

THE APPLICATION OF MODAL CONTROL THEORY
TO THE SINGLE ROTOR HELICOPTER

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ABSTRACT

Future requirements for military helicopters necessitate the application of multivariable control techniques in the design of flight control systems. Modal control provides a fast and visible design method which can meet reasonable handling quality criteria. The application of principal angles in the design method greatly enhances the decision making by aiding the choice of both the closed-loop eigenvalues and the corresponding eigenvectors. A design methodology is outlined incorporating these techniques.

INTRODUCTION

The role of the military helicopter imposes important and difficult requirements in terms of control engineering. The introduction of hingeless rotors, although improving the agility of the aircraft, introduces unstable modes in the responses of the uncontrolled helicopter [1]. The inherent cross-coupling effects between the inputs controlled by the pilot can also cause difficulties in handling, and can limit the desired flight path. This is of paramount importance when considering the tasks involved [2,3,4]. In its anti-armour role the combat helicopter has three main modes of operation, high speed flight to the contact area, nap-of-earth flight (NOE) on arrival and sustained hover for surveillance, target acquisition and engagement. The most critical of these is probably NOE flight, which involves very low level flying often only a few metres above the surface and close to obstacles. Thus a very agile aircraft is needed with a high integrity flight control system (FCS).

The requirement, then, on any combat helicopter FCS is to stabilise the aircraft and decouple the controlled inputs, thus reducing pilot workload, but still to allow fast responses to pilot demands. The design philosophy currently employed in this field has, until recently, concentrated on the use of local feedback loops designed using classical linear control theory. This provides improved overall handling characteristics without changing the basic control methods. On the other hand to achieve an efficient combat helicopter, the aircraft should be treated as a highly coupled multivariable system, thus requiring modern control techniques in the design of the FCS.

One of the major problems with modern control theory, is the lack of visibility in the design process. For example, LQG (Linear Quadratic Gaussian) or optimal control is the most widely used of all the modern techniques, but lacks visibility, in that the choice of the weighting function is difficult to relate to the physical system, unless this weighting function is very simple. This paper investigates the use of modal control and presents a design methodology which allows the specification of parameters, eigenvalues and eigenvectors, which relate directly to certain handling qualities criteria. This design methodology introduces a novel application of principal angles, i.e. angles

between subspaces, as an aid in the choice of the closed-loop eigenvalues and eigenvectors.

MODAL CONTROL

Handling quality criteria are often specified in terms of eigenvalues (frequencies, damping ratios, and time constants) and associated measures of the degree of coupling or decoupling of responses. The use of modal control provides a method of calculating a set of feedback gains, using full state feedback or output feedback, that give closed-loop eigenvalues and eigenvectors which satisfy desired handling qualities [5]. The choice of each closed-loop eigenvalue/eigenvector pair is not an arbitrary one as each eigenvector is limited to a subspace of the system subspace but this gives enough freedom to make suitable choices of those pairs.

Consider an n 'th order m -input, k -output linear time invariant system with state space description

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (1)$$

$$\mathbf{y} = \mathbf{Cx} \quad (2)$$

where \mathbf{x} is the $n \times 1$ state vector, \mathbf{u} is the $m \times 1$ input vector, \mathbf{y} is the $k \times 1$ output vector, \mathbf{A} is the $n \times n$ system matrix, \mathbf{B} is the $n \times m$ input matrix and \mathbf{C} is the $k \times n$ output matrix, where both \mathbf{B} and \mathbf{C} have full rank, i.e. $r(\mathbf{B})=m$ and $r(\mathbf{C})=k$.

Applying a feedback law of the form

$$\mathbf{u} = \mathbf{r} - \mathbf{Fy} \quad (3)$$

where \mathbf{F} is an $m \times k$ feedback matrix of constant gains and \mathbf{r} is an $m \times 1$ external controlling vector, gives the closed-loop system

$$\dot{\mathbf{x}} = \mathbf{A}_c \mathbf{x} + \mathbf{Br} \quad (4)$$

$$\mathbf{y} = \mathbf{Cx} \quad (5)$$

where $\mathbf{A}_c = (\mathbf{A} - \mathbf{BFC})$ is the desired closed-loop system matrix.

Now given the controllable and observable system described by equations (1) and (2), and the assumptions that the matrices B and C are both of full rank, then it may be shown [5] that $\max(m,k)$ closed-loop eigenvalues can be assigned and $\max(m,k)$ eigenvectors can be partially assigned with $\min(m,k)$ degrees of freedom in the choice of each vector using output feedback.

Let $(\lambda_i, \mathbf{v}_i)$ be a desired closed-loop eigenvalue/eigenvector pair, then for $(\lambda_i, \mathbf{v}_i)$ to be an assignable pair there must exist a real F such that

$$(A - BFC)\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad (6)$$

or

$$\mathbf{v}_i = (A - \lambda_i I)^{-1} BFC\mathbf{v}_i \quad (7)$$

It is assumed here that none of the desired closed-loop eigenvalues is equal to an eigenvalue of A , causing $(A - \lambda_i I)$ to be singular and therefore non-invertible.

If we define the m -vector \mathbf{m}_i as

$$\mathbf{m}_i = FC\mathbf{v}_i \quad (8)$$

then for all $\mathbf{v}_i \neq \mathbf{0}$, \mathbf{m}_i can be thought of as an arbitrary vector of dimension $\min(m,k)$. It follows that

$$\mathbf{v}_i = (A - \lambda_i I)^{-1} B\mathbf{m}_i \quad (9)$$

This expression is fundamental to modal control theory. It states that the eigenvector \mathbf{v}_i must, in all cases, be contained within the subspace $\Gamma(\lambda_i, A, B)$ spanned by the columns of $(A - \lambda_i I)^{-1} B$. The dimension of this assignable subspace $\Gamma(\lambda_i, A, B)$ is equal to the rank of the input matrix, which in turn is equal to the number of independent control inputs. Thus in order to increase the size of this subspace and hence improve the choice of \mathbf{v}_i , the number of control inputs within the system must be increased. It is also worth noting that the orientation of the assignable subspace depends only on the open-loop matrices A and B and on the desired closed-loop eigenvalue λ_i .

Given that a suitable set of eigenvalue/eigenvector pairs have been chosen the feedback matrix F must be calculated.

From equation (6) we have

$$(A - BFC)V = V\Lambda \quad (10)$$

or

$$BFCV = (AV - V\Lambda) \quad (11)$$

where V is an $n \times k$ matrix containing the k desired closed-loop eigenvectors, and Λ is an $k \times k$ block diagonal matrix containing the k desired closed-loop eigenvalues, both in modified Jordan canonical form. Then the above equation has solution for F given by [6]

$$F = B^+(AV - V\Lambda) (CV)^{-1} \quad (12)$$

where $B^+ = (B^t B)^{-1} B^t$.

