LINEAR TIME INVARIANT APPROXIMATIONS OF LINEAR TIME PERIODIC SYSTEMS

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Abstract

Several methods for analysis of linear time periodic (LTP) systems have successfully been demonstrated using harmonic decompositions. One method recently examined is to create a linear time invariant (LTI) model approximation by expansion of the LTP system states into various harmonic state representations, and formulating corresponding linear time invariant models. Although this method has shown success, it relies on a second order formulation of the original LTP system. This second order formulation can prove problematic for degrees of freedom not explicitly represented in second order form. Specifically, difficulties arise when performing the harmonic decomposition of body and inflow states as well as interpretation of LTI velocities. Instead this paper will present a more generalized LTI formulation using a first order formulation for harmonic decomposition. The new first order approach is evaluated for a UH-60 rotorcraft model, and is used to show the significance of particular harmonic terms; specifically that the coupling of harmonic components of body and inflow states with the rotor states has a significant contribution to the LTI model fidelity in the prediction of vibratory hub loads.

1. NOMENCLATURE

A	LTI State Matrix
B	LTI Input Matrix
C	LTI Output Matrix
D	LTI Direct Transmission Matrix
$F(\psi)$	LTP State Matrix
$G(\psi)$	LTP Input Matrix
$R(\psi)$	LTP Output Matrix
u	Input
$U(\psi)$	LTP Direct Transmission Matrix
x	State
y	Output
Φ	State transition matrix
ψ	Non-dimensional time
Ω	Non-dimensional rotor speed
$()_0$	Average or 0th harmonic term
$()_{nc}$	nth cosine harmonic component
$()_{ns}$	nth sine harmonic component

2. INTRODUCTION

The analysis of linear time periodic (LTP) systems is well understood using several methods. One such method is Floquet Theory, developed by Gaston Floquet [1]. This theory has been shown to provide a thorough analysis of LTP systems through the use of modal participation factors [2]. These modal participation factors describe the relative magnitude of each harmonic component for each state.

Other methods involve using a harmonic decomposition of the LTP system. One method recently examined is to create a linear time invariant model approximation by expansion of the LTP system states into various harmonic state representations and formulating corresponding linear time invariant models. Crimi and Piarulli explore the LTP system by harmonic decomposition of periodic states [3 and 4]. One method recentely examined by Prasad et al [5 - 8] use the harmonic decomposition to formulate a corresponding linear time invariant (LTI) system. This methodology provides a convenient framework, as methods for LTI system analysis, controller synthesis and design are well developed and understood as demonstrated by Lopez et al [9-12].

Although the method developed by Prasad has shown success, it relies on a second order formulation of the original LTP system. This second order formulation can prove problematic for degrees of freedom not explicitly represented in second order form. Specifically, difficulties arise when performing the harmonic decomposition of body and inflow states as well as interpretation of LTI velocities.

The aim of this work is to develop a more generalized LTI formulation using a first order formulation for harmonic decomposition. Specific objectives are:

1) Develop and validate an LTI approximation of an LTP system using a first order formulation with closed form expressions.
2) Evaluate the significance of harmonic terms of body and inflow states using modal participation.
3) Evaluate the significance of coupling between body, inflow, and rotor harmonic terms using additive uncertainty and nu gap metric analysis.

3. LTI MODEL EXTRACTION

The main results of the LTI model extraction from an LTP model using a first order formulation are presented here. The derivation in full is presented in the appendix.

Consider an LTP model with the state equation given as

(1) $$\dot{x} = F(\psi)x + G(\psi)u$$

and the output equation of a LTP given model as

(2) $$y = P(\psi)x + R(\psi)u$$

where x, u, and y are the state, input, and output vectors respectively. An LTP model can be obtained from a nonlinear model using a perturbation scheme, linearizing about a periodic equilibrium at every azimuthal position [3]. In order to extract an approximate LTI model from Eq.(1) ~ (2) , consider the following approximation of x:

(3) $$x = x_o + \sum_{n=1}^{N} x_{nc} \cos n\psi + x_{ns} \sin n\psi$$

where x_o is the average component and x_{nc} and x_{ns} are respectively the n/rev cosine and sine harmonic components of x. Likewise, the control u is expanded in terms of harmonic components as

(4) $$u = u_o + \sum_{m=1}^{M} u_{mc} \cos m\psi + u_{ms} \sin m\psi$$

and the output y is expanded in terms of harmonic components as

(5) $$y = y_o + \sum_{l=1}^{L} y_{lc} \cos l\psi + y_{ls} \sin l\psi$$

where y_o is the average component and y_{lc} and y_{ls} are respectively the l^{th} harmonic cosine and sine components of y.

The LTI model can be represented in matrix form by defining the augmented state vector as

(6) $$X = \left[x_o{}^T .. x_{ic}{}^T \quad x_{is}{}^T .. x_{jc}{}^T \quad x_{js}{}^T .. \right]^T$$

and the augmented control vector as

(7) $$U = \left[u_o{}^T .. \quad u_{mc}{}^T \quad u_{ms}{}^T .... \right]^T$$

where x_o is the zeroth harmonic component, x_{ic}, x_{is} are the i^{th} harmonic cosine and sine components of x and u_{mc}, u_{ms} are the m^{th} harmonic cosine and sine components of u, respectively. The state equation of the resulting LTI model is

(8) $$\dot{X} = [A]X + [B]U$$

Likewise, the augmented output vector of the LTI model is defined as

(9) $$Y = \left[y_o{}^T .. \quad y_{lc}{}^T \quad y_{ls}{}^T ... \right]^T$$

Then the output equation of the LTI model can be written as

(10) $$Y = [C]X + [D]U$$

The LTI model matrices of Eqs. (8) and (10) are obtained as

$$A = \begin{bmatrix} H_{oF} \ldots H_{oF^{ic}} \quad H_{oF^{is}} \ldots H_{oF^{jc}} \quad H_{oF^{js}} \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ H_{icF} \ldots H_{icF^{ic}} \quad -i\Omega + H_{icF^{is}} \ldots H_{icF^{jc}} \quad H_{icF^{js}} \ldots\ldots \\ H_{isF} \ldots i\Omega + H_{isF^{ic}} \quad H_{isF^{is}} \ldots H_{isF^{jc}} \quad H_{isF^{js}} \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ H_{jcF} \ldots H_{jcF^{ic}} \quad H_{jcF^{is}} \ldots H_{jcF^{jc}} \quad -j\Omega + H_{jcF^{js}} \ldots \\ H_{jsF} \ldots H_{jsF^{ic}} \quad H_{jsF^{is}} \ldots j\Omega + H_{jsF^{jc}} \quad H_{jsF^{js}} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} \tag{11}$$

$$B = \begin{bmatrix} H_{oG} \ldots H_{oG^{mc}} \quad H_{oG^{ms}} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ H_{icG} \ldots H_{icG^{mc}} \quad H_{icG^{ms}} \ldots \\ H_{isG} \ldots H_{isG^{mc}} \quad H_{isG^{ms}} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ H_{jcG} \ldots H_{jcG^{mc}} \quad H_{jcG^{ms}} \ldots \\ H_{jsG} \ldots H_{jsG^{mc}} \quad H_{jsG^{ms}} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix}$$

$$C = \begin{bmatrix} H_{oP} \ldots H_{oP^{ic}} \quad H_{oP^{is}} \ldots H_{oP^{jc}} \quad H_{oP^{js}} \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ H_{lcP} \ldots H_{lcP^{ic}} \quad H_{lcP^{is}} \ldots H_{lcQ^{jc}} \quad H_{lcP^{js}} \ldots\ldots \\ H_{lsP} \ldots H_{lsP^{ic}} \quad H_{lsP^{is}} \ldots H_{lsP^{jc}} \quad H_{lsP^{js}} \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix}$$

$$D = \begin{bmatrix} H_{oR} \ldots H_{oR^{mc}} \quad H_{oR^{ms}} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ H_{lcR} \ldots H_{lcR^{mc}} \quad H_{lcR^{ms}} \ldots \\ H_{lsR} \ldots H_{lsR^{mc}} \quad H_{lsR^{ms}} \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix}$$

