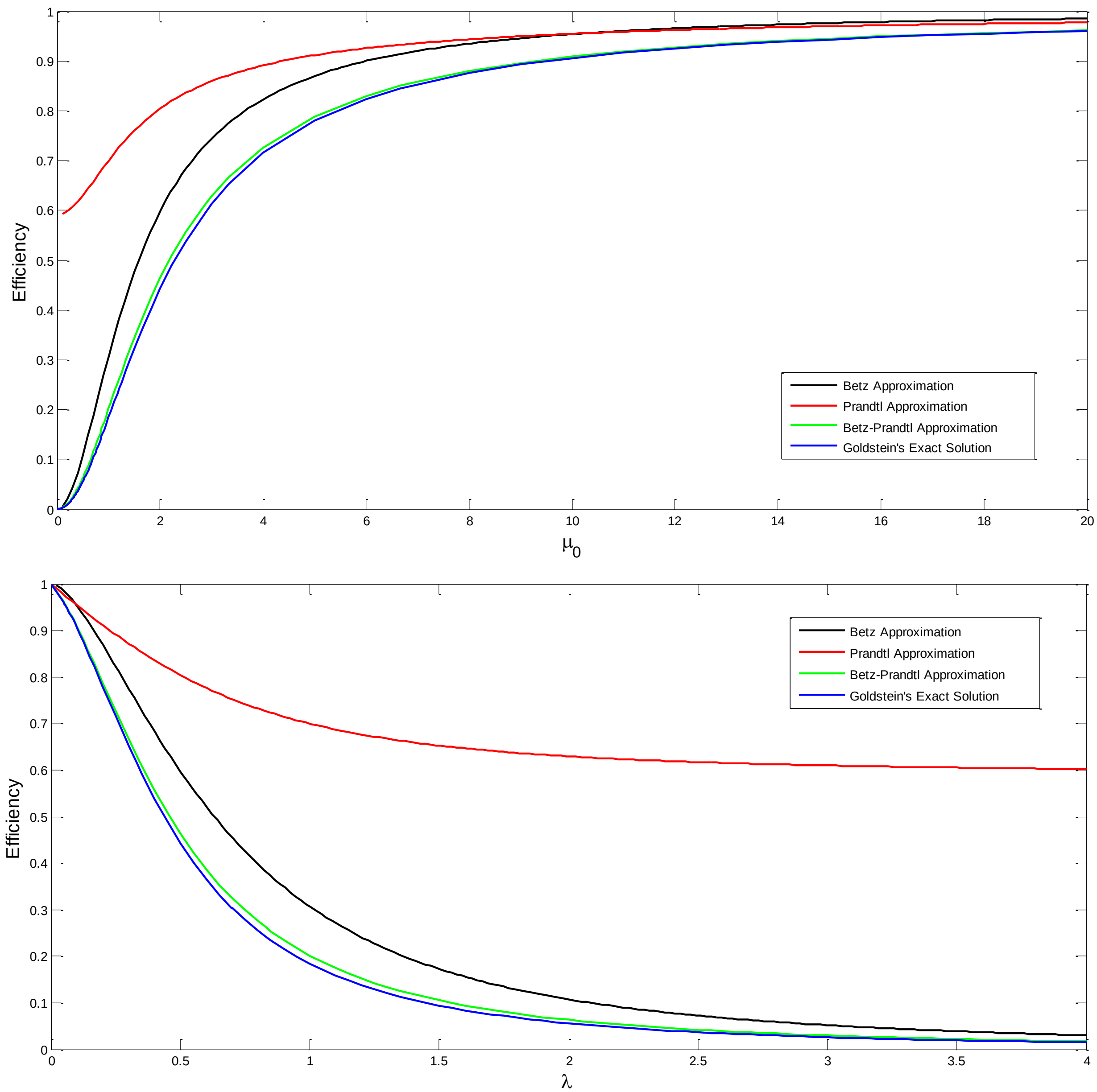

Figure 8. Induced power efficiency, Goldstein's Exact Solution vs. Other approximations, **6** bladed rotor