PRINCIPAL ANGLES

The previous section outlines a method for calculating an assignable subspace $\Gamma(\lambda_i, A, B)$ in which, given the closed-loop eigenvalue λ_i , the corresponding eigenvector v_i can be arbitrarily chosen. The question now arises, what is the best choice of the vector v_i from within this subspace, and how can one measure the amount of coupling given by this choice. Let (u_i, λ_i) be the desired closed-loop eigenvalue/eigenvector pair, then $v_i \in \Gamma(\lambda_i, A, B)$ is chosen such that the angle between that which is desired, u_i , and that which is achievable, v_i , is minimised. This solves both problems, firstly the best possible choice of the vectors v_i is made and secondly the angle between these two vectors, u_i and v_i , gives a measure of the coupling. This can be accomplished as follows.

Consider two linear subspaces F and G in R^n , with dimensions such that

$$n > p = \dim(F) > \dim(G) = q > 1$$

then the smallest angle $\theta_1 \in [0, \pi/2]$ between F and G is defined by [7]

$$\cos \theta_1 = \max_{\mathbf{v} \in F} \max_{\mathbf{u} \in G} \mathbf{u}^* \mathbf{v} \quad (13)$$

where

$$\|\mathbf{u}\| = 1, \quad \|\mathbf{v}\| = 1$$

If it is assumed that the maximum is attained for $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{u} = \mathbf{u}_1$, then θ_1 is defined as the smallest angle between the orthogonal complement of F with respect to \mathbf{v}_1 and that of G with respect to \mathbf{u}_1 . By repeating the process until one of the subspaces is empty, all the principal angles $\{\theta_k \in [0, \pi/2]; k=1, \dots, q\}$, and principal vectors $(\mathbf{v}_1, \dots, \mathbf{v}_q)$ and $(\mathbf{u}_1, \dots, \mathbf{u}_q)$ of the subspace pair (F, G) are obtained.

It follows from the above that if $F = \Gamma(\lambda_i, \mathbf{A}, \mathbf{B})$, the assignable subspace for a desired eigenvalue λ_i , and $G = \mathbf{v}_i$, the desired eigenvector for λ_i , then the first principal vector, \mathbf{v}_1 , for the subspace pair $(\Gamma(\lambda_i, \mathbf{A}, \mathbf{B}), \mathbf{u}_i)$ gives the optimal choice for the closed-loop eigenvector \mathbf{v}_i , and the principal angle θ_1 is the angle between the two vectors.

It was stated earlier that the orientation of the assignable subspace $\Gamma(\lambda_i, \mathbf{A}, \mathbf{B})$ is a function of the closed-loop eigenvalue. Thus by plotting the variation of the first principal angle θ_1 versus λ_i , the optimal value of the i th closed-loop eigenvalue can be found for minimum coupling.

DESIGN METHODOLOGY

The prerequisite to the design of any FCS, is the identification of a mathematical model describing the dynamics of the aircraft. The type of model required here, is a linear time invariant derivative model. This can be achieved directly [1] or by perturbing a trimmed non-linear model and extracting the required derivatives. Also, the control engineer must obtain a specification for the desired controlled aircraft which satisfy a set of desired handling qualities. For the case of a modal control system, this takes the form of a set of maximum time constants for the responses of each of the controlled outputs,

and some specification of the maximum amount of coupling allowed in each channel. Once this is done the design process can begin.

The first stage in the design is to associate each of the aircraft modes (eigenvalues) with the relevant states, i.e. specify the desired closed-loop eigenvectors. In some cases it may be desired to have more than one state associated with a specific mode. This can occur when the physical dynamics require a certain amount of coupling. When this situation arises, rather than choosing a specific eigenvector, a subspace is created which is spanned by the unit vectors which are given by the non-zero elements of the eigenvector. We thus obtain a desired eigenvector or eigenvector subspace for each of the modes. To find the values of the closed-loop eigenvalues which best achieve these eigenvectors the principal angles between each desired eigenvector or eigenvector subspace are calculated over a range of closed-loop eigenvalues, and the results plotted. By inspection of these plots suitable values can be chosen for each eigenvalue/eigenvector pair which both satisfy the constraints on the eigenvalues (i.e. time constants) and also give small values for the principal angles (i.e. coupling). With these pairs chosen a constant pre-compensator matrix, P , is added to reduce the coupling effect of the input matrix B . The modified input matrix is given by

$$B_c = BP$$

where P is an $m \times m$ matrix of full rank, so as not to affect the previous work by changing the orientation of the assignable subspaces. Let B_1 be an $m \times m$ matrix containing the m rows of B corresponding to the m controllable states, the P is given by

$$P = B_1^{-1}$$

thus the corresponding rows of the modified input matrix approximate the identity matrix. It is at this point that the feedback matrix is calculated using equation (12), to give the desired closed-loop system. Finally a constant scaling matrix is added to give the desired steady state responses.

EXAMPLE

To illustrate the techniques outlined in this paper, a FCS was designed for a single main rotor helicopter with rigid blades, flying with a forward speed of 80 knots. The model used was a six degree of freedom, eighth order derivative model obtained from the non-linear model Helistab [8]. The state space description is given by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

where the state vector \mathbf{x} and the input vector \mathbf{u} are

$$\mathbf{x} = \begin{bmatrix} u \\ w \\ q \\ \theta \\ v \\ p \\ \varphi \\ r \end{bmatrix} \begin{array}{l} \text{forward velocity} \\ \text{vertical velocity} \\ \text{pitch rate} \\ \text{pitch attitude} \\ \text{lateral velocity} \\ \text{roll rate} \\ \text{roll attitude} \\ \text{yaw rate} \end{array}$$

and

$$\mathbf{u} = \begin{bmatrix} \theta_{oe} \\ \theta_{ls} \\ \theta_{lc} \\ \theta_{ot} \end{bmatrix} \begin{array}{l} \text{collective cyclic} \\ \text{longitudinal cyclic} \\ \text{lateral cyclic} \\ \text{pedal} \end{array}$$

and the system matrix \mathbf{A} and the input matrix \mathbf{B} are given in table 1. It is assumed here that all the states are available for measurement either directly or through a state observer. Thus the system is a 4-input, 8-state, 8-output system, this means that it is possible to assign all 8 eigenvalues arbitrarily and all 8 eigenvectors with 4 degrees of freedom.