Where the H operators have been defined as:

$$H_{oM} = \frac{1}{2\pi} \int_0^{2\pi} M(\psi) d\psi$$

$$H_{icM} = \frac{1}{\pi} \int_0^{2\pi} M(\psi) \cos i\psi d\psi$$

$$H_{isM} = \frac{1}{\pi} \int_0^{2\pi} M(\psi) \sin i\psi d\psi$$

$$i = 1, 2, 3, \ldots, N$$

The key difference between the newly presented first order LTI formulation, Eqs. (8) ~ (10), and the previous second order LTI formulation [2] is the treatment of the velocity states. In the previous second order LTI formulation, the LTI harmonic states associated with velocities are not directly the harmonic decomposition of the LTP velocity states. Rather, they are kinematically related via terms involving powers of the rotor speed Ω. Thus, to properly determine information about the LTP velocities, one would need to perform extra work to relate the LTI harmonic states and the harmonic decomposition of the LTP velocity states. In particular, to determine modal participation [2] from the LTI appropriately [9], one would need to convert the LTI harmonic terms associated with velocities into the harmonic decomposition of the LTP velocities. Furthermore, since LTP body and inflow states do not readily come in the second order form required for the second order LTI formulation (where the time derivatives of the displacement states are exactly given by the velocity states), extra work is again needed to transform those states into a usable form.

In the first order LTI formulation presented here, there is no difference in the treatment of LTP velocity and displacement states. This allows for an overall simplified calculation, and any information about LTP velocities can be given directly by the LTI states associated with velocities (such as modal participation of velocity states). Consequently, since there is no difference in treatment between any LTP states, this formulation easily encompasses body and inflow states which are often formulated in a more generalized first order form. Thus, this first order LTI formulation directly and efficiently approximates any LTP which can be cast in first order form.

4. NUMERICAL EXAMPLE

The model examined here is a full vehicle nonlinear model (NL) in FLIGHTLAB. The full vehicle model is a UH-60 with elastic blade mode shapes and a 33 state Peters-He dynamic inflow model. The model has previously been validated [13] and been found to be consistent with trends from wind tunnel data. The NL is linearized at each azimuthal position about a periodic equilibrium at 120 knots to generate an LTP model.

4.1 LTI Model Evaluations

The full order LTI is extracted from the LTP using the first order methodology developed here, including the 0^{th} up to the 24^{th} harmonic states for each body, inflow, and rotor state resulting in a total of 3577 LTI states (LTI model referred to as LTIfull3577). The linear model is compared against the nonlinear model using error response plots between linear and nonlinear model bare airframe responses. An example error response plot is given in Figure 1 in capturing the transfer function of the LTP model from individual blade control, 4^{th} harmonic cosine (IBC4C) input to the 4/rev sine and cosine components of vertical hub shear (Fz4C and Fz4S respectively) by the LTI model. These particular transfer functions would be most relevant to vibration reduction. The error response can be measured using a single cost function as described by Tischler [14]. The average cost function over all IBC input and 4/rev output transfer functions is 4.92, meaning that the LTIfull3577 data is nearly indistinguishable from the nonlinear model data.

At this point, the LTIfull3577 model has been validated to be nearly indistinguishable from the nonlinear model in terms of IBC inputs and 4/rev output transfer functions.

4.2 LTI Modal Participation

The significance of particular harmonic states can be evaluated by determining the modal participation. [2]. For the LTP, this can be done by 1) computing the Floquet Transition Matrix, 2) computing the Floquet eigenvalues and Floquet eigenvectors, 3) computing the system eigenvalues and periodic eigenvectors, 4) decomposing each periodic eigenvector element into its corresponding Fourier coefficients, and 5) then taking the normalized magnitude of a particular harmonic.

Computing the modal participation using this methodology does pose two particular problems. First, in computing the system eigenvalues, a multi-valued complex logarithm is used, and one must therefore make a choice for integer multiple of Ω to be added. It has been shown that this choice is arbitrary, and simply shifts the resulting numbering of harmonics [2]. Secondly, computation of the state transition matrix itself as well as the solving the corresponding eigenvalue problem requires additional processing due to numerical difficulties.

The modal participation can be directly computed from the LTI itself [9]. Once an LTI has been formed, its system eigenvalues and eigenvectors can be directly solved for. The modal participation can then be determined by converting the eigenvectors from trigonometric to complex, and then taking the normalized magnitude of a particular harmonic.

The modal participations were computed for harmonic term, for each mode. Each of the 73 modes were examined and found to have similar trends. For brevity, only 5 sample modes are shown here. The modal participation is shown for rotor coning in Figure 2 as computed both by the LTP and LTI methods. As expected based on previous work [11], the harmonics with the highest modal participation are the 0, 1, and even harmonics up to 8. Also, the LTI and LTP computations result in nearly identical rotor coning modal participations, indicating that the LTI captures modal participation as accurately as the LTP.

The modal participation for average inflow is shown in Figure 3 and the modal participation for pitch attitude is shown in Figure 4. Again, the LTI and LTP computations show similar results, indicating that the LTI captures modal participation as accurately as the LTP. It is clear that similar to rotor degrees of freedom, body and inflow degrees of freedom also have contributions from harmonics 0, 1, and even numbered harmonics up to 8 (i.e. harmonics 0, 1, 2, 4, 6, and 8). These trends were observed for all modes and for every rotor, body, and inflow state. Thus, it is clear that in addition to rotor harmonics, body and inflow harmonics are important and need to be included in the LTI approximation.

4.3 LTI Input-Output Fidelity

Alternatively, the significance of particular harmonic states can then be evaluated by comparing the full model LTIfull3577 with reduced LTI models that do not include particular harmonic states. The first reduction is formed by the least significant harmonics, as shown by the modal participation evaluations. Specifically any harmonics above the 8^{th} harmonic and any odd numbered harmonics above the 2^{nd} harmonic (i.e., removing harmonics 3, 5, 7, 9 and any above 9) are removed. The resulting LTI retains the 0^{th}, 1^{st}, and 2-8 even harmonics of all body, inflow and rotor states, resulting in 803 states (referred to as LTIred803). The second reduction is formed by starting with LTIred803 and removing any body harmonic states. The resulting LTI has only 723 states (referred to as LTIred703) and contains only the 0^{th} harmonic body states, and the 0^{th}, 1^{st}, and 2-8 even harmonic inflow and rotor states. The third reduction is formed by starting with LTIred803 and removing any inflow harmonic states. The resulting LTI has only 473 states (referred to as LTIred473) and contains only the 0^{th} harmonic inflow states, and the 0^{th}, 1^{st}, and 2-8 even harmonic body and rotor states. Finally, the fourth reduction is formed by starting with LTIred803 and removing both body and inflow harmonic states. The resulting LTI has only 393 states (referred to as LTIred396) and contains only the 0^{th} harmonic body and inflow states, and the 0^{th}, 1^{st}, and 2-8 even harmonic rotor states.

