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ON THE STATIC PRESSURE IN THE WAKE OF A HOVERING ROTOR

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1. Introduction

The purpose of this paper is to challenge a basic assumption that the static pressure in the ultimate wake of a propeller or of a rotor in hovering or axial flight is the same as that of the surrounding air. The inquiry arose when trying to reconsile some results given in a paper by Theodorsen[1] with those of classical momentum theory. Theodorsen's paper showed, amongst other things, that for a hovering rotor with an infinite number of blades the ratio of the induced velocity in the wake to that at the rotor is 3/2, whereas it is well-known that a value of 2 is obtained from the momentum theory, with corresponding differences in the contraction ratio. Rather surprisingly Theodorsen does not comment on these apparent discrepancies but a perusal of his analysis shows that the difference can be attributed to the use of a pressure equation which implies that the static pressure in the slipstream of an airscrew is generally higher than that of the ambient air.

This paper considers the theoretical justification and implications of Theordorsen's pressure equation and describes some measurements to see if the predicted increase of pressure occurs in practice.

2. The pressure equation

The paper by Theodorsen referred to above is an extension of his much earlier work on propeller design and described in his book[2]. It is not necessary to go into details here of his method of propeller design except to say that it is based on consideration of the motion of the ideal helical wake which is supposed to be generated by the desired optimum propeller.

Consider such a wake, fig.1, moving relative to the air at rest with (self induced) velocity w. To an observer at rest relative to the undisturbed air the passage of the vortex wake gives rise to an unsteady flow and the appropriate pressure equation is

$$p + \tfrac{1}{2}\rho v^2 + \rho\frac{\partial\phi}{\partial t} = p_o \qquad (1)$$

where v is the local fluid velocity (not necessarily parallel to the propeller axis) and p_o is the pressure in the undisturbed flow. Now for uniformly moving helical vortex sheets the flow relative to fixed axes will appear to be periodic so that, if z is the axial displacement, we can write

$$= f(z - wt)$$

giving

$$\frac{\partial\phi}{\partial t} = -w\frac{\partial\phi}{\partial z} = -wv_z$$

where v_z is the component of fluid velocity parallel to the propeller axis. Hence

$$p + \tfrac{1}{2}\rho v^2 = p_o + \rho w v_z \qquad (2)$$

This is the pressure equation given by Theodorsen in ref.2, valid for points within and without the wake. What is usually called the "slipstream" is none other than the velocity field caused by the motion of the sheets.

The significance of this equation can be seen by considering the hovering helicopter rotor. In this case the vortex sheets are fairly close together and the helix angle is quite small, say about 10 degs. Then, except near the edges of the sheets, where the radial component of velocity may be quite large, the velocity of the air between the vortex sheets will be practically equal to the sheets themselves, i.e. v_z and v are approximately equal to w, and w can be taken as the induced velocity in the ultimate wake. Hence, equation(2) can be written as

$$p - p_o = \tfrac{1}{2}\rho w^2 \qquad (3)$$

showing that the pressure in the ultimate slipstream exceeds the ambient pressure by the slipstream dynamic head. Theodorsen calls this difference the "overpressure". It is well known that, for a given thrust, the induced velocity diminishes rapidly with rotor axial velocity so that at high axial speeds the static pressure in the wake becomes nearly equal to the ambient pressure,as is generally assumed.

The particular form of the pressure equation (5) can be obtained in another way. If the axes of reference are allowed to move with the sheets, the flow relative to these axes is now steady and the total head pressure must be the same whether within the slipstream or not. The total head pressure at points far from the wake is clearly $p_o + \frac{1}{2}\rho w^2$. For points between the sheets, and sufficiently far from the edges, the air is practically at rest relative to the sheets, and therefore to the axes, so that the total head pressure is equal to the static pressure p. Hence, as before

$$p = p_o + \tfrac{1}{2}\rho w^2$$

Since the dynamic head in the final wake is $\frac{1}{2}\rho w^2$, it follows that the static pressure is half the total head pressure relative to atmospheric.

