Paper No. 2

OPTIMIZING THE CYCLIC CONTROL RESPONSE OF HELICOPTER ROTORS*

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1. Summary

The response of a helicopter rotor to cyclic pitch control inputs is examined from the point of view of error analysis and system optimization. The transient deviations of individual blade flapping response from the desired steady state are used as the bases of calculations of a variety of performance indices. These are integral square error (ISE), integral time square error (ITSE) integral absolute error (IAE) and integral time absolute error (ITAE). It is shown in the case of conventional articulated rotors that Lock number and its influence on the blade aerodynamic flap damping ratio is the decisive parameter in minimizing and hence optimizing the various performance indices. The ideal Lock number is shown to vary between 8 and 12, depending on the performance index selected. The influence of real or virtual offset of the blade flapping hinges in the case of hingeless rotors is then examined for the case of integral square error, and the ideal trade-offs between blade flapping frequency ratio and flap damping ratio are determined. It is found that the optimum flap damping ratio increases only slightly with increasing flapping frequency ratio thus making the optimum Lock number vary directly with flapping frequency ratio.

2. Introduction

Making a system in the best possible manner, to make it optimal, to select the optimum parameters for the system or to "optimize" it is the essence of design activity. The central questions in any optimization study are what criterion, performance index, cost, penalty or payoff function is to be selected and what aspects of the design are subject to selection or rational choice? These questions are narrowed considerably by employing quantitative measures of performance (or malperformance) which are to be maximized (or minimized) through parameter selection or optimization. In this study, since time and dynamics are basic, the malperformance of the system is calculated in terms of several error indices as a function of key design parameters. Application of Laplace transformation techniques permits the calculation and minimization to be accomplished by the methods of ordinary differential calculus in a closed form.

The choice of the cyclic pitch control response of a hovering helicopter rotor is selected for optimization because it lends itself well to a formal, closed form mathematical solution while amply illustrating the principles and techniques. It is also a substantial rotor design problem in its own right. Broader and

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numerically more difficult rotor-cyclic and collective pitch-airframe dynamic optimizations would proceed in an identical manner except that the performance indices would be evaluated directly as part of the analog, digital or hybrid computation routines.

In this specific study the key dynamical parameters are quickly seen to be the equivalent viscous damping ratio and the fundamental flapping mode frequency ratio, which depend on the real or virtual offset of the blade flapping hinge, virtual spring restraint in the case of hingeless blades, and the Lock number of the rotor blade. The several malperformance indices examined and minimized are those which quantify the transient dynamical deviation of the blade response to cyclic pitch with respect to the desired steady state flapping response.

3. Analysis

The governing differential equation for determining the perturbation cyclic flapping of a helicopter rotor blade responding to a small cyclic pitch control input is derived in Appendix I and follows below as Equations (1). Equations (2) give the general solution for the case of a step change in cyclic pitch θ_c when time t or azimuth ψ equals zero. There is no flapping motion initially. By employing the concept of a virtual hinge and flapping restraint both hingeless and articulated rotor designs are included implicitly in the results.[1]

$$\beta'' + \frac{\gamma}{8}(1+\frac{4}{3}\varepsilon)\beta' + (\frac{\omega_\beta}{\Omega})^2\beta \cong \frac{\gamma}{8}\,\theta_c(1+\frac{8}{3}\varepsilon+2\varepsilon^2)\sin\psi \qquad (1)$$

$$(\frac{\omega_\beta}{\Omega})^2 \equiv [(1+e_\beta\frac{\sigma_\beta}{I_\beta}) + \frac{k_\beta}{I_\beta\Omega^2}] \qquad (1a)$$

