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RESEARCH ON THE STRESS ANALYSIS METHOD OF RUBBER STRUCTURE

— CALCULATION OF THE FREQUENCY ADAPTER STRESSES

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Abstract

In this paper, the incremental method and Newton-Raphson iteration method are used for calculating the frequency adapter stresses. The F.E. method is used to solve the stresses of the rubber structure that can be simplified as the problem of plane strain. All the formulations are based on strain energy function, considering non-linear relation of the stress-strain of rubber materials, the nonlinear relation of displacement-strain and incompressibility of the rubber materials.

1. Introduction

The frequency adapter which is made of stainless steel, aluminum alloy and silicone-rubber is a important part of the rotor-hub of helicopter. The calculations of the parts are complicated and difficult problems because of : (1) It is hard to model the parts made of several kinds of materials, (2) Rubber material is hyperelastic, its stress-strain relation can generally expressed by a strain energy function of three strain invariants I_1, I_2 and I_3 Which are very complicated nonlinear functions, (3) The deformation of structural parts of rubber associates generally with large displacements and large strains. When forces act on them, therefore, the strain-displacement relation is also a nonlinear function, (4) During deformation of rubber, the volume of rubber does not change obviously, this material is taken as incompressible. The stress tensor is not determined by the strains only. The hytrostatic pressure which does not influence the deformation must be considered when calculating the stress tensor.

Due to the nonlinearity of geometry and physics and incompressibility of rubber complicated nonlinear equations are obtained by F.E.M. after discretization. To solve them is very difficult. According to real size of frequency adapter and its working condition, a spatial problem is simplified to a plane one, All the formulations are based on the strain energy. The method of seperation of dosplacement and pressure is used, overcoming the difficulty that there are the zero elements at the diagonal in the structural tangential stiffness matrix. Combined Newton-Raphson procedure with incremental procedure is used.

2. Structure of frequency adapter and its simplification

The structure figure of frequency adapter of the hub of helicopter is shown in figure 1. Its functions which are similar to frictional adapter or oil adapter in metal hub allow to swing and damp the shake to a blade consume energy. Therefore it is subjected to shear forces associated with shear deformation along the swing direction. The maximum shear deformation is 6 ± 3.6 mm. The axial and shear forces along the direction of axis and flapping, compared with them, are small and can be neglected.

Because stiffness of stainless steel and aluminum alloy is much larger than rubber, the steel and aluminum alloy are considered as rigid, only rubber is subjected to deformation. When forces act on total adapter, structural part which is made of three kinds of materials is simplified to one that is made of rubber and the rubber is analysed.

Three ribber parts are very regular, their sizes are 10.4*72*105(mm). The sizes of two directions are considered as much larger than one of the 3rd direction. Assuming shear forces and deformations along length direction are uniform and plane strain problem is then obtained, providing convenience of calculation and saving computation time.

Simplified model is shown in Fig. 2. In order to compare with analytical results, as an example, the mesh of calculation model is shown in Fig. 3.

3. Formulation of calculation

3-1. Constitutive relation of rubber materials

Rubber materials obey Mooney's law, the stresses are described by strain energy function

$$U = C_1 \ (I_1 - 3) \ + C_2 \ (I_2 - 3) \tag{1}$$

in which I_1 and I_2 are the 1st and 2nd invariants, C_1 and C_2 are material constants obtained by experiments.

Considering the incompressibility of rubber, the modified strain energy function

$$\tilde{U} = U\ (I_1,\ I_2)\ + P\ (I_3 - 1) \tag{2}$$

is used, in which I_3 is the 3rd straininvariant, P is hydrostatic pressure (tension). Stresses are computed by

$$\{\tau\} = \frac{\partial \tilde{U}}{\partial \{\varepsilon\}} \tag{3}$$

$$\{\tau\}^T = \{\tau^{11},\ \tau^{12},\ \tau^{21},\ \tau^{22}\} \tag{4}$$

3-2. Strain-displacement relations for plane strain

Let u, v and w be the displacement components along x, y and z direction. For plane strain problem, u = u(x,y), v = v(x,y), w = 0. Green strain is written as