The open-loop dynamics of the helicopter consist of three longitudinal modes and three lateral modes, namely

longitudinal modes

- (1) the fast pitch mode = -3.199
- (2) the slow pitch mode = -0.406
- (3) the phugoid mode = $0.134 \pm j0.376$

lateral modes

- (1) the roll mode = -10.545
- (2) the spiral mode = -0.031
- (3) the dutch roll mode = $-0.654 \pm j2.255$

As can be seen from the above, the longitudinal dynamics have two real stable modes, the fast and the slow pitch modes, and one unstable mode, the phugoid mode which is a complex conjugate pair with a time to double amplitude of 5.2 seconds and a period of oscillation of 16.7 seconds. This exhibits itself in the aircraft as a slow undamped oscillation between height rate and forward speed. The lateral dynamics also have two real stable modes, the roll and the spiral modes, and a stable complex conjugate pair with a period of 2.8 seconds.

The specification for the closed-loop system is to design a feedback loop for the controlled outputs, vertical velocity, w , longitudinal velocity, u , roll rate, p , and lateral velocity, v , which satisfy the criteria given in table 2. To achieve this, the following associations were made between the modes and the states

fast pitch	-	w
slow pitch	}	u, q, θ
phugoid		
roll	-	p
spiral	-	φ
dutch roll	-	v, r

giving the desired eigenvector or eigenvector subspace for each mode shown in table 3.

For each eigenvector or eigenvector subspace the principal angles were calculated over a range of eigenvalues and plotted [fig. 1-6]. Consider the fast pitch mode, this controls the vertical velocity. It can be seen [fig. 1] that the closer the eigenvalue is to zero the less the coupling, but it is required the the mode has a time constant < 0.2 seconds, therefore the fast pitch mode was assigned a value of -5.0, which gives a principal angle of 1.1° . The slow pitch and phugoid modes have the same desired eigenvector subspace, and can therefore be dealt with collectively. Again it can be seen from fig. 2 that the smaller the eigenvalue, the smaller the coupling, thus it is the

time-constant that again limits the coupling and values of -2.0 for the slow pitch mode and $-3.0+j1.73$ for the phugoid were chosen. Now for the lateral modes. The principal angle plot for the roll mode [fig. 3] shows that the more negative the mode the less the coupling, thus a value of -11.0 was used. The choice for the spiral mode is obvious as can be seen from fig. 4, minimum coupling occurs at the desired eigenvalue, i.e. 0.0 . The last mode is the dutch roll, which is a complex conjugate pair. The principal angle plot shows that within the range of eigenvalues plotted the principal angles are all less than 0.01° , thus making the choice of this mode fairly arbitrary, the only constraint being the time-constant < 0.2 seconds. A value of $-6.0+j3.5$ was used, i.e. a damping ratio of 0.87 . With a full set of closed-loop eigenvalues chosen the corresponding eigenvectors can be obtained from the principal vectors. This gives the eigenvalue/eigenvector set given in table 4. The input decoupling pre-compensator was added allowing the feedback matrix to be calculated, giving the desired closed-loop system.

The resultant time-responses for a doublet in each channel of the closed-loop system are shown in figs. 6-9, and were obtained using the simulation language TSIM [9]. These results show a vast improvement over the open-loop system, in that the helicopter is now stable, and exhibits very little interaction between the four controlled inputs w , u , p , and v . This is very evident in the vertical velocity channel, which has a fast time response time (a time constant of 0.2 seconds), and virtually no interaction from the other three channels. Also by comparison of the longitudinal and lateral responses it can be seen that the longitudinal and lateral dynamics are highly decoupled.

CONCLUSION

Modal control theory, in conjunction with the use of principal angles offers a highly visible design method for the rapid synthesis of linear control systems. The closed-loop eigenvalues of the system are chosen for both stability and reduction in coupling. This gives a better match between the desired eigenvalues and the actual closed-loop eigenvectors. At

present this design method has only been applied to a linear derivative model. but it is hoped to extend this work over a much wider range of flight conditions, possibly incorporating some sort of gain scheduling.

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U	W	Q	THTE	V	P	PHIE	R
-0.03221	0.04038	-0.73663	-32.17915	-0.00211	-0.35668	0.00000	0.00000
-0.00954	-0.80272	134.81477	-0.69159	-0.01941	-1.48180	1.05781	0.00000
0.00825	0.00878	-2.33976	0.00000	0.00316	0.41067	0.00000	0.00000
0.00000	0.00000	0.99946	0.00000	0.00000	0.00000	0.00000	0.03287
0.00433	0.01430	-0.42178	0.02275	-0.16671	0.64606	32.16176	-133.48404
-0.01136	0.07148	-1.99888	0.00000	-0.04971	-10.52839	0.00000	-0.28668
0.00000	0.00000	-0.00071	0.00000	0.00000	1.00000	0.00000	0.02149
-0.00786	0.00075	-0.08946	0.00000	0.03095	-1.79190	0.00000	-1.35013

CONTROL MATRIX

THOE	THIS	THIC	THOTRE
14.292793	-25.034639	6.747272	0.000000
-386.932831	-99.826668	0.000000	0.000000
14.076095	28.538784	-5.851117	0.000000
0.000000	0.000000	0.000000	0.000000
4.921250	-5.009327	-30.580055	22.041023
32.097626	-25.003689	-153.225204	-1.343526
0.000000	0.000000	0.000000	0.000000
13.952953	-5.948937	-26.807583	-18.095446

Table 1. System matrix A and Control Matrix B.

Vertical velocity

Time constant < 0.25 seconds
Coupling from other channels < 1%

Longitudinal velocity

Time constant < 1.00 seconds
Coupling from other channels < 10%

Roll rate

Time constant < 0.10 seconds
Coupling from other channels < 5%

Lateral velocity

Time constant < 0.20 seconds
Coupling from other channels < 5%

Table 2. FCS requirements.

	Fast pitch	Slow pitch & phugoid	Roll	Spiral	Dutch roll
u	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

Table 3. Desired eigenvector (subspace) for each mode.

Longitudinal modes

	Fast pitch (-4.0)	Slow pitch (-2.0)	Phugoid (-3.00 +j 1.73)
u	-0.0031	0.9963	0.9825 +j -0.0001
w	0.9999	0.0000	0.0000 +j 0.0000
q	-0.0146	-0.0768	-0.1404 +j 0.1107
θ	0.0037	0.0384	0.0511 +j -0.0074
v	0.0000	0.0000	0.0001 +j 0.0000
p	0.0000	-0.0002	-0.0001 +j 0.0003
φ	0.0000	0.0001	0.0001 +j -0.0001
r	0.0001	-0.0024	-0.0042 +j 0.0040

Table 4. Closed-loop eigenvalue/eigenvector pairs.