$$\varepsilon \equiv (\frac{e_\beta}{R-e_\beta}) \qquad (1b)$$

$$\sigma_\beta \equiv \int_{e_\beta}^{R}(r-e_\beta)\,dm \qquad (1c)$$

$$I_\beta \equiv \int_{e_\beta}^{R}(r-e_\beta)^2\,dm \qquad (1d)$$

$$\gamma \equiv \frac{\rho(\frac{dC_L}{d\alpha})_\infty c(R-e_\beta)^4}{I_\beta} \qquad (1e)$$

$$\beta = \beta_1\cos\psi+\beta_2\sin\psi+(e^{-\frac{\gamma}{16}(1+\frac{4}{3}\varepsilon)\psi})\{\beta_1\cos\nu_D\psi + \frac{1}{\nu_D}[\beta_2+\frac{\gamma}{16}(1+\frac{4}{3}\varepsilon)\beta_1]\sin\nu_D\psi\} \qquad (2)$$

$$\nu_D \equiv \left[\left(\frac{\omega_\beta}{\Omega}\right)^2 - \left(\frac{\gamma}{16}\right)^2\left(1 + \frac{4}{3}\,\varepsilon\right)^2\right]^{\frac{1}{2}} \tag{2a}$$

$$\beta_1 \equiv -\theta_c\left\{\frac{\left(\frac{\gamma}{8}\right)^2\left(1+\frac{4}{3}\,\varepsilon\right)\left(1+\frac{8}{3}\,\varepsilon+2\varepsilon^2\right)}{\left[\left(\frac{\omega_\beta}{\Omega}\right)^2-1\right]^2+\left(\frac{\gamma}{8}\right)^2\left(1+\frac{4}{3}\,\varepsilon\right)^2}\right\} \tag{2b}$$

$$\beta_2 \equiv \theta_c\left\{\frac{\left(\frac{\gamma}{8}\right)\left[\left(\frac{\omega_\beta}{\Omega}\right)^2-1\right]\left(1+\frac{8}{3}\,\varepsilon+2\varepsilon^2\right)}{\left[\left(\frac{\omega_\beta}{\Omega}\right)^2-1\right]^2+\left(\frac{\gamma}{8}\right)^2\left(1+\frac{4}{3}\,\varepsilon\right)^2}\right\} \tag{2c}$$

The steady state part of the solution[2] is seen to be the desired output for the cyclic pitch input. The transient part of the general solution[2] is seen to be an error in the desired control output and leads to a cyclic pitch response control error $E(\psi)$, which is defined as

$$E(\psi) \equiv \beta(\psi)_{\substack{\text{steady}\\\text{state}}} - \beta(\psi) \tag{3}$$

Consequently the error is the negative of the transient flapping response to cyclic pitch control. This is

$$E(\psi) \equiv -e^{-\frac{\gamma}{16}\left(1+\frac{4}{3}\,\varepsilon\right)\psi}\left\{\beta_1\cos\nu_D\psi + \frac{1}{\nu_D}\left[\beta_2 + \frac{\gamma}{16}\left(1+\frac{4}{3}\,\varepsilon\right)\beta_1\right]\sin\nu_D\psi\right\} \tag{4}$$

There now arises numerous possibilities for defining performance (or more appropriately malperformance) indices whose magnitudes as a function of the system parameters can serve as a quantitative gain as much insight into the various possibilities, we first specialize the analysis to the case of a fully articulated rotor with no hinge offset or elastic flapping restraint. In this case the error $E(\psi)$ is especially simple. That is for $\varepsilon=0$, $(\omega_\beta/\Omega)=1$, $\beta_2=0$, and $\beta_1= -\theta_c$. Hence for $\varepsilon=0$

$$\frac{E(\psi)}{\theta_c} = -e^{-\frac{\gamma}{16}\psi}\left(\cos\nu_D\psi + \frac{\gamma}{16\nu_D}\sin\nu_D\psi\right) \tag{5}$$