$$\{\varepsilon\} = \{\varepsilon_0\} + \{\varepsilon_l\} \tag{5}$$

in which $\{\varepsilon_0\}$ is the linear strain, $\{\varepsilon_l\}$ is the nonlinear one

$$\{\varepsilon\}^T = \{\gamma_{11},\ \gamma_{12},\ \gamma_{21},\ \gamma_{22}\} \tag{6}$$

$$\{\varepsilon_0\} = [H]\{A\} \tag{7}$$

$$[H] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \{A\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{Bmatrix}$$

$$\{\varepsilon_l\} = \frac{1}{2}[C]\{A\} \tag{8}$$

in which

$$[C] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & 0 & 0 \\ \frac{1}{2}\frac{\partial u}{\partial y} & \frac{1}{2}\frac{\partial v}{\partial y} & \frac{1}{2}\frac{\partial u}{\partial x} & \frac{1}{2}\frac{\partial v}{\partial x} \\ \frac{1}{2}\frac{\partial u}{\partial y} & \frac{1}{2}\frac{\partial v}{\partial y} & \frac{1}{2}\frac{\partial u}{\partial x} & \frac{1}{2}\frac{\partial v}{\partial x} \\ 0 & 0 & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$$

After calculation we get

$$d\{\varepsilon_l\} = [C] d\{A\} \tag{9}$$

$$d\{\varepsilon\} = ([H] + [C]) d\{A\} \tag{10}$$

3-3. Equilibrium equations

By means of the principle of virtual displacements, equilibrium equations of an element are obtained as

$$\int_{V_e} d\{\varepsilon_e\}^T\{\tau_e\}\, dv = dW_e = d\{\psi_e\}^T\{F_e\} \tag{11}$$

in which V_e is the undeformed volume of element, dW_e is the external virtual work. $\{\psi_e\}$, $\{F_e\}$ are nodal forces, respectively. According to element interpolation functions, let

$$d\{A_e\} = [G_e] d\{\psi_e\} \tag{12}$$

in which matrix G_e is obtained by element interpolation functions. Therefore

$$d\{\varepsilon_e\} = [B_e] d\{\psi_e\} \tag{13}$$

in which

$$[B_e] = ([H_e] + [C_e])[G_e] \tag{14}$$

By substituting eq. (14) into (11) virtual displacement equations

$$\int_{V_e} [B_e]^T\{\tau_e\}\, dV = \{F_e\} \tag{15}$$

are obtained. The incompressibility in an average sense over the element is

$$\int_{V_e} (I_{3e} - 1)\, dV = 0 \tag{16}$$

3-4. Tangent stiffness matrix and residuals

Combined incremental/iterative method is chosen. The eq. (15) corresponding to load level n reads

$$\int_{V_e} [B_e]_n^T\{\tau_e\}\, dV = \{F_e\}_n = \{F_e\}_{n-1} + \{\Delta F_e\}_n \tag{17}$$

with an approximate solution at iteration step m, $\{\psi_e\}_m$ and $\{p_e\}_m$
The residual loads of eq. (17) is

$$\{R_e\} = \int_{V_e} [B_e]_{nm}^T\{\tau_e\}_{nm}\, dV - \{F_e\}_n \tag{18}$$

By using of Newton-Raphson procedure the equations

$$\{\Delta R_e\} = -\{R_e\}_m \tag{19}$$

are obtained From eq. (18) we get equations

$$\{\Delta R_e\}_m = \int_{V_e} [\Delta B_e]_m^T\{\tau_e\}_m + [B_e]_m^T\{\Delta\tau_e\}_m)\, dV \tag{20}$$