The frequency responses for the various LTI model approximations considered above are used in evaluating the individual model fidelity. For example, comparisons of frequency responses from various LTI model approximations from IBC4C input to Fx4C hub force output, IBC4C input to Fy4C hub force output and IBC4C input to Fz4C hub force output are shown in Figures 5, 6 and 7, respectively. For all of the frequency responses examined, LTIred803 is nearly indistinguishable from LTIfull3577. Thus it is clear that in this case, any harmonic terms 3, 5, 7, 9 and any above 9 do not significantly influence overall model fidelity. Comparing LTIred723 with LTIfull3577, there is a maximum of a 3 db difference in magnitude at 6 rad/s for Fx4C, and otherwise a maximum of 1 dB differences in magnitude over all frequency responses examined. Comparing LTIred473 with LTIfull3577, there is a maximum of 9.5 dB differences in magnitude below 7 rad/s, and 2.5 dB differences above 7 rad/s. Comparing LTIred393 with LTIfull3577, differences are similar to those from LTIred473 with a maximum of 10 dB differences in magnitude below 7 rad/s, and 2.5 dB differences above 7 rad/s. Thus, it is clear that inclusion of harmonics terms for both body and inflow states are important, although body harmonic terms less so than inflow harmonic terms.

The normalized additive error [10, 11, 15] for IBC4C input for each reduction is shown in Figure 8 for Fx4C, Fy4C, Fz4C, Mx4C, and My4C. Each reduction is compared with LTIfull3577, with LTIfull3577 taken as the truth model. Here it is clear that there is very small normalized additive error for LTIred803, meaning that virtually no additional robustness would be needed for designing a controller based on the LTIred803 model compared to the LTIfull3577 model. Normalized additive error for LTIred723 is on the order of 0.01~0.05 meaning that some additional robustness would be needed for designing a controller using the LTIred723 model compared to the LTIfull3577 model. Normalized additive error for LTIred473 and LTIred393 are both on the order of 0.2, meaning that additional robustness would be needed for designing a controller using either reduced model compared to the LTIfull3677 model. Thus, it is again clear that retaining harmonic terms for body and inflow states is important for reducing additional robustness needed in controller design.

The nu gap metric [10, 11, 16] for IBC4C input for each reduction is shown in Figure 9 for Fx4C, Fy4C, Fz4C, Mx4C, and My4C. Each reduction is compared with LTIfull3577, with LTIfull3577 taken as the truth model. Here it is clear that there is very small nu gap metric for LTIred803, meaning that there would be very little losses in stability margin if a controller were designed using the LTIred803 model and applied to the LTIfull3577 model. Nu gap metric for LTIred723 is at most on the order of 0.1 meaning that there would be very little losses in stability margin if a controller were designed using the LTIred723 model and applied to the LTIfull3577 model. Nu gap metric for LTIred473 and LTIred393 are both at most on the order of 0.2, meaning that there would be small losses in stability margin if a controller were designed using either model and applied to the LTIfull3577 model (small, but still larger compared to the LTIred723 and LTIred803 cases). Thus, it is again clear that retaining harmonic terms for body and inflow states is important for reducing losses in stability margin when designing controllers based on the reduced models.

5. FUTURE WORK

The results demonstrated thus far have been model fidelity evaluations of a single main rotor configuration for a moderate speed. It is recommended that the LTI models developed here be used for integrated flight and rotor control design, such as for an integrated flight and vibration controller. It is further recommended that these techniques be studied with advanced configurations such as compound, coaxial rotorcraft which travel at very high speeds and have added complexity.

6. SUMMARY

A generalized linear time invariant (LTI) approximation is developed from a linear time periodic (LTP) model using a first order formulation. Explicit formulas for LTI state space matrices are presented.

A complete numerical example is given for a UH-60 rotorcraft. The resulting LTI is validated against the original nonlinear model, and is shown to be very accurate in the frequency domain. The modal participation is calculated directly from the LTI and compared with modal participation calculated from the LTP. Modal participation, additive uncertainty, and nu gap metric analysis are used to evaluate the significance of particular harmonic terms.

7. CONCLUSIONS

The results presented here support the following conclusions:

1) A nonlinear time periodic rotorcraft model can be accurately approximated by a linear time invariant model, using harmonic decompositions and a first order representation.
2) Modal participation can be accurately and easily obtained from a linear time invariant approximation, avoiding ambiguities and numerical difficulties of obtaining modal participation from the linear time periodic model.
3) Body and inflow degrees of freedom have harmonic terms with significant modal participation. These harmonic terms for body and inflow degrees of freedom which are most significant are the same as the harmonic terms for rotor degrees of freedom which are most significant.
4) Coupling of harmonic terms for body, inflow, and rotor degrees of freedom play a significant role in the input-output fidelity for the purpose of predicting vibratory loads

8. ACKNOWLEDGEMENTS

This study is supported under the NRTC Vertical Lift Rotorcraft Center of Excellence (VLRCOE) from the U.S. Army Aviation and Missile Research, Development and Engineering Center (AMRDEC) under Technology Investment Agreement W911W6-06-2-0002, entitled National Rotorcraft Technology Center Research Program. The authors would like to acknowledge that this research and development was accomplished with the support and guidance of the NRTC. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the AMRDEC or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon.

The authors would like to thank Mr. Thomas Nathan for his software implementation of the LTI extraction methods presented here. The authors would also like to thank Ms. Caitlin Berrigan for her contributions to the software implementation of modal participation computations which were utilized here.

