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CALCULATION OF THE CROSS SECTION PROPERTIES AND THE SHEAR STRESSES OF COMPOSITE ROTOR BLADES

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Abstract

The dynamic and static behaviour of rotor blades is mainly influenced by the coordinates of the shear center and the center of elasticity as well as the bending and shear stiffnesses. However, the distribution of the shear stresses due to transverse and torsional shear, the location of the shear center and the transverse deflection resulting from transverse shear are not easily found for beams with complicated cross section configurations. The determination of these quantities depends on the solution of a two-dimensional differential equation.

This report shows a solution of this problem by means of finite elements. Special elements have been developed, the use of which permits the calculation of the warping function resulting from transverse and torsional shear. Thus the values of the cross section can be obtained. By using the finite element method, a simple usage is guaranteed for any cross section. The material can be homogeneous or inhomogeneous, isotropic or anisotropic.

By comparing the results of this numerical approximation for cross sections with known exact solutions, it can be seen that convergence is guaranteed. The calculation for inhomogeneous cross sections shows an equivalent behaviour to a three-dimensional analysis.

Contents

1. Introduction

To determine the dynamic and static properties of rotor blades it is necessary to know bending-, shear- and torsional stiffnesses, further the location of the elastic center and the shear center. Only in the case of very simply shaped cross sections, analytic solutions for the shear center, the transverse shear stress distributions and shear stiffnesses due to transverse load and torsion are known.

In the case of complicated cross section configurations, for instance, composite rotor blades, only approximate calculations or three-dimensional idealizations in finite elements are available. A numerical solution, with the help of three-dimensional finite elements often needs too much effort. Characteristic values of cross sections, for instance shear stiffness and shear center, are in that case to be determined additionally.

An analytical method for rotor blades which gives sufficiently good results and which can be performed with a minimum amount of work and calculation time is desired.

2. The equations of transverse bending and torsion for homogeneous isotropic beams

For determining the stress distributions and characteristic cross section values of rotor blades, homogeneous, isotropic, cylindrical beams are considered first (see Figure 1).

The isotropic material shall show a linear stress-strain-behaviour (Hooke's law). The effects in the local region of the load introduction are omitted (St. Venant's principle). Furthermore St. Venant's assumptions [1]:

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{1}$$

are assumed. These express that in the cross section neither transverse pressure nor inplane shear exists. By use of above assumptions, the boundary conditions can be summarized in one single term:

$$\tau_{zx} \cdot \cos(n,x) + \tau_{zy} \cdot \cos(n,y) = 0 \tag{2}$$

where "n" represents the normal of the cylindrical surface, expressing, that the resulting shear stress is tangentially directed to the boundary.

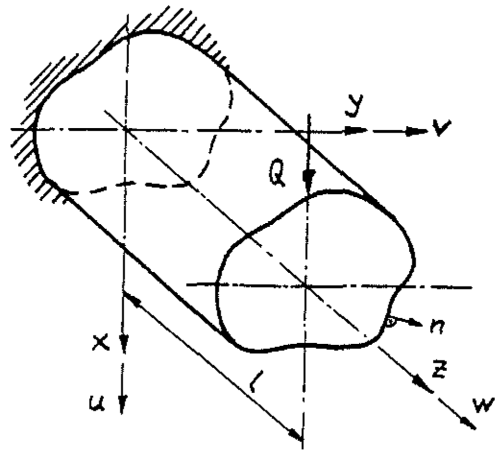


Figure 1: Shear Loaded Beam

The principle axes of inertia are represented by the x- and y-axis; the transverse load acts parallelly to the x-axis.

Assuming a linear bending stress distribution σ_z in the longitudinal direction of the beam and in the shear direction x, one obtains:

$$\sigma_z = \frac{-Q\cdot(l-z)}{J_y}\cdot x \tag{3}$$

Inserting equation (1) and (3) in the equilibrium conditions, yields:

$$\frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{zy}}{\partial z} = 0 \tag{4}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{Q\cdot x}{J_y} = 0 \quad .$$

The first two equations show that the transverse shear stresses τ_{zx} and τ_{zy} are independent of the z-axis. With the boundary equation (2) and the compatibility requirements the solution of the differential equation (4) yields, see e.g. [1,2,3]:

$$\tau_{zx} = G\cdot\Theta\cdot\left(\frac{\partial\phi}{\partial x} - y\right) - \frac{Q}{2.(1+\nu)\cdot J_y}\cdot\left[\frac{\partial\bar{\chi}}{\partial x} + \frac{\nu}{2}\cdot(x^2-y^2)\right]$$

$$\tau_{zy} = G\cdot\Theta\cdot\left(\frac{\partial\phi}{\partial y} + x\right) - \frac{Q}{2\cdot(1+\nu)\cdot J_y}\cdot\left[\frac{\partial\bar{\chi}}{\partial y} + \nu\cdot x\cdot y\right] \tag{5}$$

$$\sigma_z = \frac{-Q\cdot(l-z)}{J_y}\cdot x$$

Θ represents twist, ϕ means the torsional function and $\bar{\chi}$ represents the warping function.

The function $\bar{\chi} = \bar{\chi}\ (x,y)$ is in accordance to Poisson´s equation $\Delta\bar{\chi} = 2\cdot x$. The equations (5) are often expressed by the bending function χ, which is a harmonic function ($\Delta\chi = 0$). Thus follows:

$$\bar{\chi} = \chi + x\cdot y^2.$$

By integrating the geometrical equations it is obtained regarding Hooke´s law and the equations (5) and (3):

$$u = -\Theta\cdot z\cdot y + \frac{Q}{E\cdot J_y}\cdot[\frac{\nu}{2}\cdot(1-z)\cdot(x^2 - y^2) + \frac{1}{2}\cdot z^2 - \frac{z^3}{6}] + b_2\cdot z =$$

$$= -u_T + u_Q + u_B + b_2\cdot z$$

$$v = \Theta\cdot z\cdot x + \frac{Q}{E\cdot J_y}\cdot\nu\cdot(1-z)\cdot x\cdot y = v_T + v_Q \qquad (6)$$

$$w = \Theta\cdot\phi - \frac{Q}{E\cdot J_y}\cdot[x\cdot(1z - \frac{z^2}{2}) + \bar{\chi}] - b_2\cdot x = w_T - w_B - w_Q - b_2\ x$$

All terms of the displacement components u and v depending on (1-z) describe the influence of the cross section strains u_Q, v_Q due to the Poisson´s ratio ν and bending stress σ_z. The expression

$$u_B = \frac{Q}{E\cdot J_y}\cdot[\frac{1}{2}\cdot z^2 - \frac{z^3}{6}]$$

indicates the equation of the curved beam axis z. b_2 indicates the average shear angle, respectively $b_2\cdot z$ the additional deflection due to shear strain which is caused by transverse load.

The term $w_B = \frac{Q}{E\cdot J_y}\cdot x\cdot(1\cdot z - \frac{z^2}{2}) = x\cdot\frac{\partial u_B}{\partial x}$ takes into account the slope of the cross section, whereas the term $b_2\cdot x$ represents the angle of the cross section against the elastic line due to shear stress.

The expression $$w_Q = \frac{Q}{E\cdot J_y}\cdot\bar{\chi}$$

characterizes the cross section warping due to shear caused by transverse load. The transverse load Q must be independent of z and therefore

$$\frac{\partial w_Q}{\partial z} = 0.$$

This condition is satisfied because of $\bar{\chi} = \bar{\chi}\ (x,y) \neq \bar{\chi}\ (z)$.

The term $w_T = \Theta\cdot\phi$ represents the cross section warping due to torsion. The displacements u_T and v_T are derived from St. Venant´s torsional theory.

As a result of the expressions (5), the shear stresses due to transverse load Q can be split up into a term which takes into account warping and into one which is independent of warping.

The expressions:

$$\tau^{\circ}_{zx} = \frac{-Q}{2\cdot(1+\nu)\cdot J_y}\cdot\frac{\nu}{2}\cdot(x^2-y^2) = G\cdot\gamma^{\circ}_{zx} \quad ,$$

$$\tau^{\circ}_{zy} = \frac{-Q}{2\cdot(1+\nu)\cdot J_y}\cdot\nu\cdot x\cdot y = G\cdot\gamma^{\circ}_{zy}$$

do not include warping.

The displacements u_Q and v_Q and the shear stresses τ_{zx} and τ_{zy} which can be derived from the displacements u_Q and v_Q are shown in Figure 2.

For the solution of the equations (6) it is necessary to find the torsional function ϕ and the warping function $\bar{\chi}$, which depend on the cross section shape. The functions ϕ and $\bar{\chi}$ are only known for few shapes.

The average shear angle b_2 due to the shear load Q can be determined by taking the mean value of the cross section warping displacement w_Q and the shear angle γ°_{zx}.

The unknown cross section warping functions resulting from the shear stresses are calculated by means of the finite-element-method [4].