By calculation the 1st term in eq. (20) is

$$[\Delta B_e]_m^T\{\tau_e\}_m=[G_e]^T[M_e]_m\{\Delta A_e\}_m=[G_e]^T[M_e]_m[G_e]\{\Delta\psi_e\}_m \tag{21}$$

in which

$$[M_e]=\begin{vmatrix} \tau^{11} & 0 & (\tau^{12}+\tau^{21})/2 & 0 \\ 0 & \tau^{11} & 0 & (\tau^{12}+\tau^{21})/2 \\ (\tau^{12}+\tau^{21})/2 & 0 & \tau^{22} & 0 \\ 0 & (\tau^{12}+\tau^{21})/2 & 0 & \tau^{22} \end{vmatrix} \tag{22}$$

The 2nd term in eq. (20) involves $\{\Delta\tau_e\}$, from eq.(2) and (3) we get

$$\{\Delta\tau_e\}=[E_e]\{\Delta\varepsilon_e\}+\{L_e\}\Delta p_e \tag{23}$$

with the matrix E_e and vector L_e defined by

$$[E_e]=\frac{\partial^2 U}{\partial[\varepsilon_e]^2}+p\frac{\partial^2 I_3}{\partial[\varepsilon_e]^2}, \tag{24}$$

$$[L_e]=\frac{\partial I_3}{\partial\{\tau_e\}} \tag{25}$$

At last incremental residual is written as

$$\{\Delta R\}_m=(\int_{Ve}[G_e]^T[G_M]_m[G_e]dV+\int_{Ve}[B_e]_m[E_e][B]dV)\{\Delta\psi\} + (\int_{Ve}[B_e]_m^T\{L_e\}_m dV)(\Delta P_e)_m \tag{26}$$

The incompressibility condition has also to be accounted for iteratively. From residual

$$(r_e)_m=\int_{Ve}(I_{3e})_m dV-V_e \tag{27}$$

we get equation

$$(\Delta r_e)_m=\int_{Ve}(\Delta I_3)_m dV \tag{28}$$

in which

$$(\Delta I_{3e})_m=\{\Delta\varepsilon_e\}_m^T\cdot\frac{\partial I_{3e}}{\partial\{\varepsilon_e\}}=\{\Delta\psi_e\}_m^T[B_e]_m^T\{L_e\}_m \tag{29}$$

The element tangent stiffness matrix equation is

$$\begin{pmatrix}[K_e]_m & \{T_e\}_m \\ \{T_e\}_m^T & 0\end{pmatrix}\begin{Bmatrix}\{\Delta\psi_e\}_m \\ \{\Delta P_e\}_m\end{Bmatrix}=-\begin{Bmatrix}\{R_e\}_m \\ (r_e)_m\end{Bmatrix} \tag{30}$$

in which

$$[K_e]_m=\int_{Ve}([G_e]^T[M_e]_m[G_e]+[B_e]_m^T[E_e]_m[B_e]_m)dV \tag{31}$$

$$\{T_e\}_m=\int_{Ve}[B_e]_m^T\{L_e\}_m dV \tag{32}$$

4. Iterative solution

From eq. (30) the total tangent stiffness equation

$$\begin{pmatrix}[K]_m & [T]_m \\ [T]_m^T & [O]\end{pmatrix}\begin{Bmatrix}\{\Delta\psi\} \\ \{\Delta P\}\end{Bmatrix}_m=-\begin{Bmatrix}\{R^I\} \\ \{R^I\}\end{Bmatrix}_m \tag{33}$$

are formed. The indexes l and i on the right-hand-side refer to load residuals and incompressibility residuals.

With proper boundary conditions, eq. (33) can be solved iteratively

$$\left.\begin{aligned}\{\psi\}_{m+1} &= \{\psi\}_m + \{\Delta\psi\}_m \\ \{P\}_{m+1} &= \{P\}_m + \{\Delta P\}_m\end{aligned}\right\} \tag{34}$$

until the residuals R^l and R^i are sufficiently small.