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10. FIGURES

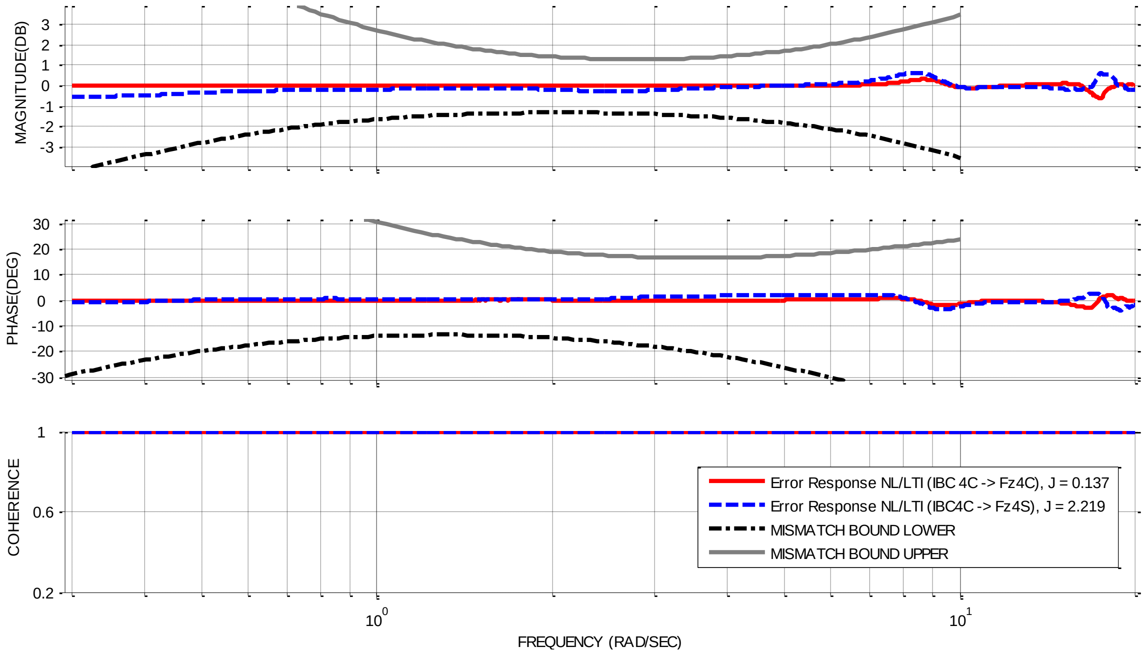

Figure 1. Error Response Plot for Fz4C and Fz4S

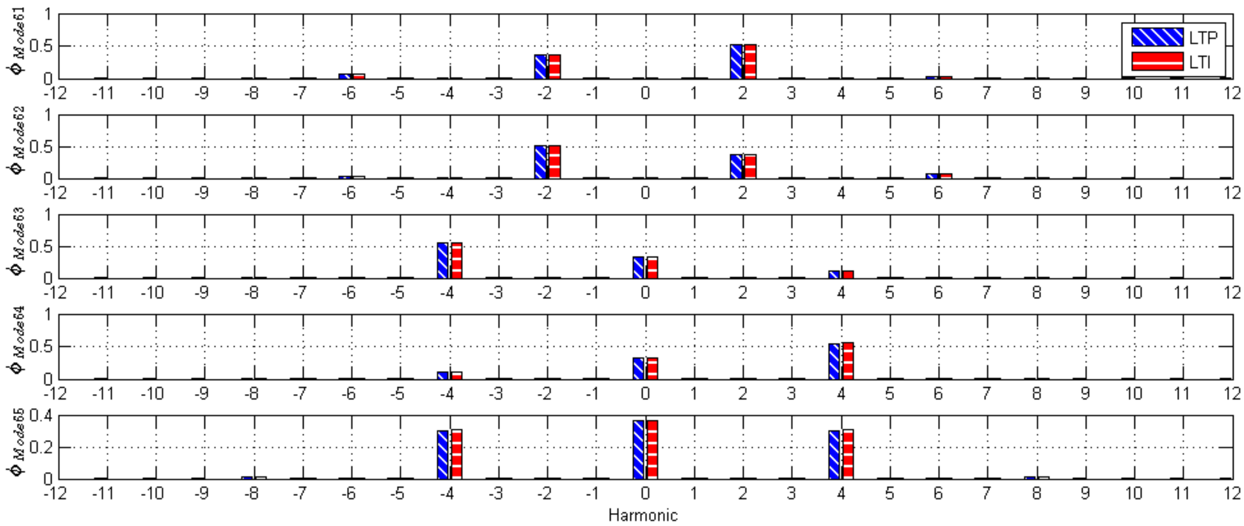

Figure 2. Modal Participation for Rotor Coning

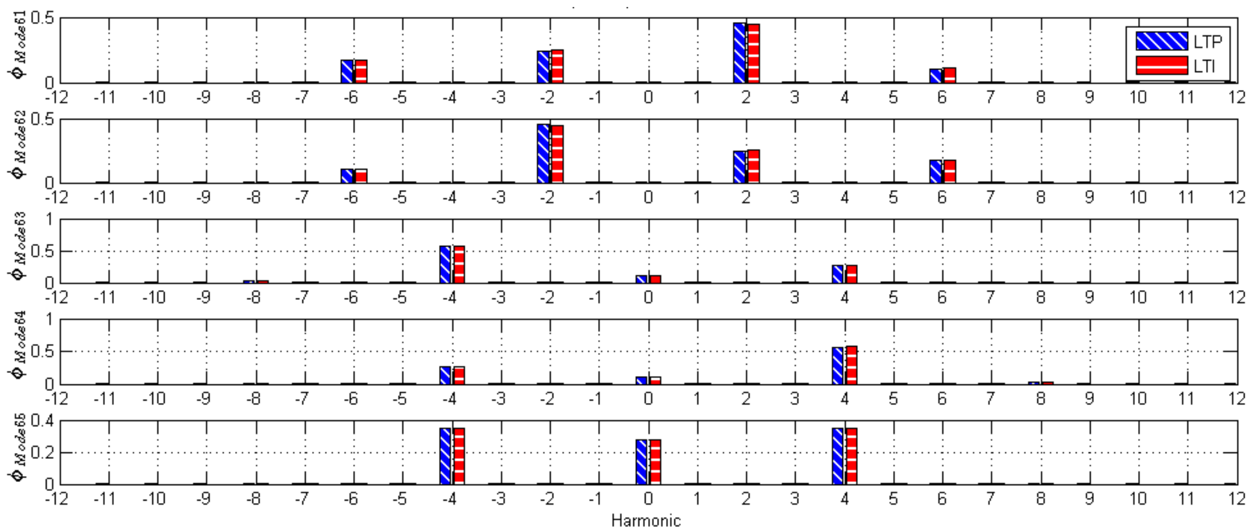

Figure 3. Modal Participation for Average Inflow

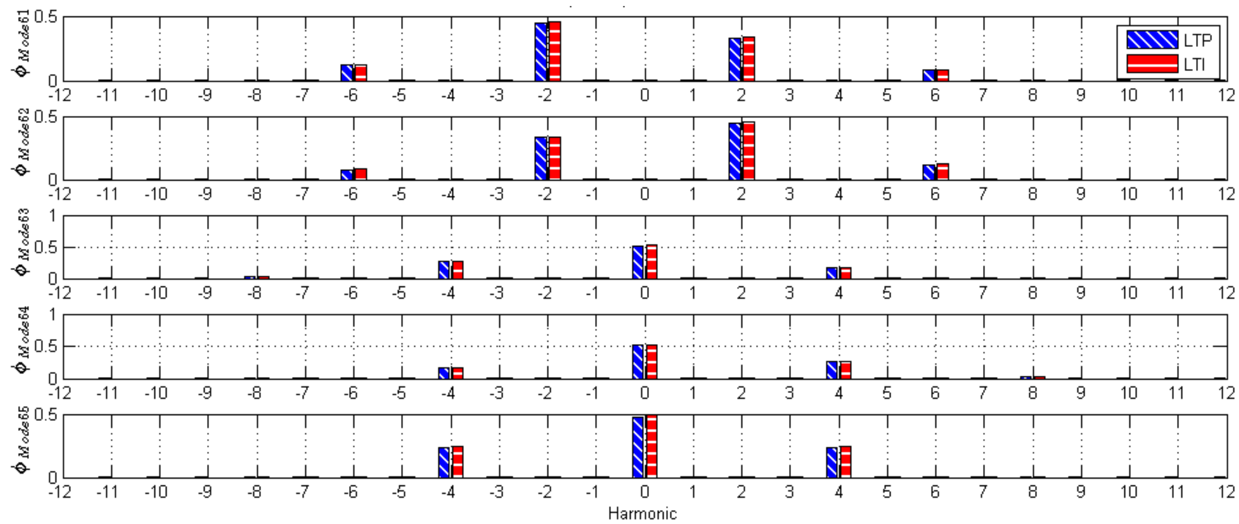

Figure 4. Modal Participation for Pitch Attitude

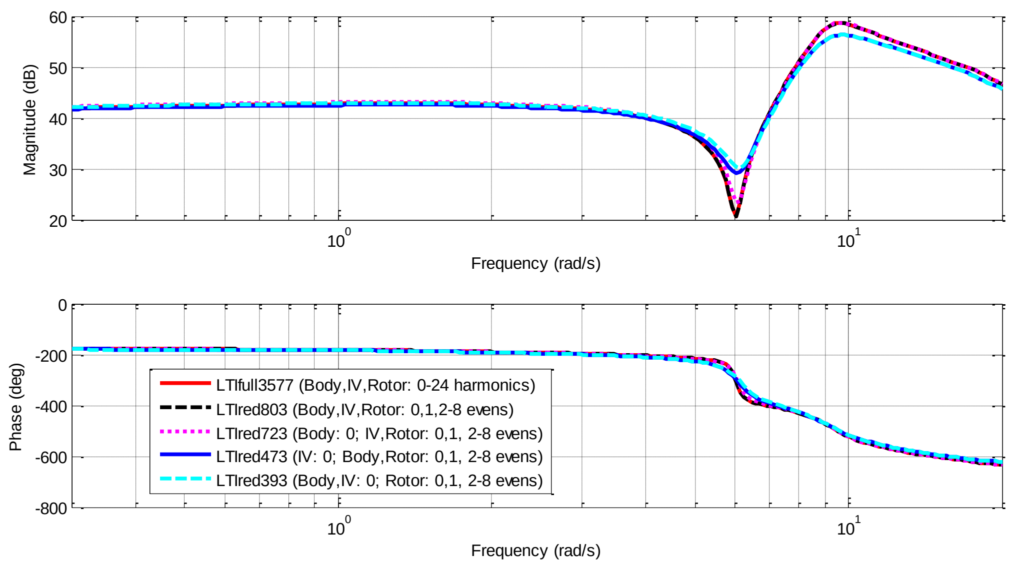

Figure 5. Frequency Response Comparison for IBC4C to Fx4C

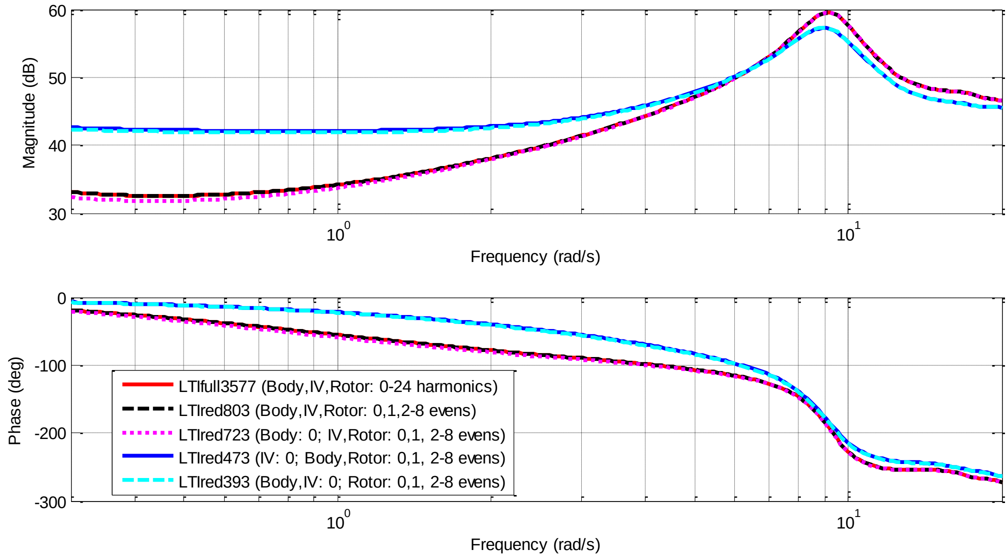

Figure 6. Frequency Response Comparison for IBC4C to Fy4C

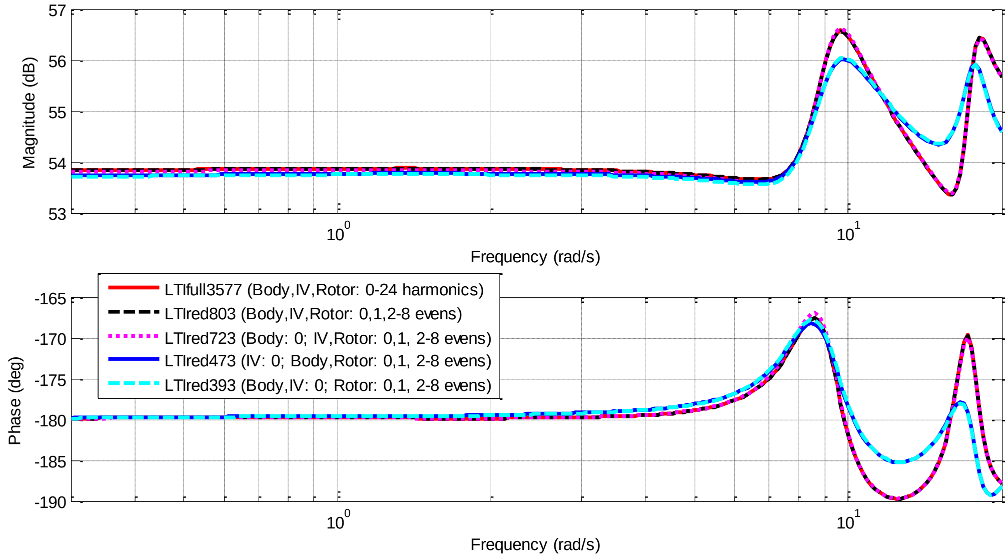

Figure 7. Frequency Response Comparison for IBC4C to Fz4C

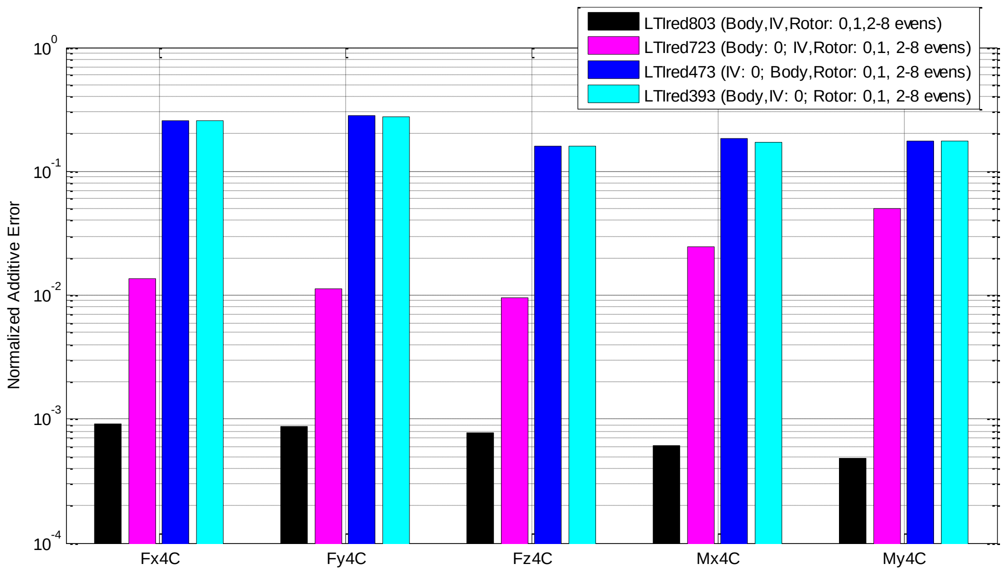

Figure 8. Normalized Additive Error Comparison for IBC4C to 4C Outputs

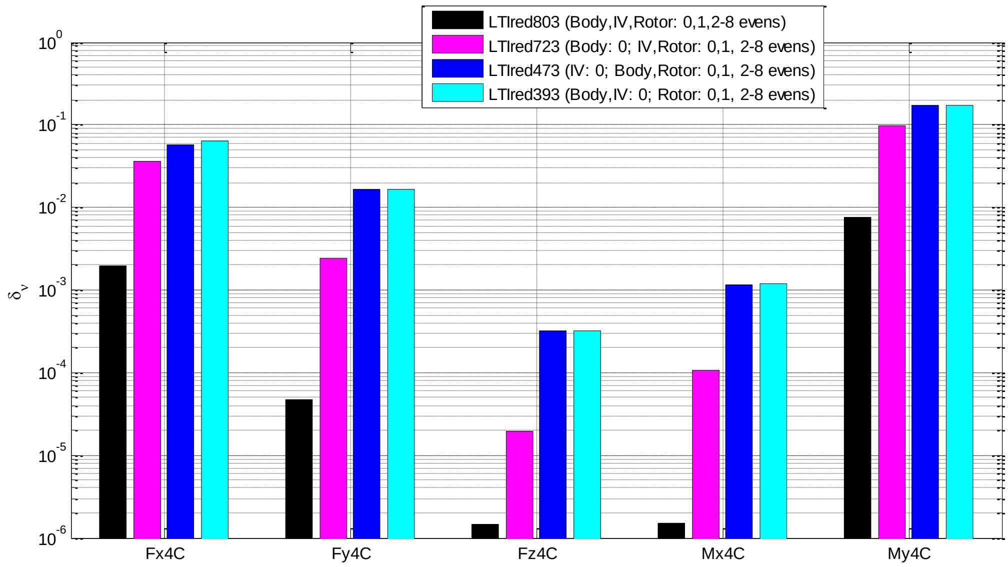

Figure 9. Nu Gap Metric Comparison for IBC4C to 4C Outputs

11. APPENDIX

The LTI model extraction for an LTP model using a first order formulation is shown in full here.

11.1 State Equation of LTI Model

Consider a Linear Time Periodic (LTP) Model with the state equation given as