Since there is no inherent preference between negative and positive errors and early and late errors, the integral of the square of the error is a useful malperformance index when evaluated over a long interval of time or over many rotor revolutions. Thus we define

$$P_o \equiv \lim_{\psi\to\infty} \int_o^{\psi} E^2 d\psi = \theta_c^2 \lim_{\psi\to\infty} \int_o^{\psi} e^{-\frac{\gamma}{8}\psi} (\cos\nu_D\psi + \frac{\gamma}{16\nu_D}\sin\nu_D\psi)^2 d\psi \quad (6)$$

It is shown in Appendix II that P_o can be evaluated by Laplace transform techniques and is given by

$$P_o = \lim_{s\to o} L[E^2(\psi)] = \theta_c^2(\frac{\gamma}{16} + \frac{4}{\gamma}) \quad (7)$$

The optimum value of Lock number γ follows as the value which minimizes P_o. That is setting

$$\frac{1}{\theta_c^2}\frac{dP_o}{d\gamma} = \frac{1}{16} - \frac{4}{\gamma^2} = 0 \quad (8)$$

results in $\gamma_{optimum} = 8$. It is further seen that P_o is not very sensitive to Lock numbers in the range $6 \leq \gamma \leq 12$. This is in the nature of the particular malperformance index selected. An alternative index might weight early errors less heavily such as

$$P_1 = \lim_{\psi\to\infty} \int_o^{\psi} \psi E^2 d\psi \quad (9)$$

It is shown in Appendix II that P_1 can also be evaluated by Laplace transform techniques as

$$P_1 = \lim_{s\to o} \{-\frac{d}{ds} L[E^2(\psi)]\} \quad (10)$$

It follows that P_1 is given by

$$P_1 = \theta_c^2[(\frac{\gamma}{16})^2 + \frac{1}{8}(\frac{16}{\gamma})^2] \quad (11)$$

and that in this case $\gamma_{optimum} \cong 9.6$. The index P_1 is seen to be slightly more sensitive than P_o with an acceptable range between eight and twelve.

Performance indices analogous to P_o and P_1, respectively, are the integral of the absolute error and the integral of the time multiplied by the absolute error. These follow below as P_3 and P_4, respectively.

$$P_3 = \lim_{\psi\to\infty} \int_o^{\psi} |E| d\psi \quad (12)$$

$$P_4 = \lim_{\psi\to\infty} \int_o^{\psi} |E| d\psi \quad (13)$$

These have been evaluated using the Laplace transform methods for rectified sine waves[3] and are found to be more sensitive to γ

than in the cases of P_0 and P_1. The optimum values of γ for P_3 and P_4 are found to be approximately ten and twelve, respectively.

The more general case of real or virtual offset of the flapping hinge is found to be much more complicated numerically but to follow similar trends. In this analysis we limit our consideration to the malperformance index P_0, the integral of the square of the error. Proceeding as in the case of a fully articulated rotor, but with a flapping frequency ratio ν greater than or equal to unity, P_0 is found to be given by

$$P_o = \frac{\theta_c{}^2}{4} \left\{ \frac{1}{\zeta} + \zeta + \frac{[1+(2\zeta^2-1)\nu^2]^2}{4\zeta^3\nu^4} + \frac{[1+(2\zeta^2-1)\nu^2]}{\zeta\gamma^2} \right\} \qquad (14)$$

$$\zeta \equiv \frac{\gamma}{16\gamma}\left(1 + \frac{4}{3}\varepsilon\right), \quad \nu \equiv \left(\frac{\omega_\beta}{\Omega}\right) \qquad (14a)$$

Proceeding as in the earlier case of $\nu=1$, the value of ζ which minimizes P_0 is given by the real solutions to the biquadratic equation

$$\zeta^4 + \frac{1}{4\nu^4}(1-5\nu^2+3\nu^4)\zeta^2 - \frac{3}{16\nu^4}(1-\nu^2)^2 = 0 \qquad (15)$$

These follow from the quadratic formula as

$$\zeta^2_{optimum} = \frac{-(1-5\nu^2+3\nu^4)+(1-10\nu^2+37\nu^4-54\nu^6+27\nu^8)^{\frac{1}{2}}}{8\nu^4} \qquad (16)$$