From eq. (33), omitting the iteration index m, we get

$$[K]\{\Delta\psi\} + [T]\{\Delta P\} = -\{R^l\} \tag{35}$$

$$[T]^t\{\Delta\psi\} = -\{R^i\} \tag{36}$$

The solution to eq. (35) is

$$\{\Delta\psi\} = -[K]^{-1}(\{R^l\} + [T]\{\Delta P\}) \tag{37}$$

Which, after being inserted in (36), gives

$$[D]\{\Delta P\} = \{Q\} \tag{38}$$

with

$$[D] = [T]^t[K]^{-1}[T] \tag{39}$$

$$-[Q] = [T]^t[K]^{-1}\{R^l\} + \{R^i\} \tag{40}$$

Eq. (38) can be solved if the conditions given in ref. (3) are avoided.

In summary, by solving eq. (38) we get $\{\Delta P\}$ which is then substituted into (37) to find $\{\Delta\psi\}$.

5. Test example and numerical solution

In order to compare with analytical solution, the test example which is simplified as plane strain and is modeled by 9 quadrilateral elements is shown in Fig. 3. The lower boundary nodes 1009-1012 are fixed. A known displacement only along x direction is given at the upper boundary nodes 1001-1004. After deformation the rest nodes on boundary are modev to make right and left boundary straight.

The element shape functions are

$$N_1 = (a-x)(b-y)/(4ab), \qquad N_2 = (a+x)(b-y)/(4ab)$$
$$N_3 = (a+x)(b+y)/(4ab), \qquad N_4 = (a-x)(b+y)/(4ab)$$

and then

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u' \\ v' \end{Bmatrix}$$

$$\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \{u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4\}^T$$

$$\{A\}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{bmatrix}^T$$

$$=\frac{1}{4ab}\begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & (b+y) & 0 & -(b+y) & 0 \\ 0 & -(b-y) & 0 & (b-y) & 0 & (b+y) & 0 & -(b+y) \\ -(a-x) & 0 & -(a+x) & 0 & (a+x) & 0 & (a-x) & 0 \\ 0 & -(a-x) & 0 & -(a+x) & 0 & (a+x) & 0 & (a+x) \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$[E_e]=4(C_2+p)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \{L_e\}=\begin{Bmatrix} 2+r_{22} \\ -4r_1 \\ -4r_1 \\ 2+r_{11} \end{Bmatrix}$$

$$\{\tau\}=\begin{Bmatrix} \tau_{11} \\ \tau_{12} \\ \tau_{21} \\ \tau_{22} \end{Bmatrix}=\begin{Bmatrix} 2(C_1+2C_2+p)+4(C_2+p)r_{22} \\ -4(C_2+p)r_{21} \\ -4(C_2+p)r_{12} \\ 2(C_1+2C_2+p)+4(C_2+p)r_{11} \end{Bmatrix}$$

The results of F.E.M. compared with analytical solution are shown in table 1. From the table the maximum error 1.36% of stresses can be seen. The result accuracy is satisfactory.

When the convergency is reached the absolute value of unbalanced nodal forces is less than 10^{-15} kg. The equilibrium condition is also satisfied.

In order to illustrate general purpose of the programme compared calculations have been done for the structure. One is conducted with given displacements and the other with given forces shown in Fig. 4 and 5. The maximum departure of displacements at the corresponding nodes is less than 1%, the pressure departure in the corresponding element is less than 2%.