To calculate the relationship between the thrust and induced velocity for a rotor having an infinite number of blades we can apply the usual method of the momentum theory but with the static pressure in the ultimate wake given by eqn. (3). Let p_1 and p_2 denote the pressures just in front of and just behind the disc and let p_3 be the pressure in the far wake. Let v_i be the induced velocity at the disc and w be the induced velocity in the far wake which, since the helix angle of the sheets should be small, can be taken as the velocity of the sheets also. Then,if p_o is the pressure far from the rotor, Bernoulli's equation for points far upstream of the disc is

$$p_o = p_1 + \tfrac{1}{2}\rho v_i^2 \qquad (4)$$

and for points downstream

$$p_3 + \tfrac{1}{2}\rho w^2 = p_2 + \tfrac{1}{2}\rho v_i^2 \qquad (5)$$

Subtracting (4) from (5) gives

$$p_2 - p_1 = p_3 - p_o + \tfrac{1}{2}\rho w^2 \qquad (6)$$

If p_3 is given by equation (3), (6) becomes

$$p_2 - p_1 = \rho w^2 \qquad (7)$$

Let A be the rotor disc area and A_3 the cross-section area of the final wake. The forces acting on the air passing through the disc are the thrust (p_2-p_1) A and that due to the "overpressure" p_3A_3 acting in the upstream direction. The momentum equation is therefore

$$(p_2 - p_1)A - p_3A_3 = \rho A v_i w \qquad (8)$$

But, from continuity,

$$A_3 = A\frac{v_i}{w}$$

Therefore

$$p_2 - p_1 = (p_3 - p_o)\frac{v_i}{w} + \rho v_i w$$

$$= \tfrac{1}{2}\rho v_i w + \rho v_i w$$

$$= \frac{3}{2}\rho v_i w \qquad (9)$$

Equating (9) and (7) gives

$$w = \frac{3}{2} v_i \qquad (10)$$

that is, the induced velocity in the final wake is 3/2 times the value at the rotor disc instead of twice the value as given by the conventional momentum theory. The result given by equation(10) agrees with that of Theodorsen[1] for the special case of an infinite number of blades. Since the thrust can be written

$$T = (p_2 - p_1)A$$

we also have from (9) and (10) that

$$T = \frac{9}{4} \rho A v_i^2 \qquad (11)$$

instead of $T = 2\rho A v_i^2$. The contraction ratio $\frac{A_3}{A}$ is equal to $\sqrt{\frac{2}{3}}$ (=0.816..) instead of $\frac{1}{\sqrt{2}}$ (= 0.707). According to equation (11) the effect of the "overpressure" is to reduce the induced velocity, for a given thrust, by about 6 per cent compared with the value given by the conventional momentum theory.

It should be emphasised that Theodorsen's pressure equation has been derived on the assumption that the wake is uniform and moving steadily. The flow studies of Gray[3] and Landgrebe[4], however, show that the wake is far from uniform, the vortex sheets becoming inclined to the rotor axis at quite large angles, fig.2. The figure also shows strong tip vortices which,according to the flow studies, move away from the rotor more slowly than the inner vortex sheets. Landgrebe[4] also finds that the wake becomes unstable within a distance of less than half the rotor radius. It is not known whether this is due to viscous dissipation or if it indicates vortex breakdown (bursting). The wake is usually fully contracted before the instability occurs.

3. Measurements of the static pressure

Since the theory of the vortex wake predicts a wake "overpressure", even when the number of blades is infinite, it is natural to enquire if measurements have been made which confirm this theoretical expectation. A search through the available literature, which was confined mainly to British and American government publications and journals, revealed only one paper, ref.5, in which static pressures had been measured. This early paper (1919) described wind tunnel tests on a propeller which were made for the express purpose of testing the validity of the Froude actuator disc theory. However, since the tests were made at a tunnel speed corresponding to normal propeller action, the theoretical static pressures would be expected to be much smaller than for the static case at the same thrust, which is the case of interest in this note. Further, a static pressure in the wake was expected in any case because of the constraint of the tunnel walls. An increase of static pressure due to the vortex wake, as discussed in the previous section, was not considered.

A small static "over pressure" was indeed measured in the tests of ref.5, but because of the difficulties mentioned above it was not possible to determine that contribution, if any,which could be attributed to the motion of the vortex sheets.

To obtain some direct measurements of the static pressure under hovering conditions tests have been made on two rotors which we shall designate as rotors A and B. Rotor A (already available) was a small two-bladed see-saw rotor of 0.37m (1.21 ft) diameter. The blades were untwisted, of constant chord, and of aspect ratio 7.4. A rotor speed of 43.3 Hz gave a tip speed of 50.2 ms^{-1} which, with a blade collective pitch of 14 deg.,gave induced velocities of up to about 7.5 ms^{-1}. Rotor B was hingless and of 0.78m (2.5 ft) diameter. In order to achieve a more uniform induced velocity distribution then that of a constant chord blade, the blades of rotor B had a tip/chord ratio of 0.5, the tip chord being 51 cm (0.167 ft). It was hoped that the more uniform induced velocity distribution would lead to a wake structure more like that of the ideal wake assumed by Theodorsen.