A convenient method to determine the required force-displacement relations for finite elements is the principle of minimum potential energy. For this purpose the whole potential energy Π has to be expressed by the unknown warping funktion $\bar{\chi}$ and the torsional function ϕ. Here only non torsional warping is considered, which means, that the transverse load acts on the shear center. In the expressions (6) Θ is considered to be zero. Therefore only a polynomial function in x and y is chosen for the unknown warping function $\bar{\chi}$.

The potential energy Π_a is described by the displacement u at the point $z = 1$. In determining this energy, the average shear angle b_2 must be known. b_2 is deduced by taking the average of the cross section warping w_Q and the shear angle γ°_{zx}.

The average shear angle b_2, which can also be expressed by the shear coefficient k, can be obtained from the following expression:

$$b_2 = \frac{k\,Q}{G\,A} = \frac{dU}{dz} + \phi$$

$\frac{dU}{dz}$ represents the average shear angle of the cross section A which is caused by the displacement u_Q. Thus

$$U = \frac{1}{A}\cdot\iint_A u_Q\cdot dx\cdot dy = \frac{1}{A}\cdot\iint_A \underbrace{\frac{Q}{E\cdot J_y}\cdot\frac{\nu}{2}\cdot(1-z)\cdot(x^2-y^2)}_{u_Q}\cdot dA$$

respectively

$$\frac{dU}{dz} = \frac{1}{A}\cdot\iint_A \underbrace{\frac{Q}{E\cdot J_y}\cdot\frac{\nu}{2}\cdot(-1)\cdot(x^2-y^2)}_{\gamma^{\circ}_{zx}}\cdot dA = \frac{1}{A}\cdot\frac{Q}{E\cdot J_y}\cdot\frac{\nu}{2}\cdot(J_x-J_y)$$

where

$$\gamma^{\circ}_{zx} = \frac{\partial u_Q}{\partial z}$$

ϕ is an average cross section twist due to warping.

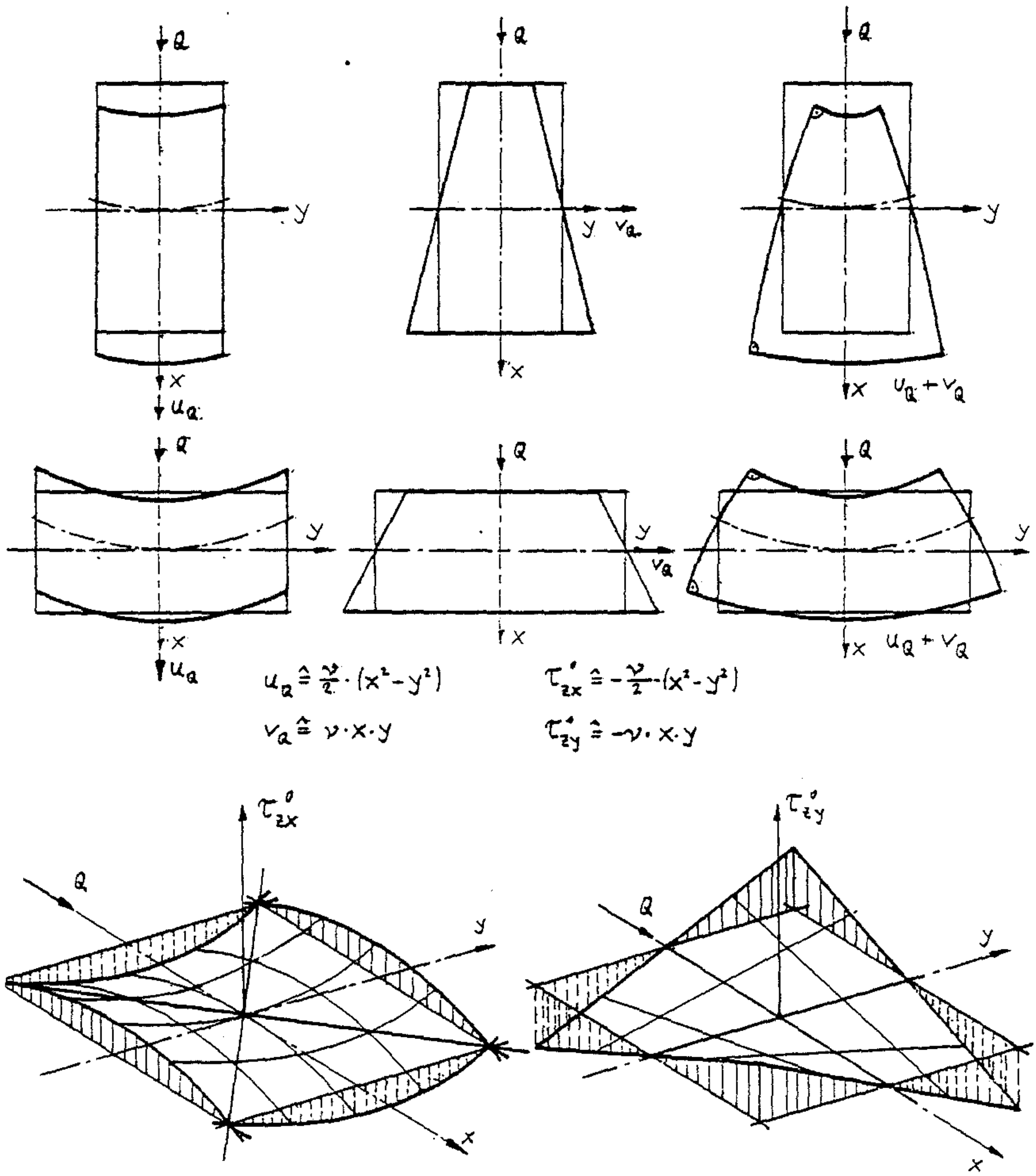


Figure 2: Quadrilateral Cross Sections: Cross Section Displacements and the Distribution of the Shear Stresses τ°_{zx} , τ°_{zy}

From the expression $\frac{dU_Q}{dz} = \frac{dU}{dz} + \Phi = \frac{k \cdot Q}{G \cdot A} \neq f(z)$

follows for k (with: $G = \frac{E}{2(1+\nu)}$):

$$k = \underbrace{\frac{\nu}{4 \cdot (1+\nu)} \cdot \left(\frac{J_x}{J_y} - 1\right)}_{k_1} - \underbrace{\frac{G \cdot A}{Q \cdot J_y} \cdot \iint_A x \cdot w_Q \cdot dA.}_{k_2} \tag{7}$$

k_1 includes the average shear deformation of the non-warping component of the cross-section strain, see [5].
The shear stresses, which depend on warping, are included by the term k_2.
The shear coefficient k is found in another way in [6]. With expression (7) it is possible to determine the shear coefficient k, respectively the average shear angle b_2, by means of the warping function χ, respectively warping w_Q (which is independent of z because of τ_{zx} and $\tau_{zy} \neq f(z)$).
Thus the energy can be determined.

The potential energy of the external loads amounts to

$$\pi_a = - Q \cdot \bar{U}_{(z=1)}$$

where the average cross section displacement at the beam end (z=1) is expressed by:

$$\bar{U} = \frac{1}{A} \cdot \iint_A u_{(z=1)} \cdot dA.$$

Regarding the expressions (6) one obtains:

$$\pi_a = - Q \cdot \left(\frac{Q}{E \cdot J_y} \cdot \frac{1^3}{3} + b_2 \cdot 1\right). \tag{8}$$

Considering St. Venant's assumptions ($\sigma_x = \sigma_y = \tau_{xy} = 0$) the deformation energy is obtained:

$$\pi_i = \frac{1}{2} \cdot \iiint_V (a_{33} \cdot \sigma_z^2 + a_{44} \cdot \tau_{yz}^2 + a_{55} \cdot \tau_{xz}^2) \cdot dV \tag{9}$$

where:

$$a_{33} = \frac{1}{E}, \quad a_{44} = a_{55} = \frac{1}{G}.$$

Finally, considering equation (8) and (9), the whole potential energy of an isotropic, cylindrical cantilever beam of the total length 1 loaded by a single load at its end, is obtained as:

$$\pi = \pi_i + \pi_a = \frac{1}{2} \cdot \iint_A \Big\langle \frac{Q^2}{E \cdot J_y^2} \cdot x^2 \cdot \frac{l^3}{3} + G \cdot l \cdot \{ \frac{Q}{E \cdot J_y} \cdot [- \frac{\partial \bar{\chi}}{\partial x} - \frac{\nu}{2} \cdot (x^2 - y^2)] \}^2 + G \cdot l \cdot \{ \frac{Q}{E \cdot J_y} \cdot [- \frac{\partial \bar{\chi}}{\partial y} - \nu \cdot x \cdot y] \}^2 \Big\rangle \cdot dA - Q \cdot \{ \frac{Q}{E \cdot J_y} \cdot \frac{l^3}{3} + \frac{Q \cdot l}{G \cdot A} [\frac{\nu}{4 \cdot (1+\nu)} \cdot (\frac{Jx}{Jy} - 1) - \frac{A}{2 \cdot (1+\nu) \cdot J_y^2} \cdot \iint_A x \cdot \bar{\chi} \cdot dA] \}. \quad (10)$$

Inserting the warping function $\bar{\chi}$ (respectively w_Q), expressed by a polynomial function, in the term of the whole potential energy (10) and integrating over the whole area of a certain finite element with its subscript k, Π^k is obtained as a portion of the whole potential energy $_k$. Considering the partial derivative of Π^k to be zero, where Π^k is differentiated with respect to the unknown deflection w_{Qi}, the unknown load-warping-function is obtained:

$$[k] \cdot \{w_Q\} + \{c\} = \{0\}.$$

$[k]$ represents the element stiffness matrix. $\{c\}$ represents the element load vector due to warping. The warping vector $\{w_Q\}$ includes the node displacements w_{Qj}.

