On a Simplified Strain Energy Function for Geometrically Nonlinear Behavior of Anisotropic Beams

Dewey H. Hodges,* Ali R. Atılgan,** Carlos E. S. Cesnik† and Mark V. Fulton‡
School of Aerospace Engineering
Georgia Institute of Technology, Atlanta, Georgia, U.S.A.

Abstract

An asymptotically exact methodology, based on geometrically nonlinear, three-dimensional elasticity, is presented for analysis of prismatic, nonhomogeneous, anisotropic beams. The analysis is subject only to the restrictions that the strain is small relative to unity and that the maximum dimension of the cross section is small relative to a length parameter which is characteristic of the rapidity with which the deformation varies along the beam; thus, restrained warping effects are not considered. A two-dimensional functional is derived which enables the determination of sectional elastic constants, as well as relations between the beam (*i.e.*, one-dimensional) displacement and generalized strain measures and the three-dimensional displacement and strain fields. Since the three-dimensional foundation of the formulation allows for all possible deformations, the complex coupling phenomena associated with shear deformation are correctly accounted for. The final form of the strain energy contains only extensional, bending, and torsional deformation measures - identical to the form of classical theory, but with stiffness constants that are numerically quite different from those of a purely classical theory. Indeed, the stiffnesses obtained from classical theory may, in certain extreme cases, be more than twice as stiff in bending as they should be. Stiffness constants which arise from these various models are used to predict beam deformation for different types of composite beams. Predictions from the present reduced stiffness model are essentially identical to those of more sophisticated models and agree very well with experimental data for large deformation.

*Professor. Member, AHS.

**Post Doctoral Fellow. Member, AHS. Presently Assistant Professor, Mechanics Division, Department of Civil Engineering, Istanbul Technical University, Istanbul, Turkey.

†Fellow of Embraer-Empresa Brasileira de Aeronáutica S. A. Member, AHS.

‡Graduate Research Assistant. Member, AHS.

1. Introduction

When a flexible structure has one dimension that is larger than the other two, it can often be treated as a beam, a one-dimensional structure. Many engineering structures can be idealized as beams, leading to much simpler equations than would be obtained if complete three-dimensional elasticity were used to model the structure. Although dimensional reduction processes can be extremely simple for homogeneous, isotropic, prismatic beams, and especially for restricted cases of deformation, they are far less tractable for composite beams undergoing arbitrary deformation. It is known that, in general, all possible deformations of the three-dimensional structure must be included in the formulation [1,2].

In this paper, we refer to all three-dimensional cross-sectional deformation as "warping" - whether the displacement is in the cross-sectional plane or out of it. All the components of warping in composite beams may be coupled. Also, there may be elastic couplings among all the global deformation components. This means that instead of 6 fundamental stiffnesses (extension, bending in the 2 principal directions, torsion, and shear in the 2 principal directions), there could be as many as 21 (a fully populated, symmetric 6×6 matrix). Furthermore, simple integrals over the cross section will not suffice to determine these elastic constants for the most general case. These complexities make the determination of the elastic constants (what is termed herein as "modeling") a much more difficult task.

There are many possible approaches to this problem found in the literature. The literature prior to 1988 is reviewed in [3]. In work not cited therein, Berdichevsky [4], appears to be the first in the literature to plainly state that "the geometrically nonlinear problem of the three-dimensional theory of elasticity for a beam can be split into a nonlinear one-dimensional problem and a linear two-dimensional problem." This statement was made concerning homogeneous beams with certain material symmetries. As pointed out in [3], this decoupling of the sectional and beam analyses is often assumed to be valid in composite beam analysis. For example, Borri and

Mantegazza [5] used a linear, two-dimensional finite element analysis which is based on [6] to find the 6×6 matrix of cross-sectional elastic constants for use in a nonlinear analysis. (Note that the cross-sectional analysis [6] has been implemented by Borri and his co-workers in a desktop computer program called *N*onhomogeneous *A*nisotropic *B*eam *S*ection *A*nalysis - NABSA.)

In later work Rehfield *et al.* [7] showed that use of the complete 6×6 is not necessary in some cases. Indeed, one can *reduce* the 6×6 matrix by minimizing the strain energy with respect to the transverse shear strain measures. The numerical values of the elements of the resultant 4×4 stiffness matrix may be quite different from those of classical theory, in which shear deformation is ignored altogether. This is because bending-transverse shear elastic coupling can significantly reduce the effective bending stiffness of a beam – possibly by more than 50%!

In this paper, to further explore these issues, we present an anisotropic beam theory from geometrically nonlinear, three-dimensional elasticity. The kinematics are derived for arbitrary warping based upon the general framework of [8]. Next, the three-dimensional strain energy based on this strain field is dimensionally reduced via the variational-asymptotical analysis of [4]. The resulting equations govern both sectional and global deformation, as well as provide the three-dimensional displacement and strain fields in terms of beam deformation quantities. The formulation also naturally leads to geometrically exact, one-dimensional kinematical and intrinsic equilibirum equations for the beam deformation [9].