Lateral modes

	Roll (-11.00)	Spiral (0.00)	Dutch roll (-6.00 +j 3.50)
u	0.0015	0.0000	-0.0001 +j -0.0001
w	-0.0001	-0.0011	0.0000 +j 0.0000
q	-0.0011	-0.0076	-0.0004 +j 0.0000
	-0.0002	0.0008	-0.0002 +j 0.0000
v	0.9960	-0.0032	0.9986 +j -0.0001
p	0.0034	-0.0050	0.0000 +j 0.0000
	-0.0005	0.9728	-0.0002 +j 0.0000
r	0.0892	0.2316	0.0451 +j -0.0289

Table 4. (Continued).

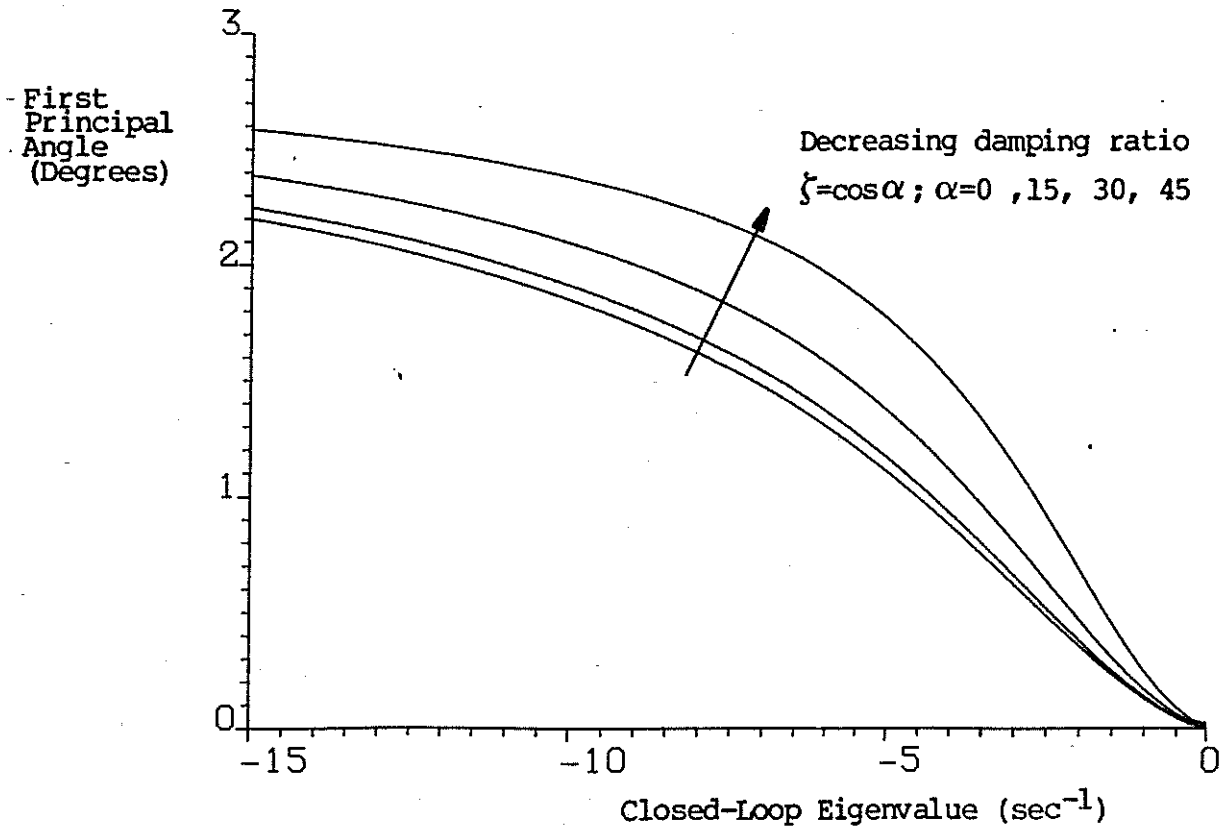


Fig. 1. Principal angle plot for fast pitch mode.

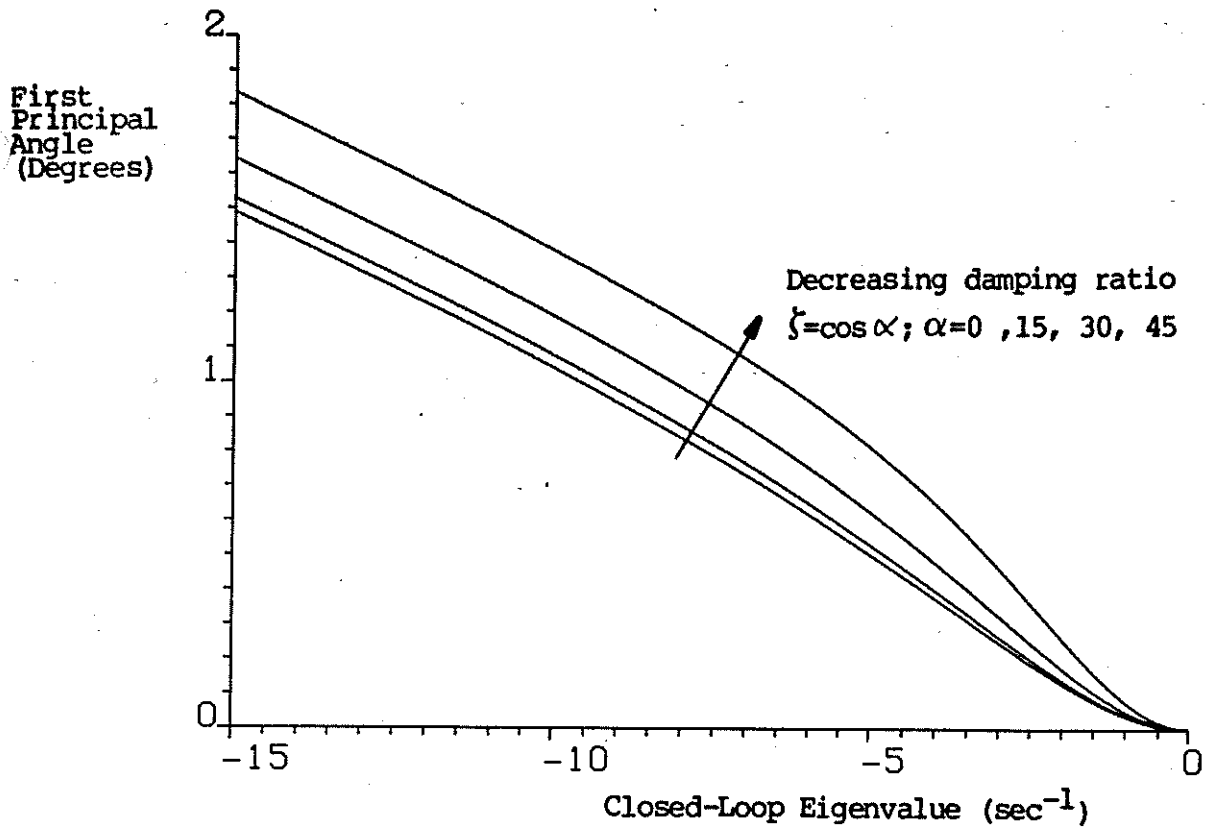


Fig. 2. Principal angle plot for slow pitch and phugoid modes.