It is seen that the general expression for optimum flap damping ratio yields the value one-half as the flapping frequency ratio approaches unity. Sample calculations for flapping frequency ratios greater than unity and as great as 1.20 show that there is no significant increase in the optimum flap damping ratio. It follows from the definition of Equation (14a) that the optimum Lock number for a hingeless rotor blade or other system yielding a frequency ratio greater than or equal to unity is given to a close approximation by

$$\gamma_{optimum} \cong \frac{8\nu}{1 + \frac{4}{3}\varepsilon} \qquad (17)$$

when the malperformance index P_0 is minimized or optimized. In the case of a virtual flapping hinge offset $\varepsilon \cong .15$ and a flapping frequency ratio $\nu \cong 1.15$, the optimum Lock number decreases slightly to approximately 7.7. Noting that the definition of Lock number employed in Equation (1e) implies a tendency to decrease with virtual offset of the flapping hinge, it follows that optimum Lock number tends to be primarily a blade property and is not sensitive to the hinge location or constraint. It is easy to show that Lock number tends to be proportional to blade span and that the medium sized rotors of twenty to thirty foot span tend to fall naturally into the optimum range of Lock numbers. On the other hand the relatively small and large rotors tend naturally to fall outside the optimum Lock number range. However the very large rotors generally require significant tip weight to reduce steady coning to acceptable levels and thereby will have smaller Lock numbers than indicated by rotor span alone.

4. Conclusions

It is concluded that the deviation or error between the desired steady state flapping response and the actual instantaneous flapping motion provides a practical quantitative measure on which to base an optimization analysis. The malperformance indices P_o, P_1, P_3 and P_4 are seen to be at a minimum when the blade Lock number is in the range 8 to 12 in the case of a conventional, fully articulated rotor. In the case of practical hingeless rotors, where the fundamental flapping frequency ratio is of the order of 1.1 to 1.2 cycles per revolution, the optimum flap damping ratio differs negligiblyfrom that for the articulated rotor. It then follows from the relationship between flap damping ratio, flapping frequency ratio and Lock number that the optimum Lock number varies directly with the flapping frequency ratio of the hingeless rotor blade.

The foregoing optimization of the cyclic pitch response of a helicopter rotor serves to illustrate the concepts and techniques which are available for enhancing and ultimately optimizing the transient and steady state dynamics of an entire helicopter. The dynamics of the rotor, control system and airframe are cascaded or coupled. A suitable malperformance index (such as the integral time absolute error) is selected. A machine or analytical evaluation of the index is carried out during the computation of the total helicopter system transient response to one or more standardized disturbances. The design parameters are then adjusted over their permissible range to minimize the index.

5. References

1. Maurice I. Young, A simplified theory of hingeless rotors. Proceedings of The American Helicopter Society Annual Forum. Washington, D.C. (May 1962).

2. Jacob P. Den Hartog, Mechanical Vibrations, 4th Edn. McGraw-Hill Book Co., New York. pp. 47-55 (1956).

3. W. T. Thomson, Laplace Transformation, Prentice-Hall, Inc., New York. pp. 18-22 (1950).

6. Notation

c	=	blade chord, ft
e_β	=	real or virtual offset of flapping hinge, ft
k_β	=	real or virtual spring rate, ft-lb/rad
m	=	blade mass, slugs
r	=	radial position of blade element, ft
s	=	subsidiary variable in Laplace transformation
E	=	flapping error, rad
I_β	=	second mass moment about flapping hinge, slug-ft^2
P	=	malperformance index, rad
R	=	radius of rotor, ft