Joining the element matrices together, respectively the element load vectors to the total stiffness matrix, respectively to the total load vector, and solving the equation set, the node displacements w_{Qj} are obtained. With the chosen polynomial function of one finite element the warping deflection within this element is known. Finally, the shear coefficient k (see equation (7)) and the average shear angle b_2 is determined. Furthermore it is possible to calculate the deflections u, v, w (see equations (6)). By means of the known deflections and the material law the shear stresses τ_{zx} and τ_{zy} (see equations (5)) are computed. Under the requirement that a cross section should be displaced without any torsion and by knowing the shear stresses resulting from transverse bending, the shear center coordinates can be obtained by means of the moment relations in the following way:

$$y_M = \frac{1}{Q_x} \cdot \iint_A (-x \cdot \tau_{zy} + y \cdot \tau_{zx}) \cdot dA$$

$$x_M = \frac{1}{Q_y} \cdot \iint_A (x \cdot \tau'_{zy} - y \cdot \tau'_{zx}) \cdot dA.$$

τ_{zx} and τ_{zy} are the shear stresses resulting from the loads $Q_x \neq 0$ and $Q_y = 0$; τ'_{zx} and τ'_{zy} result from the loads $Q_y \neq 0$, $Q_x \neq 0$.

For solving the torsion problem, the total potential energy π is calculated by using the terms depending on Θ of the equations (6) for u, v and w. The unknown torsional funktion ϕ of a cross section is approximated by a polynomial in analogy to the warping function $\bar{\chi}$, see [7]. Apart from the element load vector, the same element stiffness matrix is obtained as by using the solution described in this paper. When the shear stresses τ_{zx}^T and τ_{zy}^T due to torsion have been determined, the torsional stiffness GI_T can be calculated by:

$$G \cdot J_T = \frac{M_t}{\Theta} = \iint_A (\tau_{zy}^T \cdot x - \tau_{zx}^T \cdot y) \cdot dA .$$

3. The characteristic cross section properties due to transverse bending and torsion for inhomogeneous, orthotropic rotor blades

Rotor blades composed of fiber-reinforced materials are orthotropic structures in which the three symmetry axes are orthogonal to each other.

The longitudinal axis z of the blade is considered to be an orthotropic direction. Normally, both orthotropic axes in the cross section plane do not coincide with the cross section axes x and y. This special symmetry model corresponds to a monoclinic cristal [8]. The above mentioned kind of material anisotropy, however, is the most significant case in technical applications. In order to demonstrate the computation of these properties a cantilever beam composed of orthotropic material is considered.

To begin with, Hooke's law in its generalized form is stated again:

$$\{\varepsilon\} = [a] \cdot \{\sigma\}$$

For monoclinic bodies the relation between stress and strain is given in the following form:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} \\ & a_{22} & a_{23} & 0 & 0 & a_{26} \\ & & a_{33} & 0 & 0 & a_{36} \\ & & & a_{44} & a_{45} & 0 \\ & \text{symmetric} & & & a_{55} & 0 \\ & & & & & a_{66} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \quad (11)$$

A significant feature of the equations (11) is the coupling of stress and strain by the terms a_{16}, a_{26} and a_{36}.

Regarding expression (11) it follows:

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} a_{44} & a_{45} \\ a_{45} & a_{55} \end{bmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \qquad (12)$$

In order to obtain the relation

$$\{\sigma\} = [a]^{-1} \cdot \{\varepsilon\} = [A] \cdot \{\varepsilon\}$$

the compliance matrix [a] has to be inverted. Its structure is the same as in the matrix [a]. Therefore the coefficients A_{44}, A_{45}, A_{55} of the applicated material stiffness matrix can be found by solving the set of equations (12).

Now it follows:

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \cdot \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

If some materials of the inhomogeneous beam possess different Poisson´s ratios, there will be a mutual cross sectional restraint of each layer during the cross section deformation. Hence, St. Venant´s assumptions (1) are no longer valid.

If such a three-dimensional problem shall be reduced to a two-dimensional one, the dependence of all displacements and stresses on the beam´s longitudinal axis z must be known. For instance, if St. Venant´s assumptions are still valid, the displacements u, v and w are given, excepting the unknown cross section warping. As a result of the conditions outlined above, a solution by means of the warping function $\bar{\chi}$ is proposed and this solution shall be described now.

First of all, the cross section shall consist of "n" homogeneous partial areas, then one obtains for the displacements [5]:

$$u^m = \frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \left[\frac{1}{2}(b_{23}^m \cdot y^2 - b_{13}^m \cdot x^2) \cdot (1-z) + \frac{1}{2} \cdot z^2 - \frac{z^3}{6}\right] + b_2 \cdot z$$

$$v^m = \frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \left[- b_{23}^m \cdot x \cdot y \cdot (1-z) - \frac{b_{36}^m}{2} \cdot x^2 \cdot (1-z)\right]$$

$$w^m = \frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \left[- \bar{\chi}^m - x(1 \cdot z - \frac{z^2}{2})\right] - b_2 \cdot x$$

with

$$b_{13}^m = \frac{a_{13}^m}{a_{33}^m}, \quad b_{23}^m = \frac{a_{23}^m}{a_{33}^m}, \quad b_{36}^m = \frac{a_{36}^m}{a_{33}^m} \quad \text{und} \quad \frac{1}{E_z^m} = a_{33}^m$$

The subscript "m" is related to the homogeneous partial area m, where the cross section A is subdivided into "n" partial areas A^m.

The moments of inertia I^m_y of each homogeneous partial area are related to the principle axis of inertia y, which has its origin in the elastic center. This point is defined as follows:

$$x_s = \frac{\sum_{m=1}^{n} E_z^m \cdot x_s^m \cdot A^m}{\sum_{m=1}^{n} E_z^m \cdot A^m} \quad , \quad y_s = \frac{\sum_{m=1}^{n} E_z^m \cdot x_s^m \cdot A^m}{\sum_{m=1}^{n} E_z^m \cdot A^m}$$

This law results from the force equilibrium of all normal bending stresses in direction of z

$$\iint_A \sigma_z \, dA = 0$$

After shifting the Cartesian coordinate system parallelly into the elastic center, the unknown principle axes x, y yield:

$$tg\ (2 \cdot \alpha) = \frac{2 \cdot \sum_{m=1}^{n} E_z^m \cdot J_{xy}^m}{\sum_{m=1}^{n} E_z^m \cdot J_y^m - \sum_{m=1}^{n} E_z^m \cdot J_x^m}$$

In the relations for x_s, y_s und α a linear stress distribution in each homogeneous partial section is assumed. Possible mutual deformation restraints are neglected.

For any energy consideration the product of force multiplied by the displacement has to be taken. Therefore to determine both the average cross section displacement u and the cross section warping ϕ, one must know all shear and longitudinal stiffnesses of each partial (homogeneous) area. We have:

$$U = \frac{1}{\sum_{m=1}^{n} A_{55}^m \cdot A^m} \cdot \sum_{m=1}^{n} \iint_{A^m} u_Q^m \cdot A_{55}^m \cdot dA^m \qquad A_{55}^m = G_{zx}^m$$

$$\Phi = \frac{1}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \sum_{m=1}^{n} \iint_{A^m} x \cdot w_Q^m \cdot E_z^m \cdot dA^m$$

With that it follows:

$$b_2 = \frac{1}{\sum\limits_{m=1}^{n} A_{55}^m \cdot A^m} \cdot \frac{Q}{\sum\limits_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \frac{1}{2} \cdot \sum_{m=1}^{n} \iint_{A^m} A_{55}^m \cdot (b_{13}^m \cdot x^2 - b_{23}^m \cdot y^2) \cdot dA^m -$$

$$- \frac{1}{\sum\limits_{m=1}^{n} E_z^m \cdot J_y^m} : \sum_{m=1}^{n} \iint_{A^m} E_z^m \cdot x \cdot w_Q^m dA^m$$

and

$$k = b_2 \cdot \frac{\sum\limits_{m=1}^{n} A_{55}^m \cdot A^m}{Q} .$$

In case of inhomogeneity the term $\sum\limits_{m=1}^{n} A_{55}^m \; A^m$ has to be taken into account for the calculation of the shear coefficient.