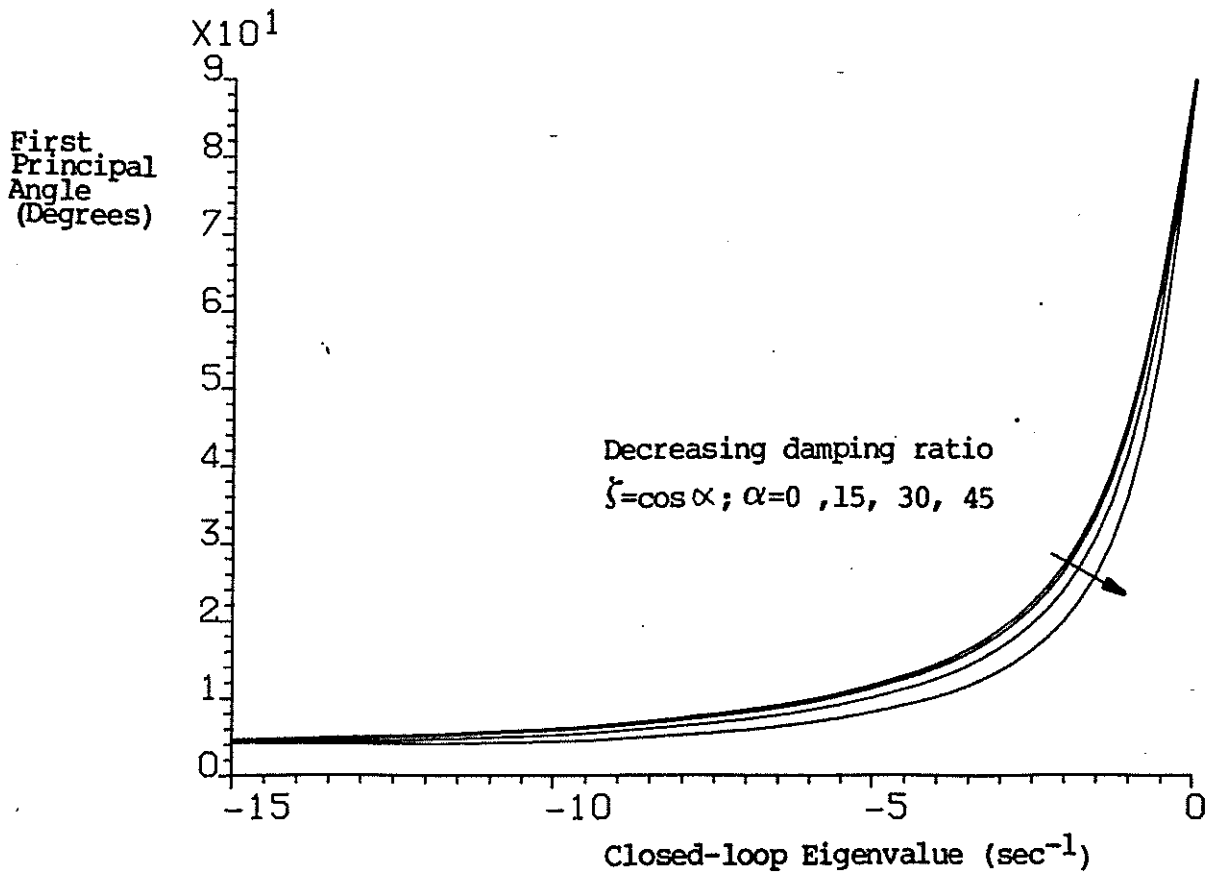


Fig. 3. Principal angle plot for roll mode.

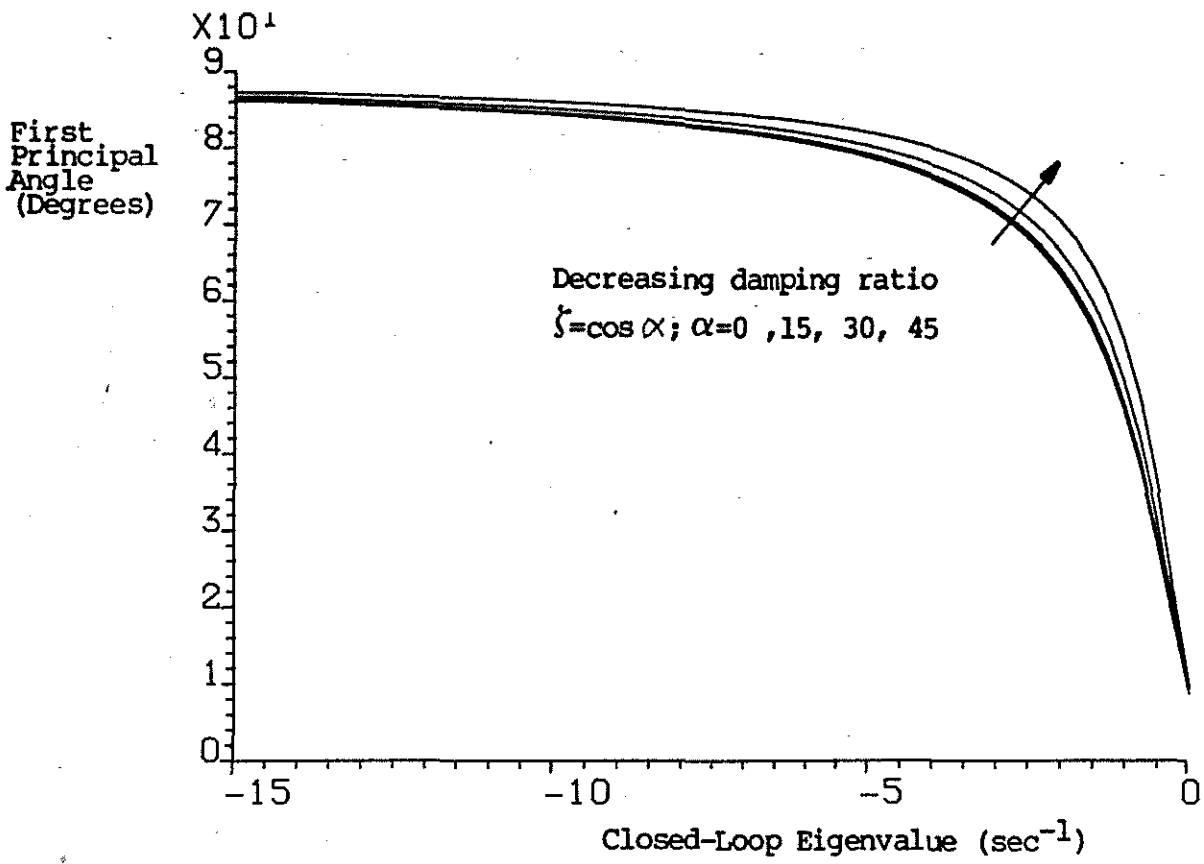


Fig. 4. Principal angle plot for spiral mode.