(12) $$\dot{x} = F(\psi)x + G(\psi)u$$

An LTP model can be obtained from a nonlinear model using a perturbation scheme, linearizing about a periodic equilibrium at every azimuthal position [3]. In order to extract an approximate LTI model from Eq. (12), consider the following approximation of x:

(13) $$x = x_o + \sum_{n=1}^{N} x_{nc}\cos n\psi + x_{ns}\sin n\psi$$

where x_o is the average component and x_{nc} and x_{ns} are respectively the n/rev cosine and sine harmonic components of x. Likewise, control (u) is expanded in terms of harmonic components as

(14) $$u = u_o + \sum_{m=1}^{M} u_{mc}\cos m\psi + u_{ms}\sin m\psi$$

Differentiation of Eq. (13) with respect to time results in

(15) $$\dot{x} = \dot{x}_o + \sum_{n=1}^{N} x'_{nc}\cos n\psi + x'_{ns}\sin n\psi$$

where

(16) $$x'_{nc} = \dot{x}_{nc} + n\Omega x_{ns} \qquad n = 1,2,.....,N$$

(17) $$x'_{ns} = \dot{x}_{ns} - n\Omega x_{nc} \qquad n = 1,2,.....,N$$

Substituting Eqs. (13), (14), (15) and (16) into Eq. (12) results in

(18)

$$\dot{x}_o + \sum_{n=1}^{N}(x'_{nc}\cos n\psi + x'_{ns}\sin n\psi) =$$
$$[F(\psi)](x_o + \sum_{n=1}^{N}(x_{nc}\cos n\psi + x_{ns}\sin n\psi))$$
$$+[G(\psi)](u_o + \sum_{m=1}^{M}(u_{mc}\cos m\psi + u_{ms}\sin m\psi))$$