The potential energy of all external forces becomes:

$$\pi_a = - Q \cdot [\frac{1}{\sum\limits_{m=1}^{n} A_{55}^m \cdot A^m} \cdot \frac{Q}{\sum\limits_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \frac{l^3}{3} \cdot \sum_{m=1}^{n} \iint_{A^m} A_{55}^m \cdot dA^m + b_2 \cdot l]$$

with

$$\pi_i = \frac{1}{2} \cdot \iiint_V (A_{33} \cdot \varepsilon_z^2 + A_{44} \cdot \gamma_{yz}^2 + 2 \cdot A_{45} \cdot \gamma_{yz} \cdot \gamma_{xz} + A_{55} \cdot \gamma_{xz}^2) \cdot dV$$

respectively

$$\pi_i = \frac{1}{2} \cdot \iiint_V (a_{33} \cdot \sigma_z^2 + a_{44} \cdot \tau_{yz}^2 + 2 \cdot a_{45} \cdot \tau_{yz} \cdot \tau_{xz} + a_{55} \cdot \tau_{xz}^2) \cdot dV$$

and with reference to the conditions outlined above the expression of the whole potential energy with regard to non-torsional transverse bending yields:

$$\pi = \pi_i + \pi_a = \frac{1}{2}\cdot\left(\frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m}\right)^2 \cdot \frac{l^3}{3} \cdot \sum_{m=1}^{n} \iint_{A^m} E_z^m \cdot x^2 \cdot dA^m +$$

$$+ \frac{1}{2} \cdot \sum_{m=1}^{n} \iint_{A^m} \{A_{44}^m \cdot \left(\frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m}\right)^2 \cdot [-\frac{\partial\bar{\chi}}{\partial y} + b_{23}^m \cdot x \cdot y + \frac{b_{36}^m}{2} \cdot x^2]^2 +$$

$$+ 2 \cdot A_{45}^m \cdot \left(\frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m}\right)^2 \cdot [-\frac{\partial\bar{\chi}}{\partial y} + b_{23}^m \cdot x \cdot y + \frac{b_{36}^m}{2} \cdot x^2]$$

$$\cdot [-\frac{\partial\bar{\chi}}{\partial y} - \frac{1}{2} \cdot (b_{23}^m \cdot y^2 - b_{13}^m \cdot x^2)] +$$

$$+ A_{55}^m \cdot \left(\frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m}\right)^2 \cdot [-\frac{\partial\bar{\chi}}{\partial x} - \frac{1}{2} \cdot (b_{23} \cdot y^2 - b_{13} \cdot x^2)]^2 \} \cdot dA^m -$$

$$- Q \cdot \{\frac{1}{\sum_{m=1}^{n} A_{55}^m \cdot A^m} \cdot \frac{Q}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \frac{l^3}{3} \cdot \sum_{m=1}^{n} \iint_{A^m} A_{55}^m \cdot dA^m + \frac{Q \cdot l}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot [\frac{1}{\sum_{m=1}^{n} A_{55}^m A^m} \cdot$$

$$\cdot \frac{1}{2} \cdot \sum_{m=1}^{n} A_{55}^m \cdot (b_{13}^m \cdot J_y^m - b_{23}^m \cdot J_x^m) - \frac{1}{\sum_{m=1}^{n} E_z^m \cdot J_y^m} \cdot \sum_{m=1}^{n} \iint_{A^m} x \cdot \bar{\chi} \cdot E_z^m \cdot dA^m]\}$$

With this expression the unknown force-deformation relation can be set up (see chapter 2).

4. Element stiffness matrices and element load vectors

The purpose of this chapter is to deduce the element stiffness matrices and element vectors of three element types. The approximate quality of the solution shall be discussed.

We shall consider a triangular element with a linear warping displacement function and an isoparametric triangular, respectively quadrilateral element with a second order respectively third order parabola (see Figure 3).

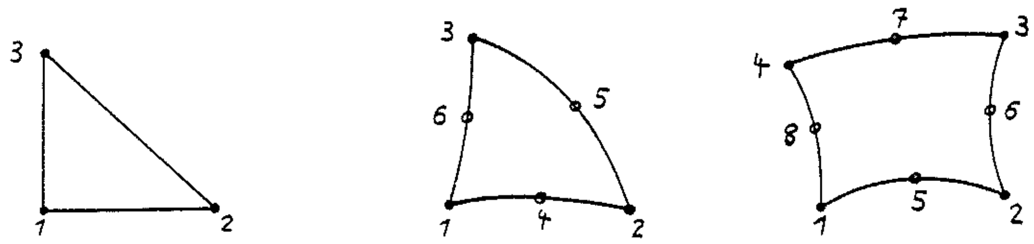


Figure 3: Triangular element and isoparametric triangular and quadrilateral element

Thus, the coefficients of both the element stiffness matrix k_{ij} and the element load vector c_j for the triangle with 3 nodes are:

$$k_{11} = \frac{1}{4a}\cdot[A^m_{44}\cdot x^2_{32} + A^m_{45}\cdot 2\cdot x_{32}\cdot y_{23} + A^m_{55}\cdot y^2_{23}]$$

$$k_{12} = \frac{1}{4a}\cdot[A^m_{44}\cdot x_{13}\cdot x_{32} + A^m_{45}\cdot(x_{13}\cdot y_{23} + x_{32}\cdot y_{31}) + A^m_{55}\cdot y_{23}\cdot y_{31}]$$

$$k_{13} = \frac{1}{4a}\cdot[A^m_{44}\cdot x_{21}\cdot x_{32} + A^m_{45}\cdot(x_{21}\cdot y_{23} + x_{32}\cdot y_{12}) + A^m_{55}\cdot y_{12}\cdot y_{23}]$$

$$k_{22} = \frac{1}{4a}\cdot[A^m_{44}\cdot x^2_{13} + A^m_{45}\cdot 2\cdot x_{13}\cdot y_{31} + A^m_{55}\cdot y^2_{31}]$$

$$k_{23} = \frac{1}{4a}\cdot[A^m_{44}\cdot x_{13}\cdot x_{21} + A^m_{45}\cdot(x_{13}\cdot y_{12} + x_{21}\cdot y_{31}) + A^m_{55}\cdot y_{12}\cdot y_{31}]$$

$$k_{33} = \frac{1}{4a}\cdot[A^m_{44}\cdot x^2_{21} + A^m_{45}\cdot 2\cdot x_{21}\cdot y_{12} + A^m_{55}\cdot y^2_{12}]$$

and

$$c_1 = T\cdot\{(x_2\cdot y_3 - x_3\cdot y_2)\cdot\bar{x}\cdot a + J^k_y\cdot y_{23} + J^k_{xy}\cdot x_{32} - $$
$$- A_{44}\cdot x_{32}\cdot R - A_{45}\cdot(y_{23}\cdot R + x_{32}\cdot S) - A_{55}\cdot y_{23}\cdot S\}$$

$$c_2 = T\cdot\{(x_3\cdot y_1 - x_1\cdot y_3)\cdot\bar{x}\cdot a + y_{31}\cdot J^k_y + x_{13}\cdot J^k_{xy} - $$
$$- A_{44}\cdot x_{13}\cdot R - A_{45}\cdot(y_{31}\cdot R + x_{13}\cdot S) - A_{55}\cdot y_{31}\cdot S\}$$

$$c_3 = T\cdot\{(x_1\cdot y_2 - x_2\cdot y_1)\cdot\bar{x}\cdot a + y_{12}\cdot J^k_y + x_{21}\cdot J^k_{xy} - $$
$$- A_{44}\cdot x_{21}\cdot R - A_{45}\cdot(y_{12}\cdot R + x_{21}\cdot S) - A_{55}\cdot y_{12}\cdot S\}$$

where for the abbrevations:

$$x_{ij} = x_i - x_j \ , \quad y_{ij} = y_i - y_j$$

$$R = b^m_{23} \cdot J^k_{xy} + \frac{b^m_{36}}{2} \cdot J^k_y$$

$$S = \frac{1}{2} \cdot (b^m_{13} \cdot J^k_y - b^m_{23} \cdot J^k_x)$$

$$T = \frac{Q}{\sum_{m=1}^{n} E^m_z \cdot J^m_y} \cdot \frac{E^m_z}{2a}$$

The terms of the warping c_j , which depend on R and S and therefore, with regard to the material coefficients b_{13}, b_{23} and b_{36}, contain the Poisson's ratios ν_{31} and ν_{32} , from the internal energy Π^k_i . Therefore this warping term is obtained exclusively by the cross section strains u_Q , v_Q , which vary in z-direction. At the cross section edge the boundary conditions are fullfilled and hence, the shear stresses and the shear angles are zero in the normal direction to the cross section edge. Thus, the cylinder surface slope due to a varying cross section deflection in z-direction is regarded. (For information about element load vectors in case of torsion see [7,8]).