Total head and static pressures were measured relative to atmospheric pressure by means of a standard pitot-static tube and simple inclined liquid and projection memometers. A traversing gear enabled a range of radial measurements to be made at fixed distances below the rotor. In the tests on rotor A some induced velocity measurements were made by means of a hot wire anemometer.

Figures 3 to 6 show that there is a considerable static pressure in the slipstream of both rotors although the magnitude is less than that given by Theordorsen's wake equation. The static pressure in each case shows a marked falling off towards the edge of the slipstream. This is to be expected since for a finite number of blades, there is a large variation of local velocity near the edges of the vortex sheets and a corresponding variation of local pressure. Thus, the static pressures shown in the figures are the time averaged values recorded by the pitot-static tube.

To get a rough idea of the pressure variation we can use Prandtl's representation of the vortex sheets[6]. Prandtl replaced the curved sheets of the ideal helical wake by a system of straight, semi-infinite, parallel lines having a gap of distance s between them, fig.7. Assuming that the time-averaged pressure is the same as the distance-averaged pressure between the sheets, it can be shown[7] that

$$\frac{(p - p_o)_{mean}}{(p - p_o)_\infty} = 1 - \frac{2k}{\pi} K(k) \tag{12}$$

where $(p-p_o)_\infty$ is the pressure between the sheets for an infinite number of blades and $K(k)$ is the complete elliptic integral of the first kind of modulus k and where $k = e^{-2\pi(R_o - r)/s}$, R_o being the radius of the contracted wake and s the distance between the sheets. The value of s can be obtained from an estimation of the final wake velocity. Thus, an idea of the static pressure $(p-p_o)_\infty$ when the number of blades is infinite ($s \to o$) can be obtained from eqn.(12) by writing it as

$$(p - p_o)_\infty = \frac{(p - p_o)_{mean}}{1 - \frac{2k}{\pi} K(k)} ,$$

$(p - po)_{mean}$ being the measured value. This relation has been applied to the measurements shown in figs. 3 and 5 and it can be seen that the "correction" extends the measured values roughly linearly.

It was shown in sect.2 that Theodorsen's equation implies that the static pressure in hovering flight is half the total head pressure. As remarked above, the measurements show that the actual static pressure is less than this and this is still the case when corrected for an infinite number of blades. It can also be seen from figs. 3 to 6 that the ratio of the static to total head pressure varies considerably along the blade, the ratio decreasing with distance from the hub. It might have been expected that the results for rotor B would have shown larger ratios since the blades were designed to give a fairly uniform velocity distribution in an attempt to approximate to the "ideal" wake. However, it was noticed in the tests on this rotor that there were quite large pressure fluctuations in the region 0.3R to 0.4R which may have indicated strong vortices being shed from the rather wide blade roots (c = 10cm at r = 3.5cm). The lack of agreement, generally, is thought to be due to the large difference between the actual and ideal wake geometries, as described in sect.2.

Although pressure measurements do not define the slipstream boundaries very clearly (there were large pressure fluctuations where the pressure began to fall near the blade tips), it appears from tests on both rotors that the slipstream radius is about 0.8 times the rotor radius. This figure is in good agreement with the value of 0.78 found by Landgrebe from his smoke tests[4]. Both these figures are much closer to Theodorsen's value of 0.816 than the conventional momentum value of 0.707.

Some results of the hot wire anemometer measurements on rotor A are given in fig.8, showing the axial variation of maximum induced velocity. The ratio of the final value to that at the disc can be seen to be 1.51, in close agreement with Theodorsen's value of 1.5, but, as is obvious from the steep gradient at $z/R = o$, the ratio is very sensitive to the precise position of the "disc" which, because of blade bending and of the inclination of the blades due to collective pitch, is very difficult to define.

It can be seen from fig.8 that the maximum value of induced velocity is reached at a distance of only about 0.2R below the rotor. This agrees with Landgrebe's results[4] for a two bladed rotor,so that at this distance it can be supposed that the contraction is complete. Thus, the pressure measurements shown in figs. 3 to 6 correspond to the fully contracted slipstream.