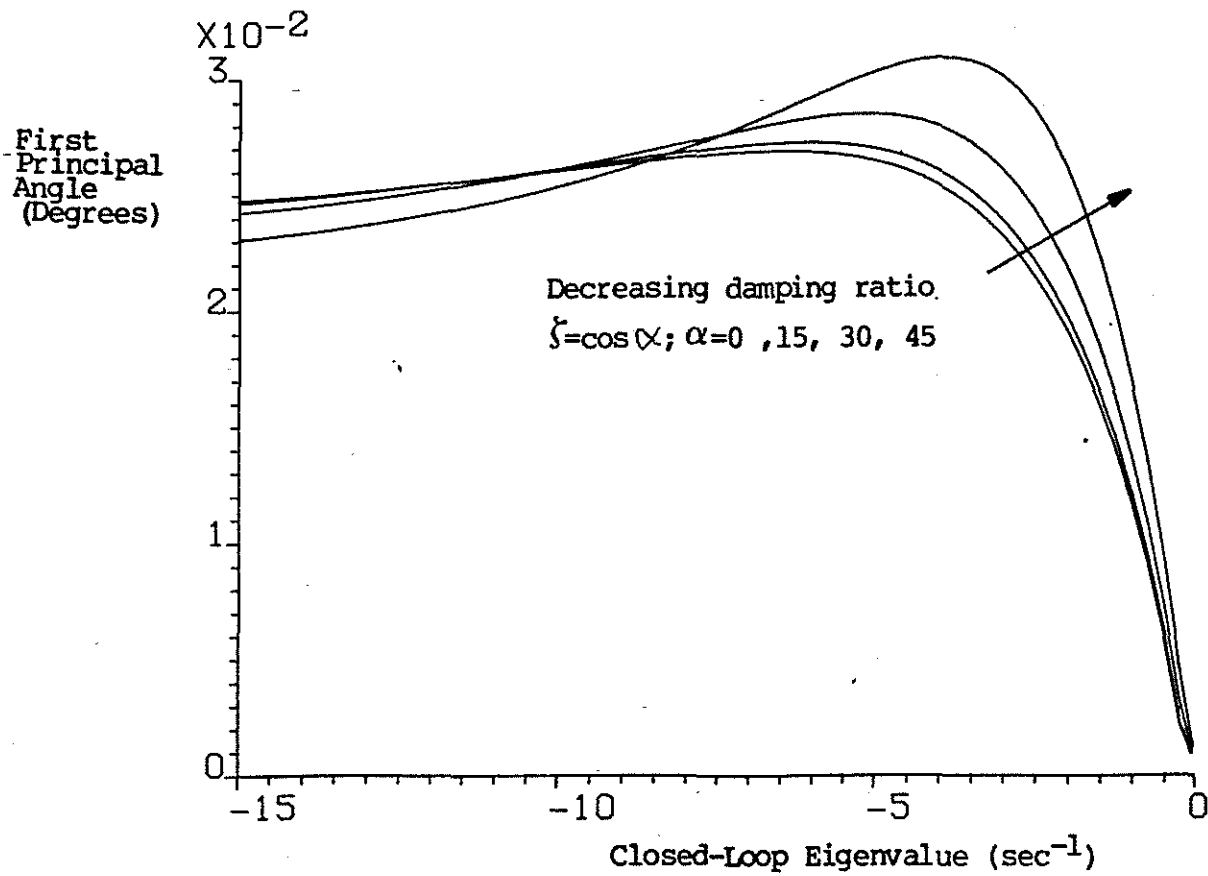


Fig. 5. Principal angle plot for dutch roll mode.