Equations for the individual harmonic components of x can be obtained by multiplying Eq. (18) on both sides by cos iψ or sin iψ, i= 1, 2, …, N, and integrating the result over one rotor revolution. The equation for the average component (x_o) is obtained by integrating Eq. (18) over one rotor revolution.

(19)

$$\dot{x}_o = \frac{1}{2\pi}\int_0^{2\pi} \{[F(\psi)](x_o + \sum_{n=1}^{N}(x_{nc}\cos n\psi + x_{ns}\sin n\psi)) + [G(\psi)](u_o + \sum_{m=1}^{M}(u_{mc}\cos m\psi + u_{ms}\sin m\psi))\}d\psi$$

Likewise, the equation for the i[th] harmonic cosine component (x_{ic}) can be obtained as

(20)

$$x'_{ic} = \frac{1}{\pi}\int_0^{2\pi} \{[F(\psi)](x_o + \sum_{n=1}^{N}(x_{nc}\cos n\psi + x_{ns}\sin n\psi)) + [G(\psi)](u_o + \sum_{m=1}^{M}(u_{mc}\cos m\psi + u_{ms}\sin m\psi))\}\cos i\psi d\psi \qquad i = 1, 2, 3, \ldots, N$$

and the equation for the i[th] harmonic sine component (x_{is}) can be obtained as

(21)

$$x'_{is} = \frac{1}{\pi}\int_0^{2\pi} \{[F(\psi)](x_o + \sum_{n=1}^{N}(x_{nc}\cos n\psi + x_{ns}\sin n\psi)) + [G(\psi)](u_o + \sum_{m=1}^{M}(u_{mc}\cos m\psi + u_{ms}\sin m\psi))\}\sin i\psi d\psi \qquad i = 1, 2, 3, \ldots, N$$