4. Conclusions

(1) The theory of the ideal vortex wake predicts a static pressure in the wake of a propeller or rotor which is higher than the ambient air pressure and that this difference is greatest in the static case (hovering flight). This contrasts with the familiar actuator disc theory which assumes these pressures to be the same. For the vortex helix angle typical of the hovering rotor, the static "over pressure" should be approximately equal to $\frac{1}{2}\rho w^2$, where w is the self induced velocity of the vortex wake.

(2) Measurements made on two model rotors (one having constant chord blades and the other with a root/tip ratio of 2) confirms that a static over pressure occurs but that its magnitude is smaller than that predicted by the ideal wake theory. This is probably due to the fact that the actual wake configuration in the hovering state departs considerably from the ideal. The tests show that the contraction ratio and the ratio of the final induced velocity in the wake to that at the rotor disc appear to be in closer agreement with the theoretical values of Theodorsen then those of the conventional momentum theory.

(3) Theoretical calculations and rotor tests question the validity of the assumption that the static pressure in the wake of a propeller or rotor in axial flight is the same as the ambient pressure. Since the wake configurations in theoretical rotor calculations are often based on the results of the momentum theory, or may even incorporate the theory in the system of equations, it is important that a fuller understanding of the conditions in the wake be obtained.

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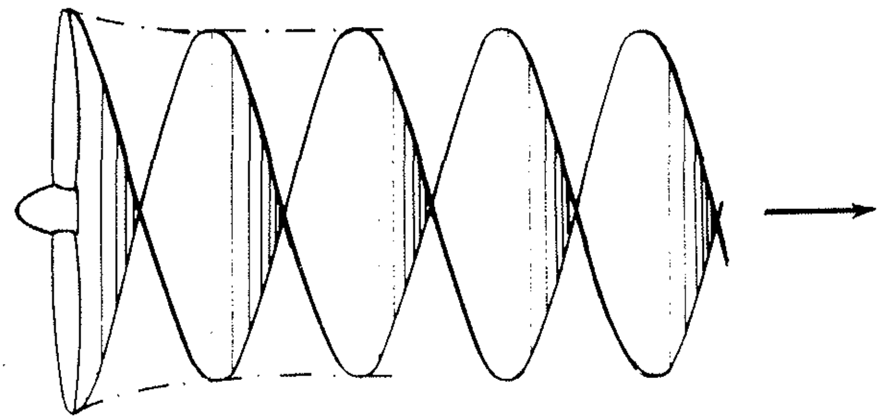