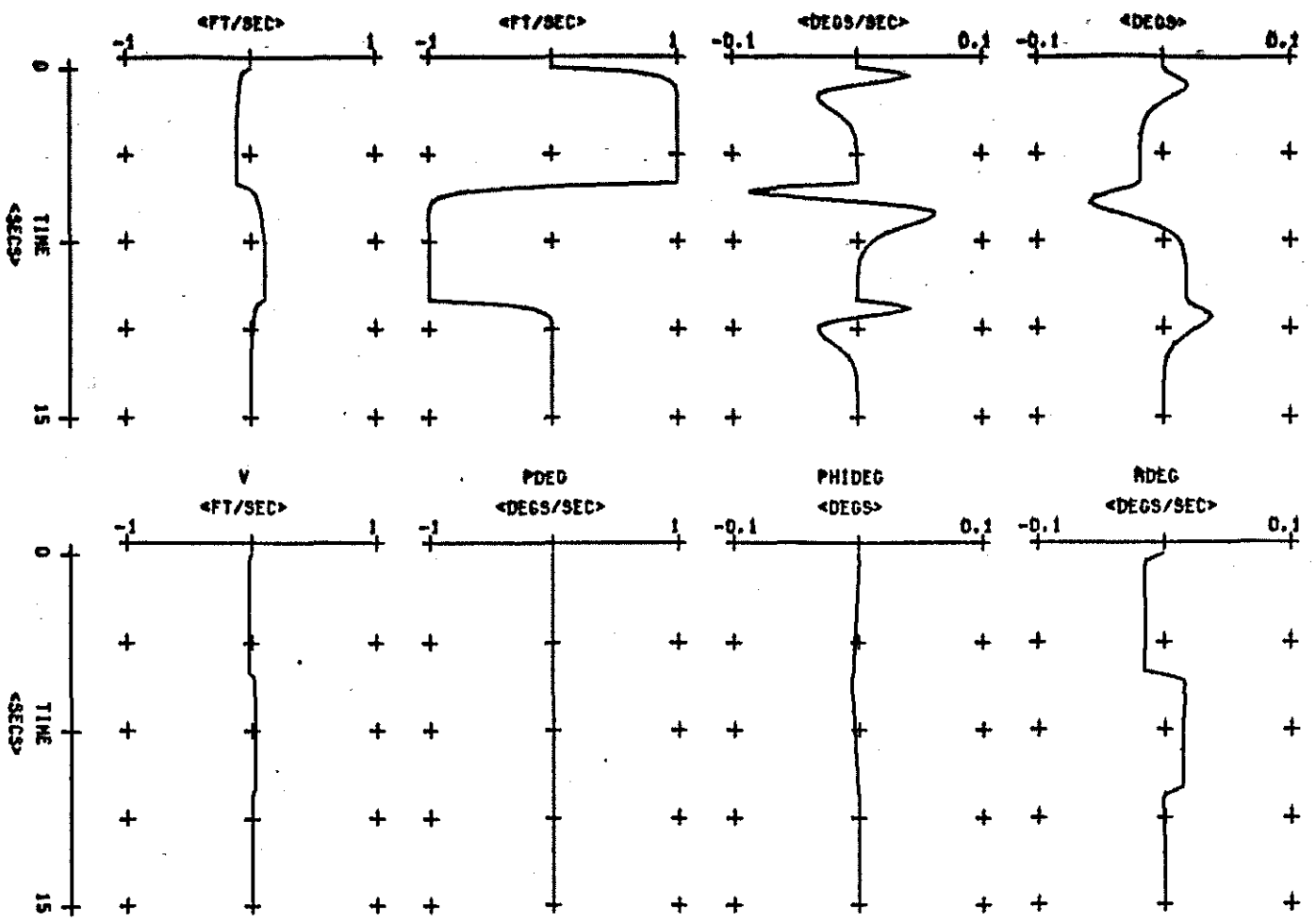


Fig. 6. Time responses for doublet in θ_{0e}

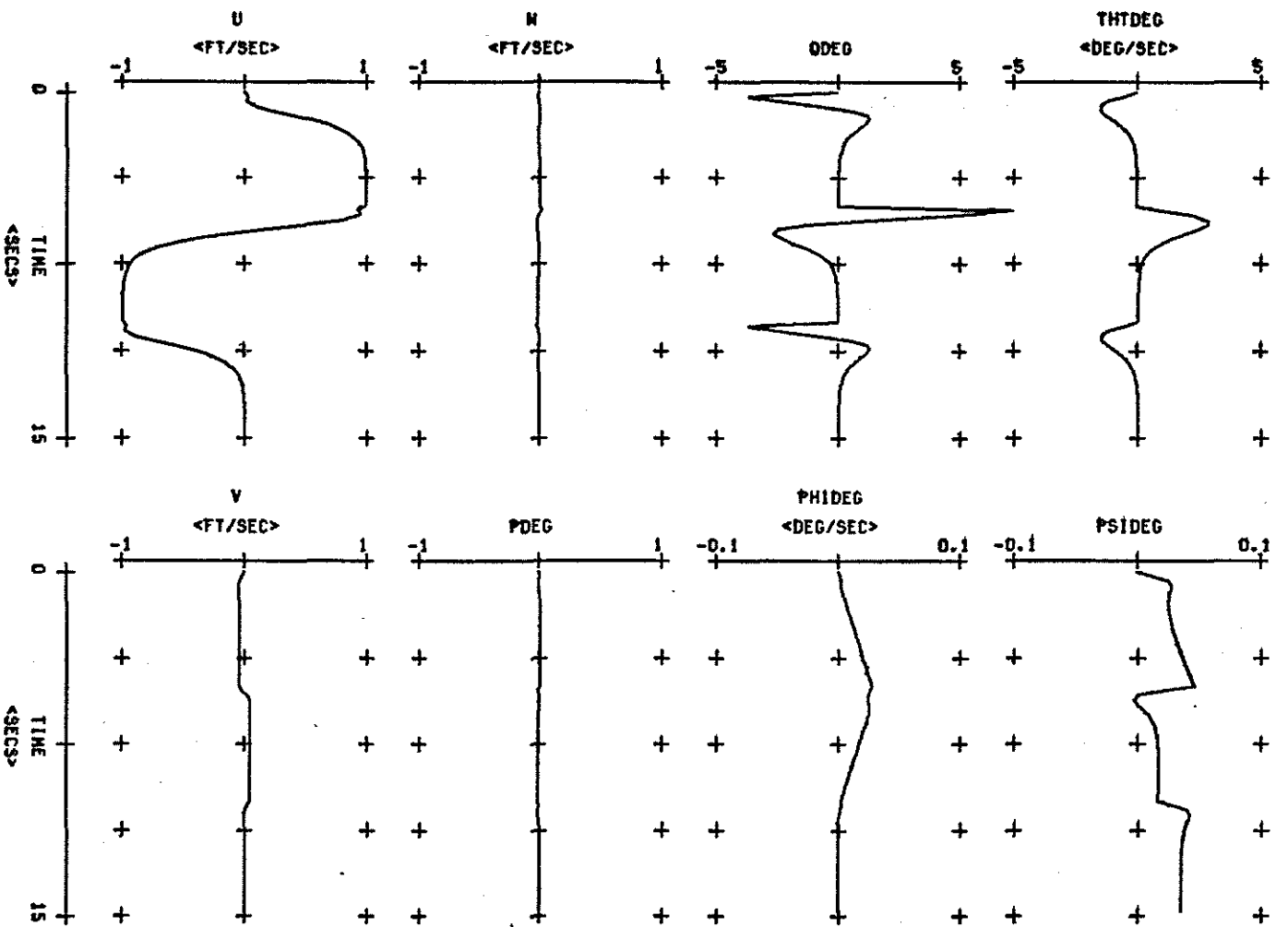


Fig. 7. Time responses for doublet in θ_{1s}