The coefficients in case of an isoparametric triangular and quadrilateral element are as follows:

$$k_{ij} = \iint_a \{A^m_{44} \cdot \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y} + A^m_{45} \cdot (\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial x}) + A^m_{55} \cdot \frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x}\} \cdot \det[J] \cdot d\xi \cdot d\eta \qquad \begin{matrix} i = 1,2,\dots 6 \ (8) \\ j = 1,2,\dots 6 \ (8) \end{matrix}$$

$$c_j = \frac{Q}{\sum_{m=1}^{n} E^m_z \cdot J^m_y} \cdot \iint_a E^m_z \cdot x \cdot N_i \cdot \det[J] \cdot d\xi \cdot d\eta + \frac{Q}{\sum_{m=1}^{n} E^m_z \cdot J^m_y} \cdot \iint_a \{\frac{\partial N_i}{\partial x} \cdot (- A^m_{55} \cdot \bar{S} \quad A^m_{45} \cdot \bar{R}) + \frac{\partial N_i}{\partial y} (- A^m_{44} \cdot \bar{R} - A^m_{45} \cdot \bar{S})\} \cdot \det[J] \cdot d\xi \cdot d\eta \qquad j = 1,2,\dots 6 \ (8)$$

where it means:

$$x = \sum_{j=1}^{6(8)} N_j \cdot x_j \ , \quad y = \sum_{j=1}^{6(8)} N_j \cdot y_j \ ,$$

$$\bar{R} = b^m_{23} \cdot x \cdot y + \frac{b^m_{36}}{2} \cdot x^2 ,$$

$$\bar{S} = \frac{1}{2} \cdot (b^m_{13} \cdot x^2 + b^m_{23} \cdot y^2)$$

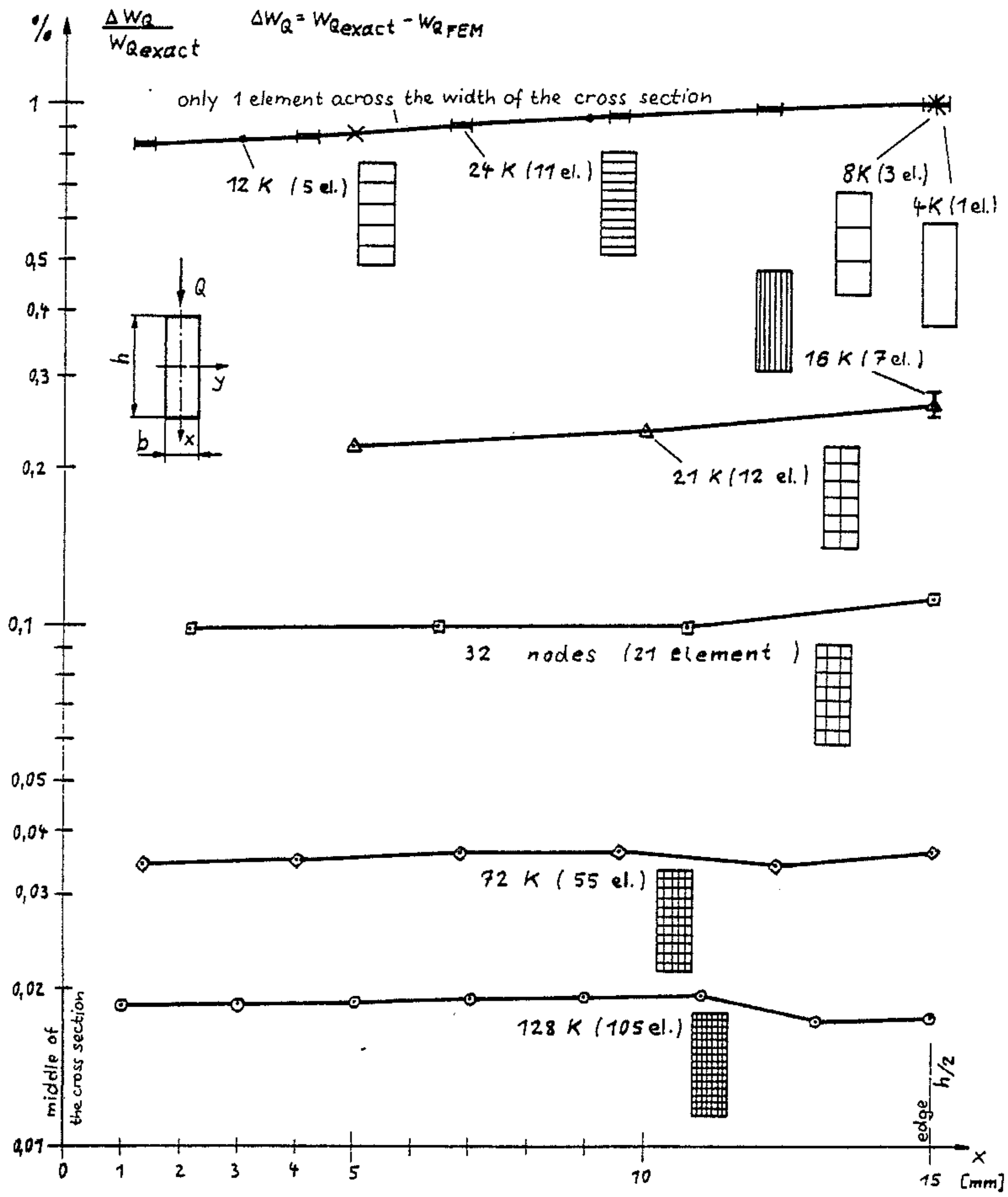


Diagram 1: Rectangular cross section (h/b=3/1), isotropic, subdivision into quadrilateral elements: relative deviation of warping w_Q (y=b/2) in relation to the analytical solution (reference value: w_Q exact)

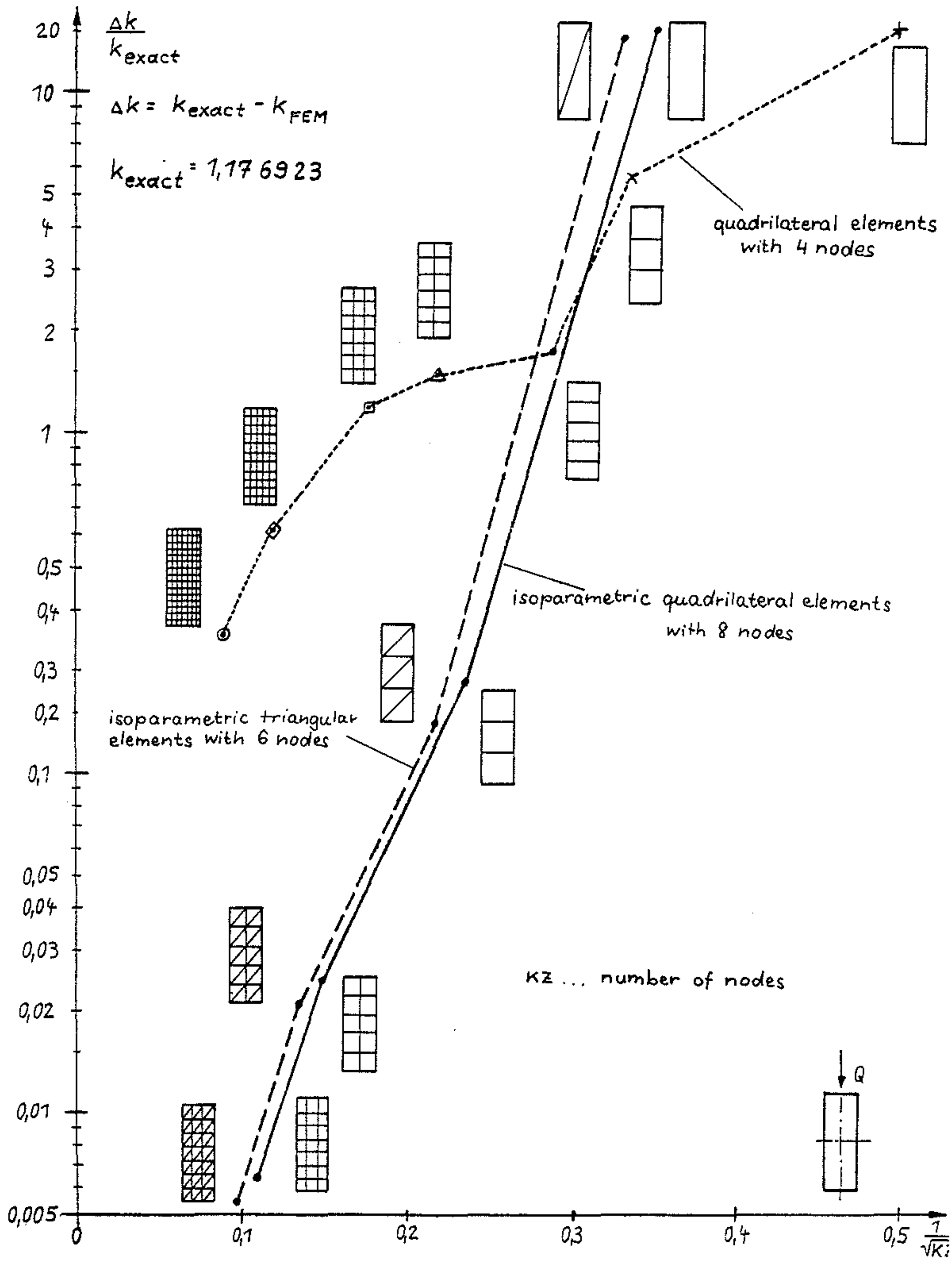


Diagram 2: Rectangular cross section (h/b=3/1), isotropic, subdivision into quadrilateral and triangular elements, relative deviation of the shear coefficient k in relation to the analytical solution (reference value: k_{exact}).