Fig. 1 Ideal wake

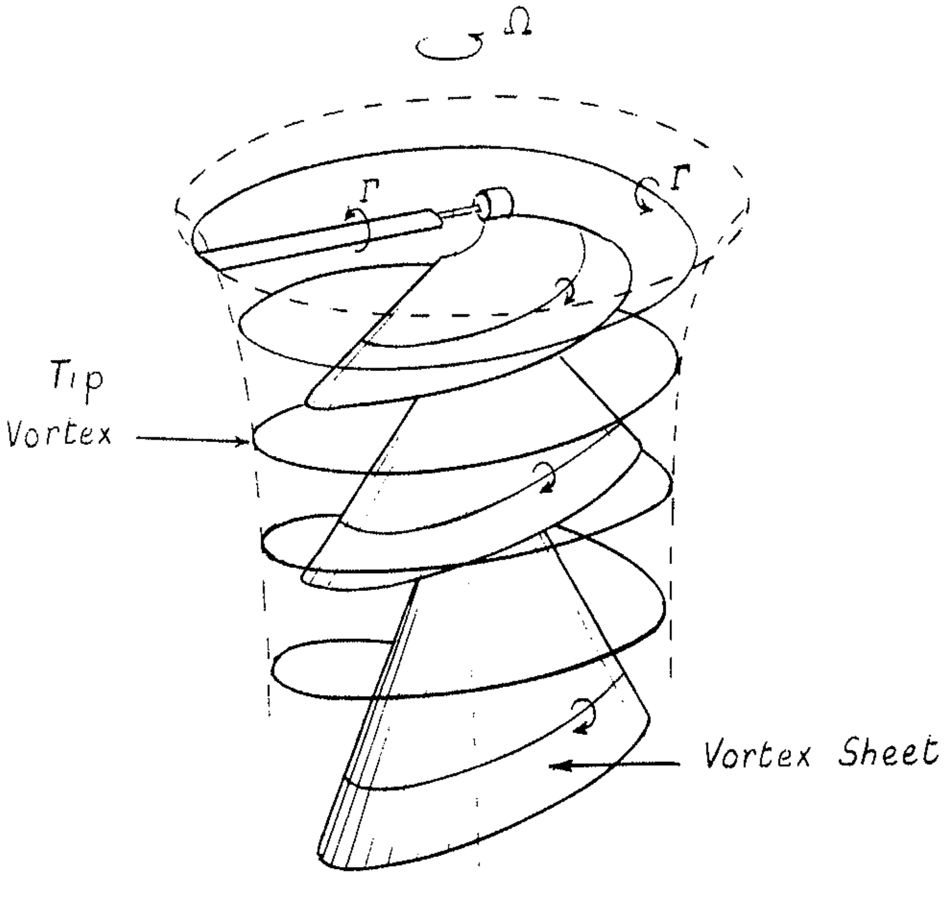


Fig. 2 Rotor Wake (Ref. 3)