$(\frac{dC_L}{d\alpha})_\infty$	=	lift curve slope for infinite aspect ratio, per rad
α	=	blade element angle of attack, rad
β	=	blade perturbation flapping angle, rad
γ	=	blade Lock number, dimensionless
ε	=	dimensionless offset of flapping hinge
ζ	=	damping ratio, fraction of critical flap damping
θ_c	=	amplitude of blade cyclic pitch perturbation, rad
ν	=	flapping frequency ratio, cycles per revolution
ρ	=	density of air, slugs/ft^3
σ_β	=	first mass moment of blade about flapping hinge, slug-ft
ψ	=	blade azimuth angle, rad
ω_β	=	blade fundamental flapping frequency, rad/sec
Ω	=	rotor rotational frequency, rad/sec
$\cdot$	=	differentiation with respect to time, per sec
'	=	differentiation with respect to azimuth, per rad

7. Appendix I - Derivation of Governing Differential Equation

The dynamic equilibrium of the inertial moments of force and the perturbation aerodynamic moments of force about the real or virtual flapping hinge of the rotor blade (together with the real or virtual spring moment of force, if any) is given approximately by the integral from the hinge to the blade tip as

$$k_\beta \beta + \int_e^R \{[r-e_\beta)^2 \ddot{\beta} + r(r-e_\beta)\Omega^2]dm - [\tfrac{1}{2}\rho(\frac{dC_L}{d\alpha})_\infty (r^2\Omega^2(r-e_\beta)\alpha]dr\} = 0 \qquad (18)$$

$$\alpha \equiv \theta_c \sin\Omega t - \frac{(r-e_\beta)\dot{\beta}}{r\Omega} \qquad (19)$$

Carrying out the indicated integrations and introducing the fundamental flapping frequency ratio

$$(\frac{\omega_\beta}{\Omega}) \equiv (1 + \frac{k_\beta}{I_\beta\Omega^2} + e_\beta \frac{\sigma_\beta}{I_\beta})^{\frac{1}{2}} \qquad (20)$$

$$\sigma_\beta \equiv \int_{e_\beta}^{R} (r-e_\beta)dm \qquad (21)$$

$$I_\beta \equiv \int_{e_\beta}^{R} (r-e_\beta)^2 dm \qquad (22)$$

the dimensionless offset distance ε

$$\varepsilon \equiv (\frac{e_\beta}{R-e_\beta}) \qquad (23)$$

and the Lock number

$$\gamma \equiv \frac{\rho(\frac{dC_L}{d\alpha})_\infty c(R-e_\beta)^4}{I_\beta} \qquad (24)$$

and azimuth ψ as independent variable, the governing differential equation can be expressed (approximately) as

$$\beta'' + \frac{\gamma}{8}(1+\frac{4}{3}\varepsilon)\beta' + (\frac{\omega_\beta}{\Omega})^2\beta \cong \frac{\gamma}{8}\theta_c(1+\frac{8}{3}\varepsilon+2\varepsilon^2)\sin\psi \qquad (25)$$

8. Appendix II - Evaluation of Performance Indices by Laplace Transformation

Applying the final value theorem of the Laplace transformation,[3]

$$P_o \equiv \lim_{\psi\to\infty} \int_o^\psi E^2(\psi)d\psi = \lim_{s\to o} sL[\int_o^\psi E^2(\psi)d\psi] \qquad (26)$$

Applying the integration theorem of the Laplace transformation

$$P_o = \lim_{s\to o} s\{(\frac{1}{s})L[E^2(\psi)]\} = \lim_{s\to o} L[E^2(\psi)] \qquad (27)$$

Applying the theorem above and the negative differentiation theorem in the s-domain which yields multiplication by ψ in the ψ-domain

$$P_1 \equiv \lim_{\psi\to\infty} \int_o^\psi \psi E^2(\psi)d\psi = \lim_{s\to o} \{-\frac{d}{ds} L[E^2(\psi)]\} \qquad (28)$$

P_3 and P_4 can be evaluated by the previous methods by making use of the Laplace transform of the rectified error $E(\psi)$ developed in Reference 3.