N_i, N_j are form functions, det [J] is the determinant of JACOBI'S matrix and ξ, η represent the field coordinates.

For detailed explanation of these coefficients see [5]. The coefficients k_{ij} and c_{ij} are, of course, also valid in case of an isotropic, homogeneous cross section. This kind of cross section is a special case of an orthotropic, inhomogeneous cross section.

For testing the element quality, a homogeneous rectangular cross section with a known analytical solution shall be considered. The cross section shall be subdivided into a number of square elements which are composed of triangular elements with three nodes (The idealizations are shown in Diagram 1). It can be shown that the approximate value of the warping calculated with help of the finite element analysis, convergates to the true value.

The convergence behaviour of the shear coefficient k is shown in Diagram 2; k also takes into account cross section warping (see equation 7). In addition to this, there are also considered some subdivisions of the cross section into triangular and quadrilateral elements with six, respectively eight nodes. These elements have higher order deflection terms for warping than the element with four nodes. It is obviously recognizable that with isoparametric elements essentially better results can be obtained. Apart from this, it is also possible to have a very good approximation in some special points (optimum stress points), see [10,11].

5. Comparison of calculated results of a three-dimensionally and a two-dimensionally idealized inhomogeneous, orthotropic beam

The quality of the "warping elements" is checked by comparing them with a three-dimensionally idealized cantilever beam. The FE-idealization of the cross-section is shown in Figure 4. The idealization of the beam in three-dimensional elements is shown in Figure 5.

The idealization of the beam in three-dimensional elements needs more effort than the idealization in two-dimensional "warping elements". For this example, the calculation time relation between three- and two-dimensional types of elements is 506 sec/5 sec.

In Figure 6 the stress distribution in z-direction is shown. It can be seen, that the disturbance of the stresses due to the load introduction is restricted to a small area. The shear stresses calculated with the two-dimensional "warping elements" coincide with those calculated with three-dimensional elements [5]. It can be stated, that for helicopter main rotor blades an idealization in "warping elements" gives sufficient results to dimension the blade. However for dimensioning tail rotor blades an analysis which includes three-dimensional elements may be necessary. Such an idealization for the BO105 tail rotor is shown in Figure 8.

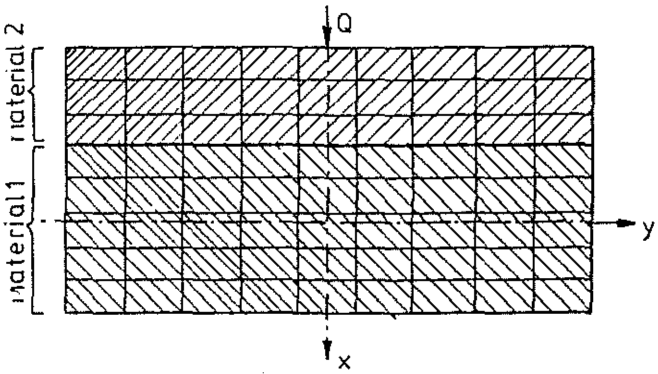
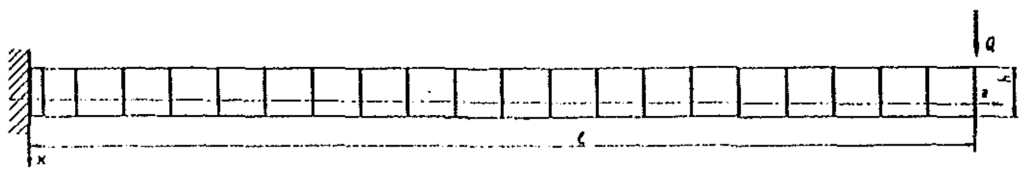


Figure 4: FE-idealization of an inhomogeneous rectangular cross section



dimensions: h = 10 mm
b = 20 mm
l = 200 mm

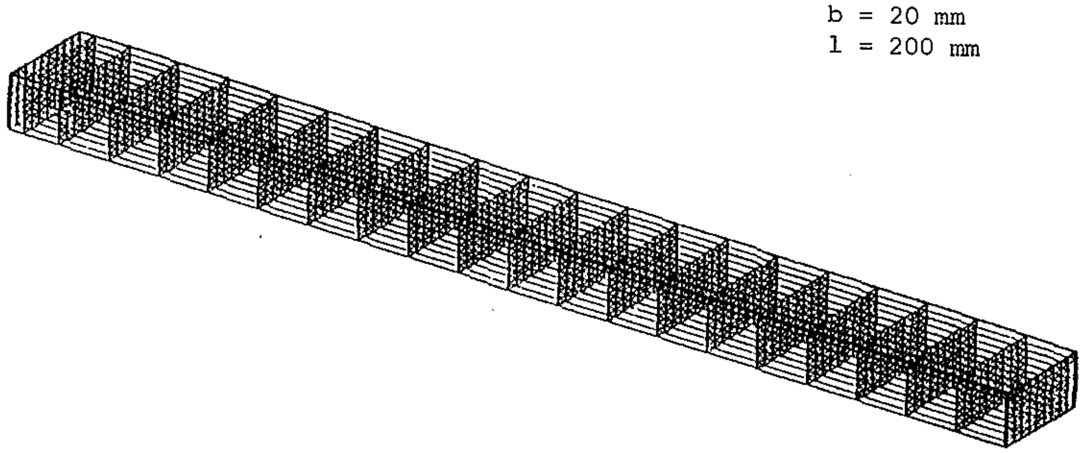
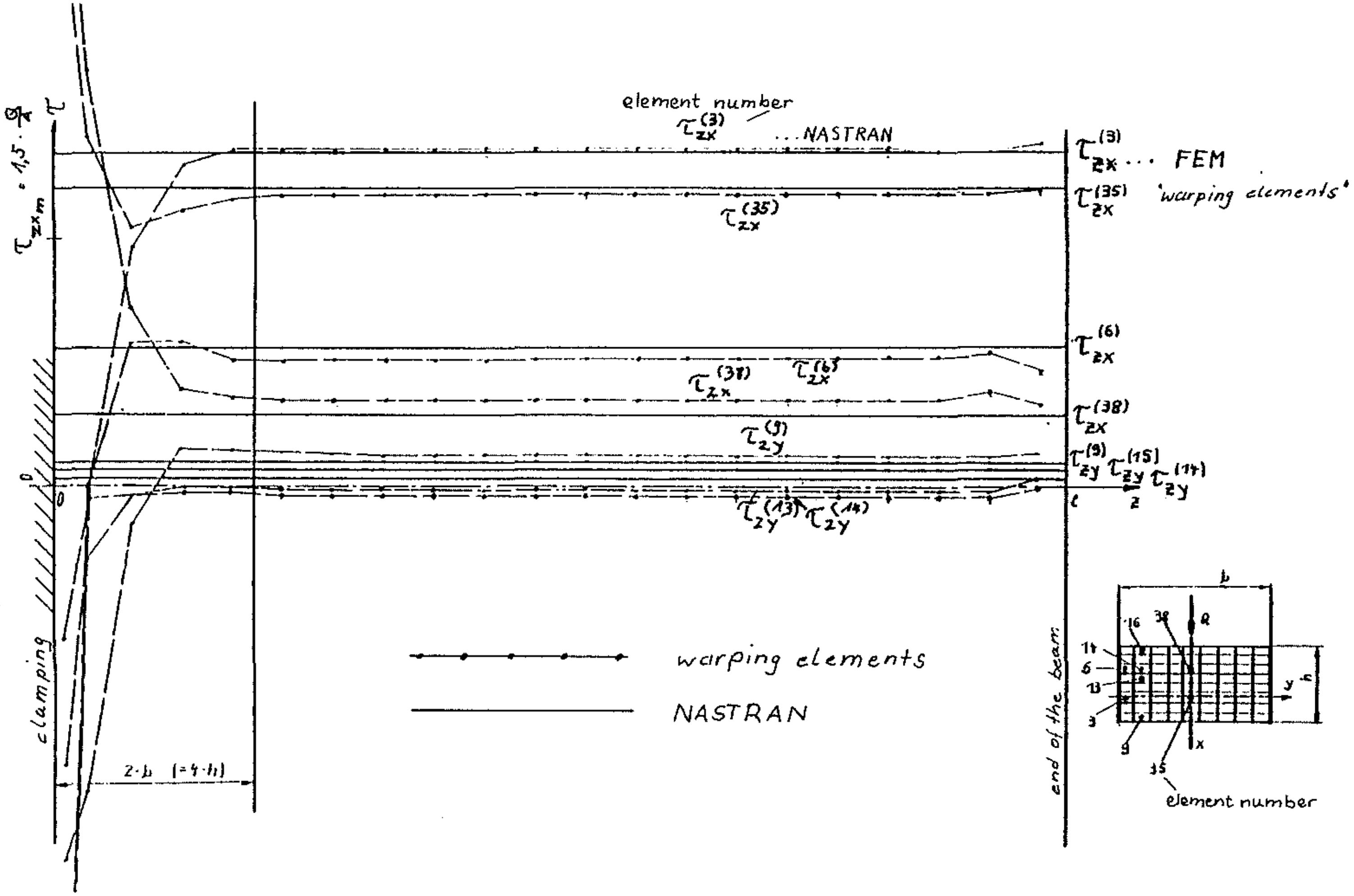


Figure 5: Element idealization of an inhomogeneous, anisotropic cantilever beam by means of three-dimensional elements (1512 elements, 1980 nodes)

In this case the disturbance of the shear stresses due to the load introduction cannot be neglected.