6. Conclusion

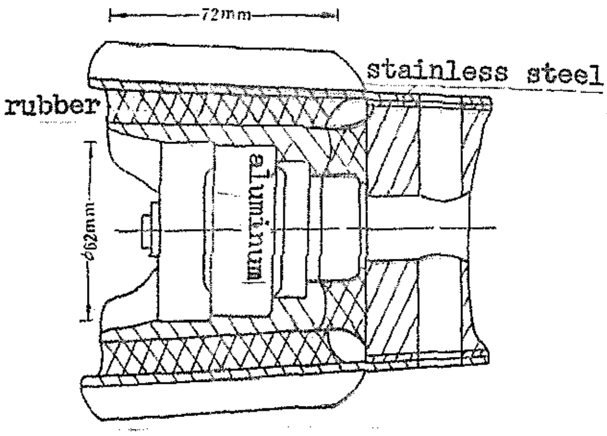

fig.1

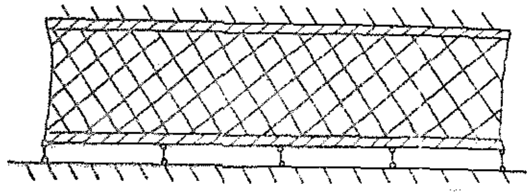

fig.2

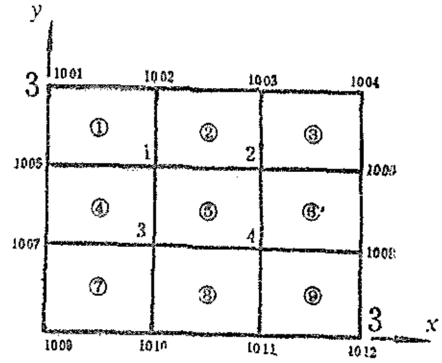

fig.3

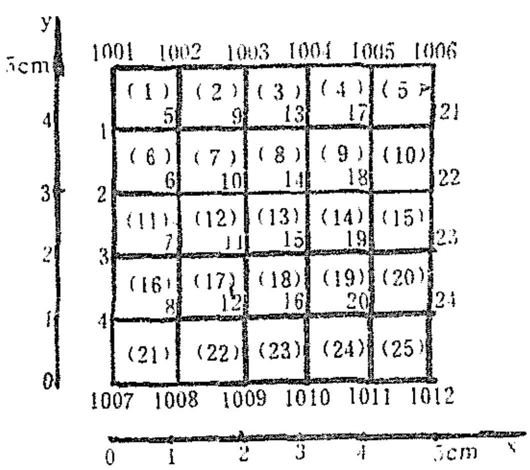

fig.4

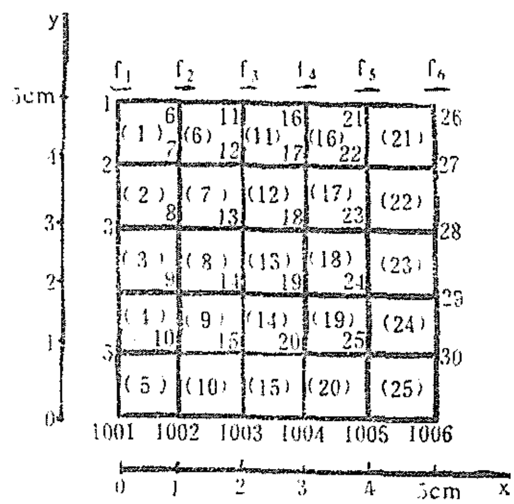

fig.5

table 1.

Δ	τ^{11}		τ^{12}		τ^{22}	
	analytical	present	analytical	present	analytical	present
0.05	16.88467	16.88467	−0.14061	−0.140616	16.87764	16.87764
0.3	16.9027	17.1305	−0.8437	−0.843696	16.87764	16.87764
0.5	17.5806	17.58072	−1.4061	−1.40616	16.87764	16.87764
0.8	18.6772	18.67752	−2.2498	−2.24986	16.87764	16.87764
1.0	19.7488	19.68986	−2.8122	−2.81232	16.87764	16.87764

By means of several examples and in comparison with analytical solution we think the computation model of rubber material is proper, the accuracy is satisfactory, the programme that we have designed is of general purpose. This method can be generalized to structural calculations with large deformation.

References

1. J. T. Oden	Finite Element of Nonlinear Continua McGraw-Hill, 1972
2. C. C. Zienkiewicz	The Finite Element Method, 3rd Edition McGraw-Hill, 1977
3. Lu Hexiang	Analysis of Axisymmetric large deformation of rubber Journal DII. No.1, 1984
4. I. M. Ward	Mechanical Properties of Solid Polymers Wiley-Inter-Science, 1971
5. E. Jankouich F. Leblane M. Durand	A Finite Element Method for the Analytical Assessment Computers & Structures, 14, 1981