Using the following notation

(22)

$$F^{nc}(\psi) = F(\psi)\cos n\psi$$
$$F^{ns}(\psi) = F(\psi)\sin n\psi$$
$$G^{mc}(\psi) = G(\psi)\cos m\psi$$
$$G^{ms}(\psi) = G(\psi)\sin m\psi$$
$$n = 1, 2, 3, \ldots, N \quad and \quad m = 1, 2, \ldots, M$$

and substituting Eq. (22) into Eqs. (19) - (21) yields

(23)

$$\dot{x}_o = \frac{1}{2\pi}\int_0^{2\pi} \{F(\psi)x_o + \sum_{n=1}^{N}(F^{nc}(\psi)x_{nc} + F^{ns}(\psi)x_{ns}) + G(\psi)u_o + \sum_{m=1}^{M}(G^{mc}(\psi)u_{mc} + G^{ms}(\psi)u_{ms})\}d\psi$$

(24)

$$\dot{x}_{ic} = -i\Omega x_{is} + \frac{1}{\pi}\int_0^{2\pi} \{F(\psi)x_o + \sum_{n=1}^{N}(F^{nc}(\psi)x_{nc} + F^{ns}(\psi)x_{ns}) + G(\psi)u_o + \sum_{m=1}^{M}(G^{mc}(\psi)u_{mc} + G^{ms}(\psi)u_{ms})\}\cos i\psi d\psi \qquad i = 1, 2, 3, \ldots, N$$

(25)

$$\dot{x}_{is} = i\Omega x_{ic} + \frac{1}{\pi}\int_0^{2\pi} \{F(\psi)x_o + \sum_{n=1}^{N}(F^{nc}(\psi)x_{nc} + F^{ns}(\psi)x_{ns}) + G(\psi)u_o + \sum_{m=1}^{M}(G^{mc}(\psi)u_{mc} + G^{ms}(\psi)u_{ms})\}\sin i\psi d\psi \qquad i = 1, 2, 3, \ldots, N$$

Now defining the following operators

(26)

$$H_{oM} = \frac{1}{2\pi}\int_0^{2\pi} M(\psi)d\psi$$
$$H_{icM} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\cos i\psi d\psi$$
$$H_{isM} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\sin i\psi d\psi$$
$$i = 1, 2, 3, \ldots, N$$

Eqs. (23), (24) and (25) can be written as

(27)

$$\dot{x}_o = H_{oF}x_o + \sum_{n=1}^{N}(H_{oF^{nc}}x_{nc} + H_{oF^{ns}}x_{ns}) + H_{oG}u_o + \sum_{m=1}^{M}(H_{oG^{mc}}u_{mc} + H_{oG^{ms}}u_{ms})$$

(28)

$$\dot{x}_{ic} = -i\Omega x_{is} + H_{icF}x_o + \sum_{n=1}^{N}(H_{icF^{nc}}x_{nc} + H_{icF^{ns}}x_{ns}) + H_{icG}u_o + \sum_{m=1}^{M}(H_{icG^{mc}}u_{mc} + H_{icG^{ms}}u_{ms}) \qquad i = 1, 2, 3, \ldots, N$$

(29)

$$\dot{x}_{is} = i\Omega x_{ic} + H_{isF} x_o + \sum_{n=1}^{N} (H_{isF^{nc}} x_{nc} + H_{isF^{ns}} x_{ns}) + H_{isG} u_o + \sum_{m=1}^{M} (H_{isG^{mc}} u_{mc} + H_{isG^{ms}} u_{ms})$$

$$i = 1, 2, 3, \ldots.., N$$

11.2 Output Equation of LTI Model

Given the output equation of a LTP model as

(30) $$y = P(\psi) x + R(\psi) u$$

an approximation to y in terms of its harmonic components is sought as

(31) $$y = y_o + \sum_{l=1}^{L} y_{lc} \cos l\psi + y_{ls} \sin l\psi$$

where y_o is the average component and y_{lc} and y_{ls} are respectively the l^{th} harmonic cosine and sine components of y. Substituting Eqs. (13) and (14) and (31) into Eq. (30) results in

(32)

$$y_o + \sum_{l=1}^{L} y_{lc} \cos l\psi + y_{ls} \sin l\psi) = [P(\psi)](x_o + \sum_{n=1}^{N} (x_{nc} \cos n\psi + x_{ns} \sin n\psi)) + [R(\psi)](u_o + \sum_{m=1}^{M} (u_{mc} \cos m\psi + u_{ms} \sin m\psi))$$

Eq. (32) is multiplied with $\cos l\psi$ or $\sin l\psi$, l=0, 1, 2,..,L and is integrated over one rotor revolution, resulting in the following expressions for y_o, y_{lc} and y_{ls}.

(33)

$$y_o = \frac{1}{2\pi} \int_0^{2\pi} \{[P(\psi)](x_o + \sum_{n=1}^{N} (x_{nc} \cos n\psi + x_{ns} \sin n\psi)) + [R(\psi)](u_o + \sum_{m=1}^{M} (u_{mc} \cos m\psi + u_{ms} \sin m\psi)\} d\psi$$

(34)

$$y_{lc} = \frac{1}{\pi} \int_0^{2\pi} \{[P(\psi)](x_o + \sum_{n=1}^{N} (x_{nc} \cos n\psi + x_{ns} \sin n\psi)) + [R(\psi)](u_o + \sum_{m=1}^{M} (u_{mc} \cos m\psi + u_{ms} \sin m\psi)\} \cos l\psi d\psi$$

$$l = 1,\ 2, 3, \ldots\ldots, L$$

(35)

$$y_{ls} = \frac{1}{\pi} \int_0^{2\pi} \{[P(\psi)](x_o + \sum_{n=1}^{N} (x_{nc} \cos n\psi + x_{ns} \sin n\psi)) + [R(\psi)](u_o + \sum_{m=1}^{M} (u_{mc} \cos m\psi + u_{ms} \sin m\psi)\} \sin l\psi d\psi$$

$$l = 1, 2, 3, \ldots.., L$$

Using similar notation as before, for example, P^{nc}=P(ψ)cosnψ, etc., and the H operator, yields

(36)

$$y_o = H_{oP} x_o + \sum_{n=1}^{N} (H_{oP^{nc}} x_{nc} + H_{oP^{ns}} x_{ns}) + H_{oR} u_o + \sum_{m=1}^{M} (H_{oR^{mc}} u_{mc} + H_{oR^{ms}} u_{ms})$$

(37)

$$y_{lc} = H_{lcP} x_o + \sum_{n=1}^{N} (H_{lcP^{nc}} x_{nc} + H_{lcP^{ns}} x_{ns}) + H_{lcR} u_o + \sum_{m=1}^{M} (H_{lcR^{mc}} u_{mc} + H_{lcR^{ms}} u_{ms})$$

$$l = 1, 2, 3, \ldots.., L$$

(38)

$$y_{ls} = H_{lsP} x_o + \sum_{n=1}^{N} (H_{lsP^{nc}} x_{nc} + H_{lsP^{ns}} x_{ns}) + H_{lsR} u_o + \sum_{m=1}^{M} (H_{lsR^{mc}} u_{mc} + H_{lsR^{ms}} u_{ms})$$

$$l = 1, 2, 3, \ldots.., L$$

11.3 LTI Models in Matrix Form

Equations (27) - (30) and (36) - (38) can be represented in matrix form by defining the augmented state vector as

$$X = \left[x_o^{\ T} .. x_{ic}^{\ T} \quad x_{is}^{\ T} .. x_{jc}^{\ T} \quad x_{js}^{\ T} .. \right]^T \tag{39}$$

and the augmented control vector as

$$U = \left[u_o^{\ T} .. \quad u_{mc}^{\ T} \quad u_{ms}^{\ T} .... \right]^T \tag{40}$$

where x_o is the zeroth harmonic component, x_{ic}, x_{is} are the i^{th} harmonic cosine and sine components of x and u_{mc}, u_{ms} are the m^{th} harmonic cosine and sine components of u, respectively. The state equation of the resulting LTI model is

$$\dot{X} = [A]X + [B]U \tag{41}$$

Likewise, the augmented output vector of the LTI model is defined as

$$Y = \left[y_o^{\ T} .. \quad y_{lc}^{\ T} \quad y_{ls}^{\ T} ... \right]^T \tag{42}$$