<u>Figure 6:</u> Shear stress distribution in direction of the beam´s longitudinal axis z

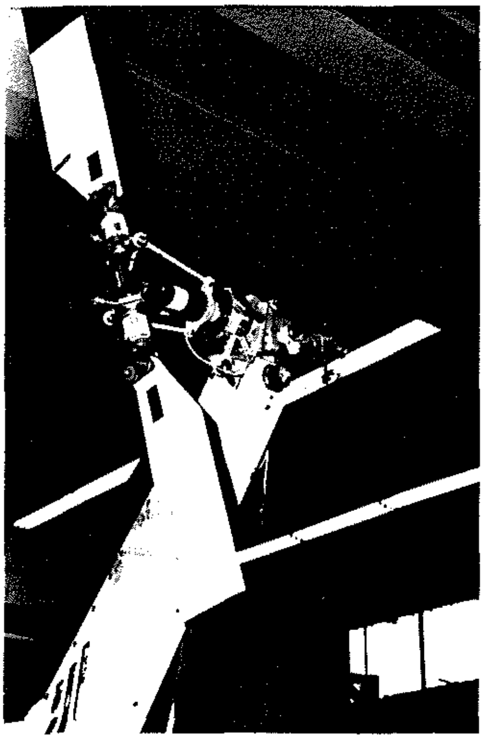


Figure 7: Tail-rotor of the BO105 helicopter

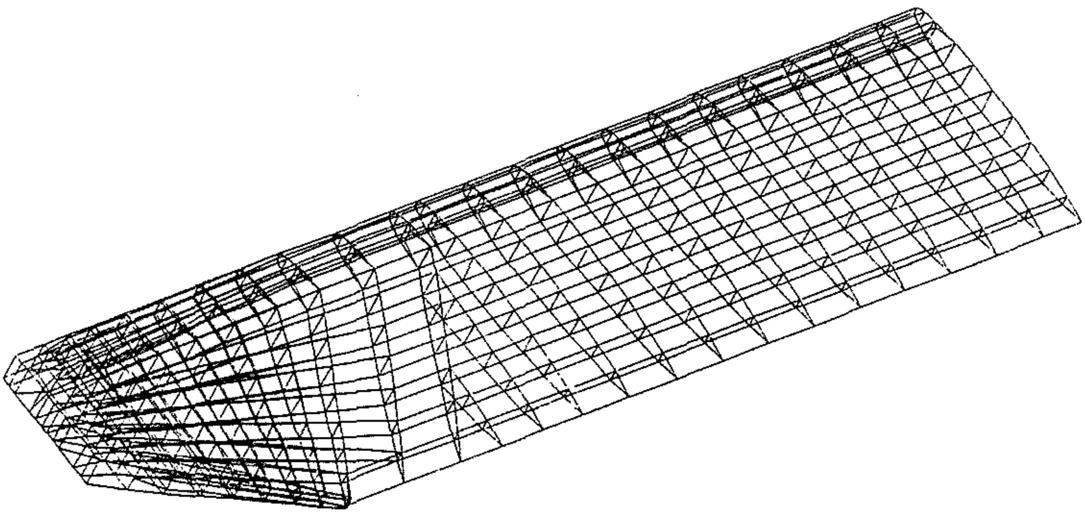


Figure 8: Idealization of the tail-rotor in three-dimensional elements

6. Analysis of rotor blade cross sections

For the helicopters BO105 and BK117, main and tail rotors were analyzed. These blades have inhomogeneous cross sections composed of anisotropic materials. A cross section of the BO105 main rotor is shown in Figure 9.

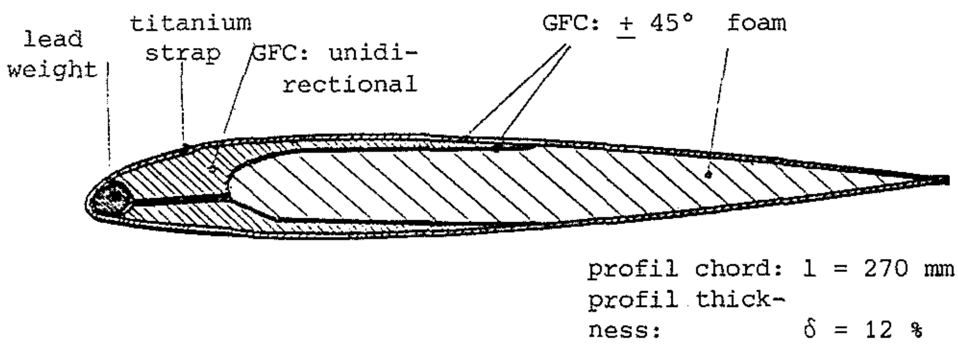


Figure 9: Cross section of the BO105 main rotor blade

The rotor blade is mainly composed of E-glass unidirectional fibers and ± 45°-fabrics and of Conticell C60 foam. The unidirectional layers react both bending moments and centrifugal force which acts on the mass center whereas the ± 45°-fabrics form a stiff closed shell for transfering the torsional moments. The foam supports the glass fiber fabrics and prevents the cross section from beeing deformed. The protection against erosion consists of a titanium strap. By using a lead weight the mass center can be shifted cordally.

The dynamic and static behaviour is mainly influenced by the coordinates of the shear center and the center of elasticity. Other important features are:

- mass center
- main axes
- bending stiffnesses
- torsional stiffness
- longitudinal stiffness

- shear stiffnesses

- distribution of the shear stresses due to torsion and shear forces

The bending stiffnesses as well as the main axes of anisotropic cross sections will not be discussed here. The analysis for calculating the shear center and shear stiffnesses were discribed in detail in the preceding chapters. There is the possibility to calculate shear center by means of the shear stresses resulting from torsion or by means of the shear stresses resulting from transverse bending. This fact is used to control the results given by transverse shear or torsion.

Of much importance is the calculation of the torsional and shear stiffnesses. Usually the chordwise shear stiffness is more important than the flapwise, because of the influence on the dynamic behaviour of the rotor.

To discribe the calculation of the shear stresses due to torsion and transverse bending is the main purpose of this paper. The shear stresses are fundamental for the correct design of rotor blades, especially if bearingless rotors with torsional elastic elements are used. The idealized cross section is shown in Figure 10 with the position of the elastic, shear and mass center.

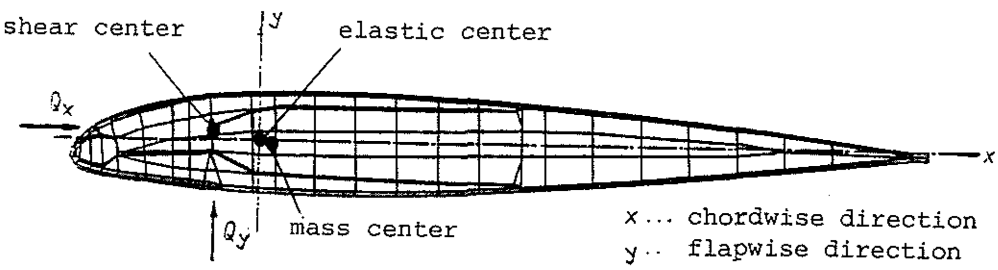


Figure 10: Rotor blade cross section: idealization in triangular and quadrilateral elements

In Figure 11 and 12 the warping due to transverse load in flapwise and chordwise direction is shown.

The stress distribution due to transverse load in flapwise direction is shown in Figure 13 to 16. The maximum shear stress occurs in the flapwise direction in the ± 45° fabric, see Figure 13.