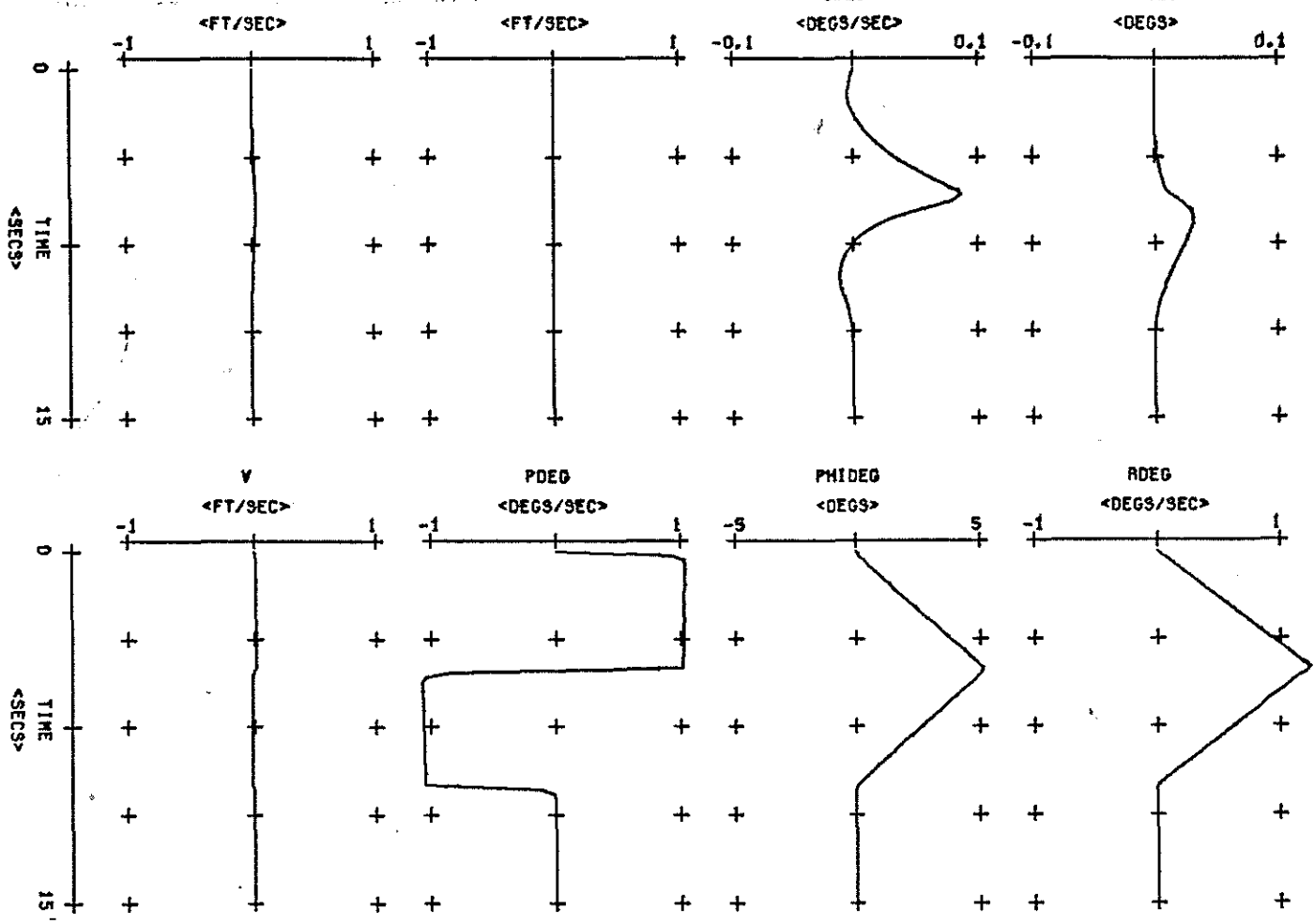


Fig. 8. Time responses for doublet in θ_{1c}

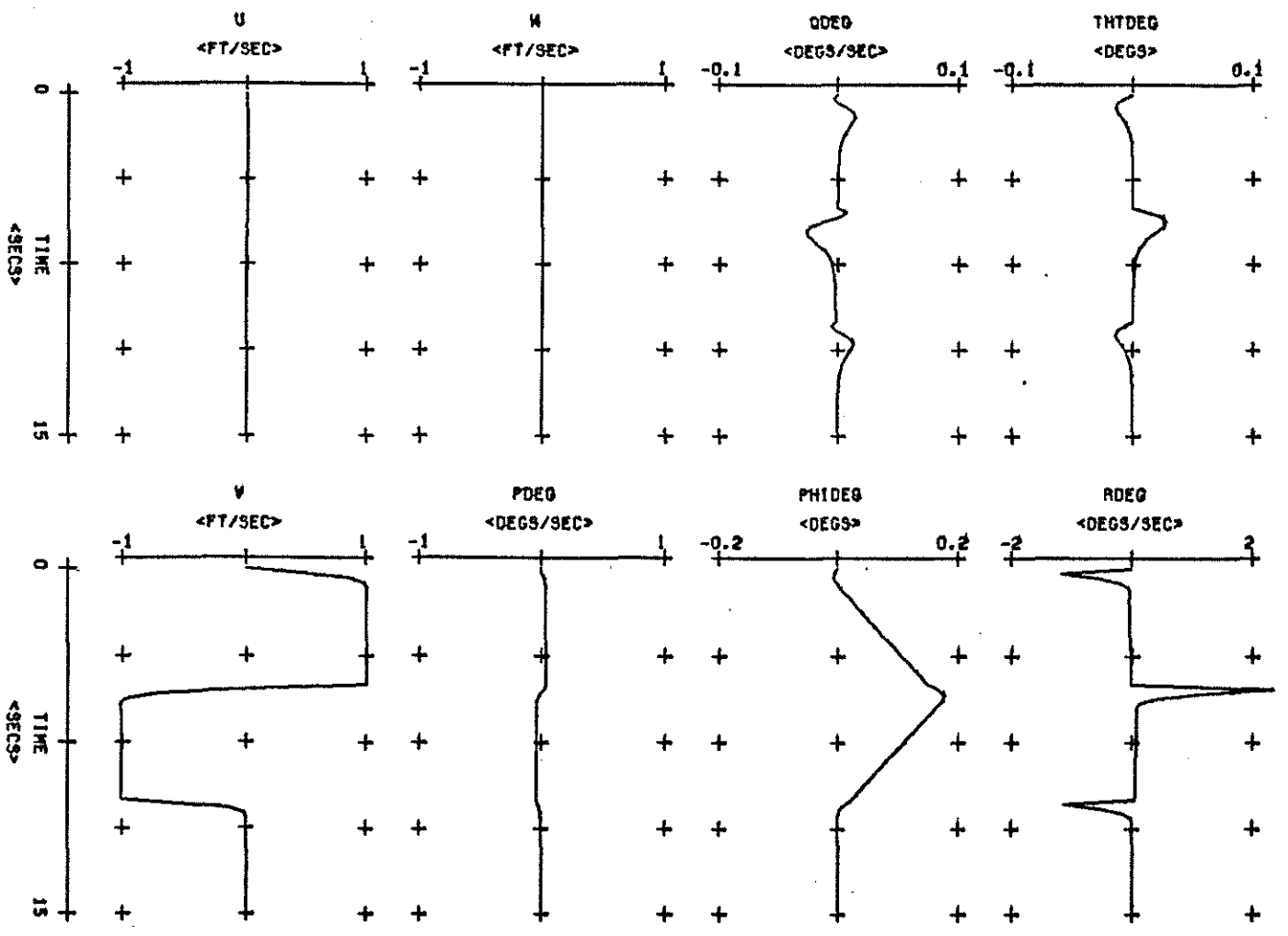


Fig. 9. Time responses for doublet in θ_{0t}