Then the output equation of the LTI model can be written as

$$Y = [C]X + [D]U \tag{43}$$

The LTI model matrices of Eqs. (41) and (43) are obtained as

$$A = \begin{bmatrix} H_{oF} ... H_{oF^{ic}} \quad H_{oF^{is}} ... H_{oF^{jc}} \quad H_{oF^{js}} .... \\ .... \\ H_{icF} ... H_{icF^{ic}} \quad -i\Omega + H_{icF^{is}} ... H_{icF^{jc}} \quad H_{icF^{js}} .... \\ H_{isF} ... i\Omega + H_{isF^{ic}} \quad H_{isF^{is}} .... H_{isF^{jc}} \quad H_{isF^{js}} .... \\ .... \\ H_{jcF} ... H_{jcF^{ic}} \quad H_{jcF^{is}} ... H_{jcF^{jc}} \quad -j\Omega + H_{jcF^{js}} ... \\ H_{jsF} ... H_{jsF^{ic}} \quad H_{jsF^{is}} ... j\Omega + H_{jsF^{jc}} \quad H_{jsF^{js}} ... \\ .... \end{bmatrix}$$

$$B = \begin{bmatrix} H_{oG} ... H_{oG^{mc}} \quad H_{oG^{ms}} ... \\ .... \\ H_{icG} ... H_{icG^{mc}} \quad H_{icG^{ms}} ... \\ H_{isG} ... H_{isG^{mc}} \quad H_{isG^{ms}} ... \\ .... \\ H_{jcG} ... H_{jcG^{mc}} \quad H_{jcG^{ms}} ... \\ H_{jsG} ... H_{jsG^{mc}} \quad H_{jsG^{ms}} ... \\ .... \end{bmatrix}$$

$$C = \begin{bmatrix} H_{oP} ..... H_{oP^{ic}} \quad H_{oP^{is}} .... H_{oP^{jc}} \quad H_{oP^{js}} .... \\ .... \\ H_{lcP} ... H_{lcP^{ic}} \quad H_{lcP^{is}} .. H_{lcQ^{jc}} \quad H_{lcP^{js}} ...... \\ H_{lsP} .... H_{lsP^{ic}} \quad H_{lsP^{is}} ... H_{lsP^{jc}} \quad H_{lsP^{js}} ...... \\ .... \end{bmatrix}$$

$$D = \begin{bmatrix} H_{oR} ... H_{oR^{mc}} \quad H_{oR^{ms}} ... \\ .... \\ H_{lcR} ... H_{lcR^{mc}} \quad H_{lcR^{ms}} ... \\ H_{lsR} ... H_{lsR^{mc}} \quad H_{lsR^{ms}} ... \\ .... \end{bmatrix}$$

11.4 Closed Form Expressions for LTI Model

Closed form expressions for various terms in the A, B, C and D matrices above can be obtained if one considers harmonic expansions of the LTP model matrices. If a time periodic matrix M(ψ) is expanded in terms of its harmonic components as

$$M(\psi) = M_o + \sum_{k=1}^{\infty} (M_{kc} \cos k\psi + M_{ks} \sin k\psi) \tag{44}$$

then it can be shown that

(45)

$$H_{oM} = \frac{1}{2\pi}\int_0^{2\pi} M(\psi)d\psi = M_o$$

$$H_{oM^{ic}} = \frac{1}{2\pi}\int_0^{2\pi} M(\psi)\cos i\psi\, d\psi = \frac{M_{ic}}{2}$$

$$H_{oM^{is}} = \frac{1}{2\pi}\int_0^{2\pi} M(\psi)\sin i\psi\, d\psi = \frac{M_{is}}{2}$$

$$i = 1,2\,3,.......$$

(46)

$$H_{icM} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\cos i\psi\, d\psi = M_{ic}$$

$$H_{isM} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\sin i\psi\, d\psi = M_{is}$$

$$i = 1,2\,3,.......$$

(47)

$$H_{icM^{jc}} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\cos j\psi \cos i\psi\, d\psi$$

$$= M_o + \frac{M_{kc}}{2} \quad \text{for } i = j \text{ where } k = i + j$$

$$= \frac{M_{kc} + M_{lc}}{2} \quad \text{for } i \neq j \text{ and } i > j$$

$$\text{where } k = i + j,\ l = i - j$$

$$= \frac{M_{kc} + M_{mc}}{2} \quad \text{for } i \neq j \text{ and } j > i$$

$$\text{where } k = i + j,\ m = j - i$$

$$i = 1,2,..\ \text{and}\ j = 1,2,..$$

(48)

$$H_{icM^{js}} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\sin j\psi \cos i\psi\, d\psi$$

$$= \frac{M_{ks}}{2} \quad \text{for } i = j \text{ where } k = i + j$$

$$= \frac{M_{ks} - M_{ls}}{2} \quad \text{for } i \neq j \text{ and } i > j$$

$$\text{where } k = i + j,\ l = i - j$$

$$= \frac{M_{ks} + M_{ms}}{2} \quad \text{for } i \neq j \text{ and } j > i$$

$$\text{where } k = i + j,\ m = j - i$$

$$i = 1,2,..\text{and}\ j = 1,2,....$$

(49)

$$H_{isM^{jc}} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\cos j\psi \sin i\psi\, d\psi$$

$$= \frac{M_{ks}}{2} \quad \text{for } i = j \text{ where } k = i + j$$

$$= \frac{M_{ks} + M_{ls}}{2} \quad \text{for } i \neq j \text{ and } i > j$$

$$\text{where } k = i + j,\ l = i - j$$

$$= \frac{M_{ks} - M_{ms}}{2} \quad \text{for } i \neq j \text{ and } j > i$$

$$\text{where } k = i + j,\ m = j - i$$

$$i = 1,2,...\ \text{and}\ j = 1,2,....$$

(50)

$$H_{isM^{js}} = \frac{1}{\pi}\int_0^{2\pi} M(\psi)\sin j\psi \sin i\psi\, d\psi$$

$$= M_o - \frac{M_{kc}}{2} \quad \text{for } i = j \text{ where } k = i + j$$

$$= \frac{M_{lc} - M_{kc}}{2} \quad \text{for } i \neq j \text{ and } i > j$$

$$\text{where } k = i + j,\ l = i - j$$

$$= \frac{M_{mc} - M_{kc}}{2} \quad \text{for } i \neq j \text{ and } j > i$$

$$\text{where } k = i + j,\ m = j - i$$

$$i = 1,2,..\ \text{and}\ j = 1,2,..$$