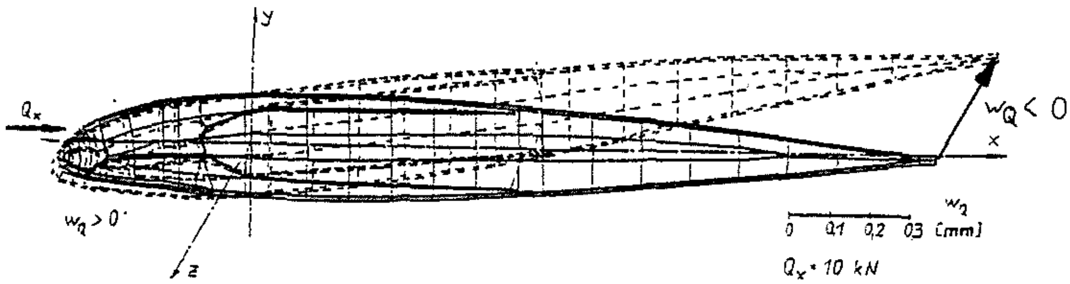


Figure 11: Rotor blade cross section: warping w_Q due to shear load in chordwise direction

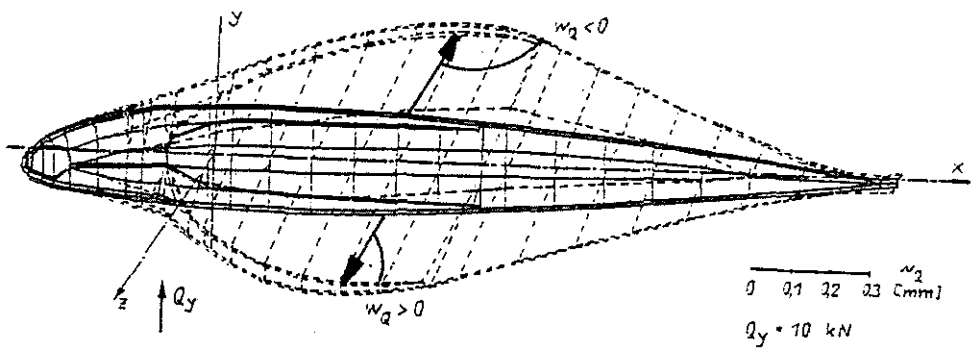


Figure 12: Rotor blade cross section: warping w_Q due to shear load in flapwise direction

The maximum stress normal to the flapwise direction is about 20 % lower, see Figure 15. This stress also occurs in the $\pm$ 45° layer. The maximum shear stresses of the unidirectional material are much lower, but are of major importance because of the low interlaminar shear strength.

The calculated shear stresses differ essentially from those calculated by the simple beam theory.

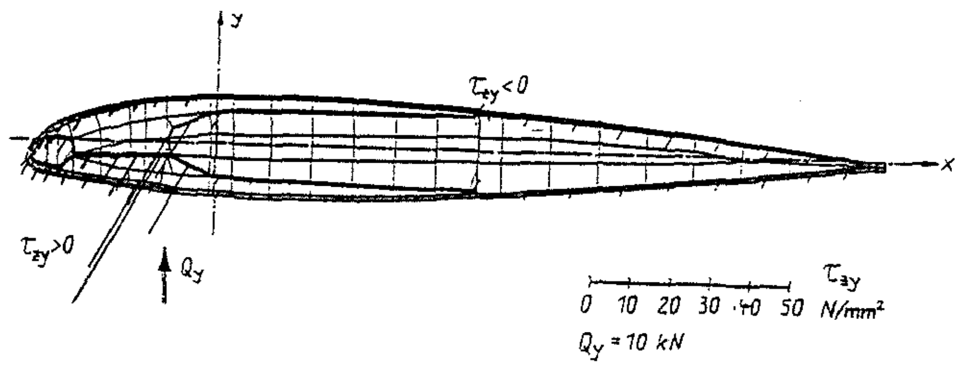


Figure 13: Rotor blade cross section: stress distribution τ_{zy} in the GFC ± 45° fabric due to transverse load in flapwise direction

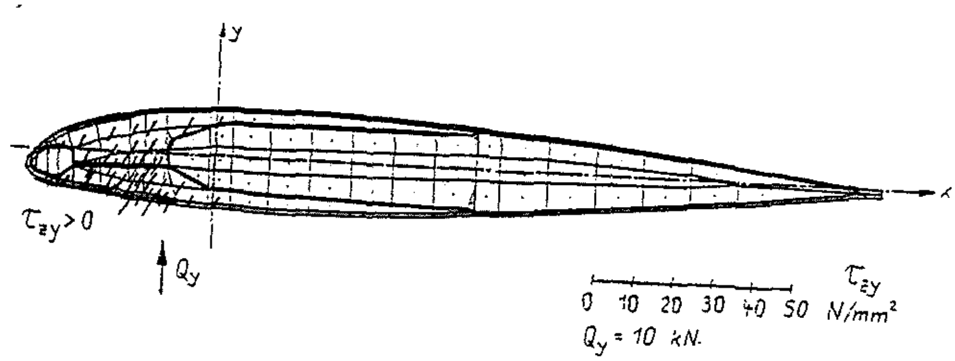
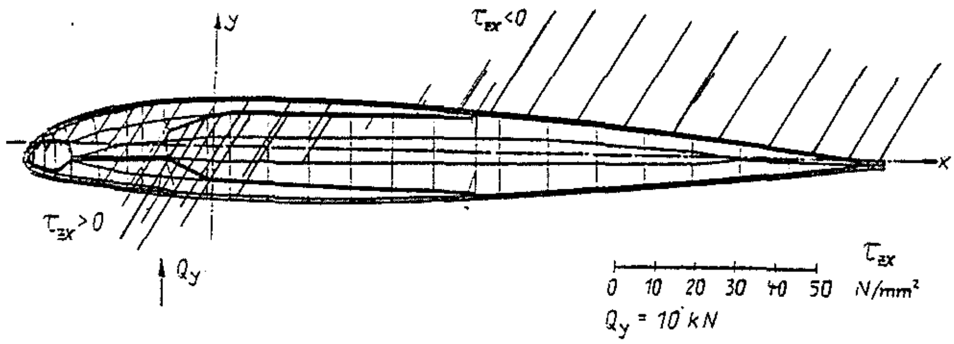


Figure 14: Rotor blade cross section: stress distribution τ_{zy} in the GFC unidirectional laminate and in the foam due to transverse load in flapwise direction.



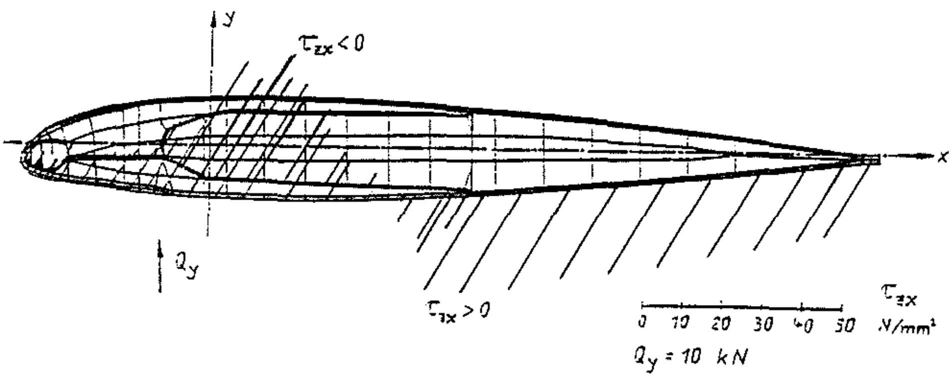


Figure 15: Rotor blade cross section: stress distribution τ_{zx} in the GFC ± 45° fabric due to transverse load in flapwise direction

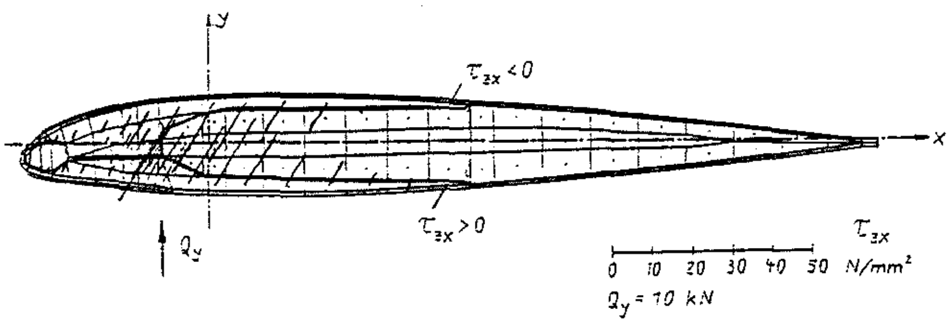


Figure 16: Rotor blade cross section: stress distribution τ_{zx} in the GFC unidirectional laminate and in the foam due to transverse load in flapwise direction

7. Summary

Shear stress distribution and shear stiffness of transverse loaded cylindrical beams of any cross section shape are determined by means of the Finite-Element-Method. For calculating warping resulting from transverse load, special two-dimensional elements were developed. If the warping of any cross section is known, shear stress distribution and some cross section properties, as shear stiffnesses and shear center, can be calculated. The most important point, however, is that these values can be obtained for any cross section by means of a two-dimensional analysis and therefore the whole structure has to be considered.

As a result of the given method it is possible to calculate the stress distribution even for complicated cross sections. The material can be homogeneous or inhomogeneous, isotropic or orthotropic.

Because only small amounts of work effort is necessary, the method can be used in early design stage. For conical structures more cross sections have to be considered. The validity of the given theory is limited to small conus angles.

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