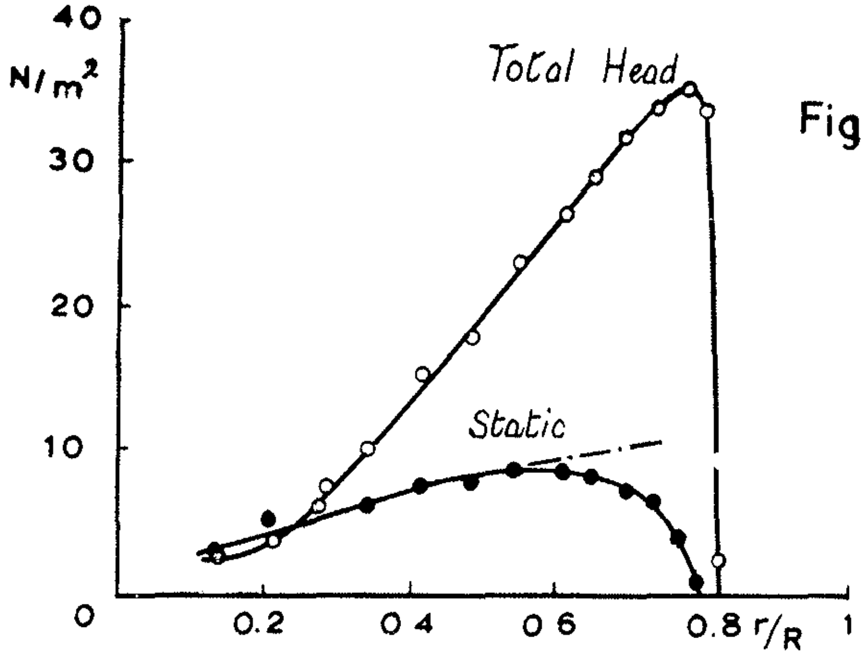


Fig. 3

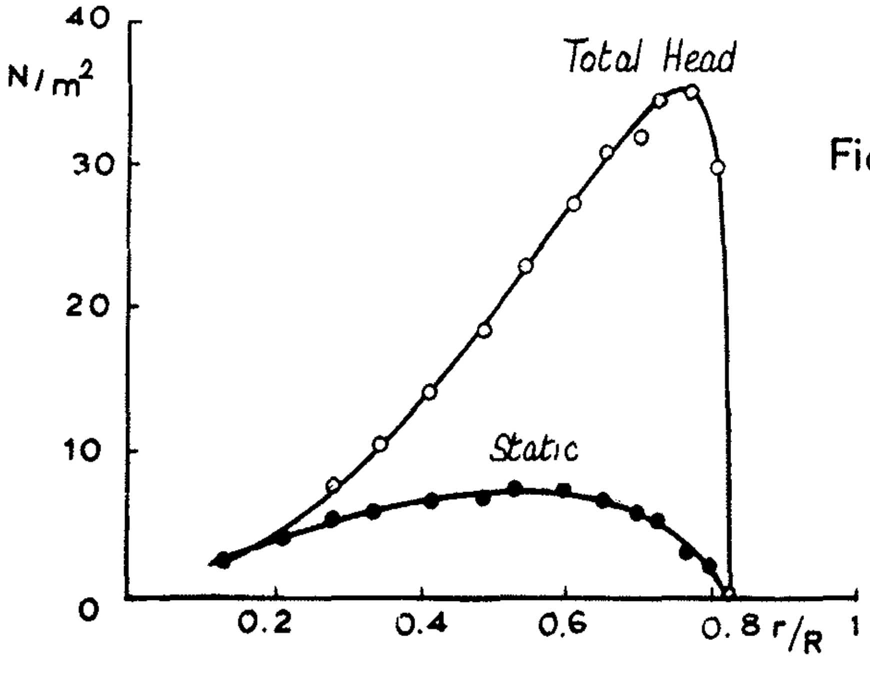
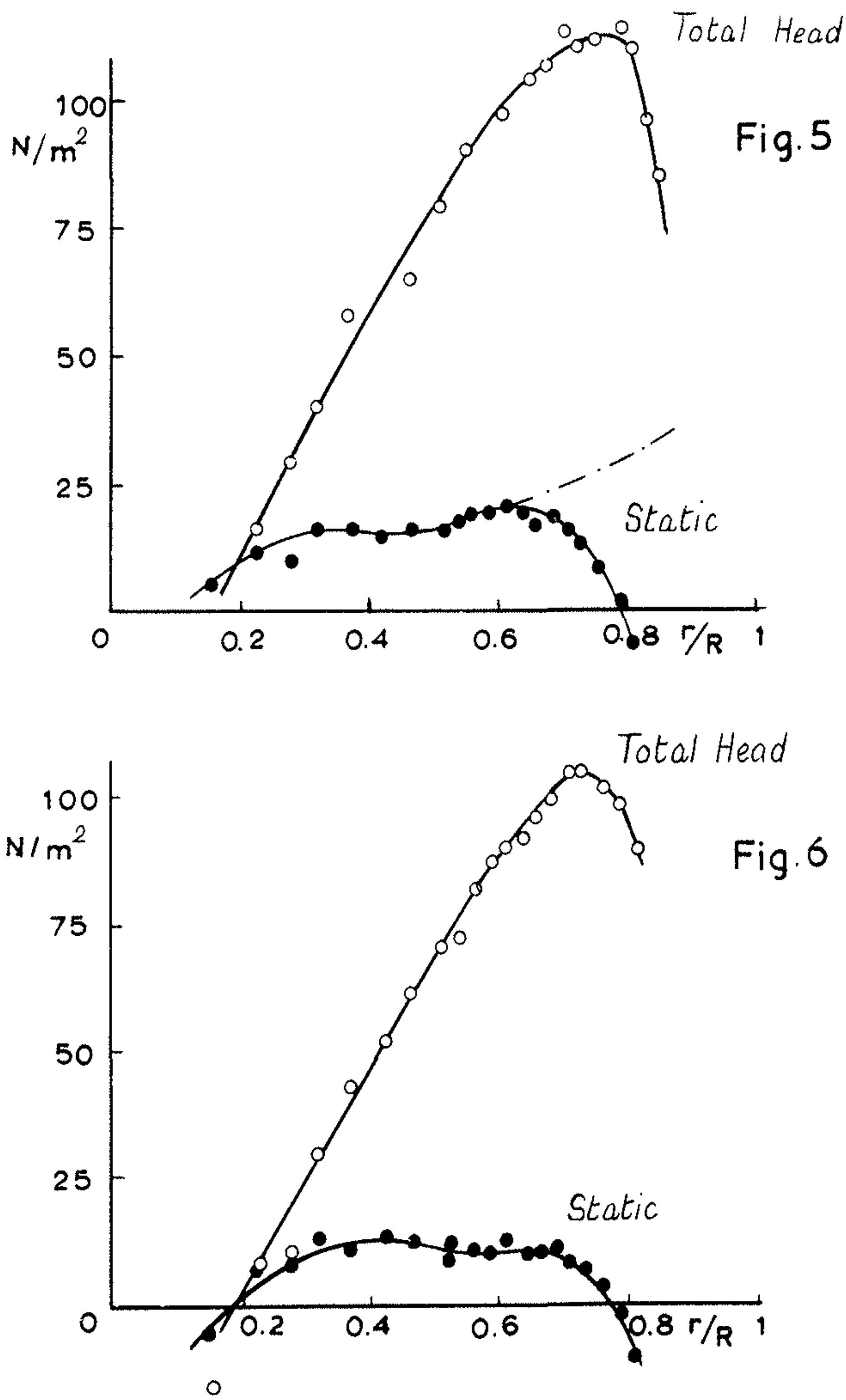


Fig. 4

Figs. 3&4 Pressure Distributions

(Rotor A)



Figs. 5 & 6 Pressure Distributions (Rotor B)