The relationship of the present extension, bending, and torsional elastic constants to those of the 6×6 stiffness matrix from NABSA is then explored. These constants are then used to calculate the nonlinear static behavior from the one-dimensional beam equations. Finally, correlations with experiments are given as a means of validation, thus testing the predictive capability of the reduced stiffness model described above.

2. Three-Dimensional Formulation

In this section, the three-dimensional displacement and strain fields are developed, giving emphasis to three-dimensional beam geometry. The present analysis can be easily extended to treat initially curved and twisted beams, but herein we will consider only straight and untwisted beams. Here and throughout this paper Greek indices assume values 2 and 3 while Latin indices assume values 1, 2, and 3 and repeated indices are always summed over their range.

Undeformed Beam Geometry

Let x_1 denote length along a reference line r within an undeformed beam. Let x_α denote lengths along lines orthogonal to the reference line r. Let $\mathbf{b}_i$ denote a dextral orthogonal reference triad along the undeformed coordinate lines. The position vector from a fixed point O to an arbitrary point is

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = x_i \mathbf{b}_i = \mathbf{r}(x_1) + x_\alpha \mathbf{b}_\alpha \qquad (1)$$

where $\mathbf{r}(x_1)$ is defined such that

$$\mathcal{A}\mathbf{r}(x_1) = \int_{\mathcal{A}} \hat{\mathbf{r}}(x_1, x_2, x_3) d\mathcal{A} \triangleq \langle \hat{\mathbf{r}}(x_1, x_2, x_3) \rangle \qquad (2)$$

where the use of angle brackets to denote the above integral will be used throughout the rest of the development and where the cross-sectional area $\mathcal{A} = \langle 1 \rangle$. From this one can infer that the reference line is chosen such that

$$\langle x_\alpha \rangle = 0 \qquad (3)$$

In other words, the reference line passes through the centroid of each cross section. Since the beam is assumed to be prismatic, this line is straight.

Deformed Beam Geometry

In a similar manner, consider the deformed beam configuration. The particle which had position vector $\hat{\mathbf{r}}(x_1, x_2, x_3)$ in the undeformed beam now has position vector $\hat{\mathbf{R}}(x_1, x_2, x_3)$. The specific form of $\hat{\mathbf{R}}$ must await the introduction of several entities related to the deformation.

To this end, we introduce another orthonormal triad $\mathbf{B}_i(x_1)$ which we call the deformed beam triad. The vectors $\mathbf{B}_i$ can be specified relative to $\mathbf{b}_i$ by an arbitrarily large rotation, and $\mathbf{B}_i$ coincides with $\mathbf{b}_i$ when the beam is undeformed. Rotation from $\mathbf{b}_i$ to $\mathbf{B}_i$ is described in terms of a matrix of direction cosines $C(x_1)$ such that

$$\mathbf{B}_i = C_{ij}\mathbf{b}_j \qquad C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j \qquad (4)$$

Further specification of the triad $\mathbf{B}_i$ must be postponed until the generalized strain measures are introduced below.

Once a specific form of the displacement field is introduced, the matrix χ whose elements are defined by

$$\chi_{ij} = \mathbf{B}_i \cdot \frac{\partial \hat{\mathbf{R}}}{\partial x_i} \tag{5}$$

can be found. Now, following Danielson and Hodges [8], the polar decomposition theorem shows that χ can be uniquely decomposed into an orthogonal rotation matrix $\hat{C}$ times a symmetric right stretch matrix $\mathcal{U}$

$$\chi = \hat{C}\mathcal{U} \tag{6}$$

Note that $\hat{C}$ is *not* the total rotation, because the global rotation has been effectively removed by resolving χ in mixed bases as implied by Eq. (5). The matrix of Jaumann strain components is then defined by

$$\hat{\Gamma} = \mathcal{U} - I_3 \tag{7}$$

where I_3 is the 3×3 identity matrix and $\hat{\Gamma}$ is a 3×3 symmetric matrix containing the (3-D) Jaumann strain components. The expression for $\hat{\Gamma}$ is quite simple once the components of the deformation gradient are known. Danielson and Hodges [8] were able to show that the strain field can be expressed approximately as

$$\hat{\Gamma} = \frac{1}{2}\left(\chi + \chi^T\right) - I \tag{8}$$

Using the estimation procedure developed in [4] (see below), it is possible to show that this expression is valid as long as the strain is small.

Specification of Displacement Field

Now, for the purpose of later obtaining the strain field in terms of generalized (*i.e.*, one-dimensional) strain measures, we introduce a vector, which is the position vector from O to the points of the reference line of the deformed beam such that

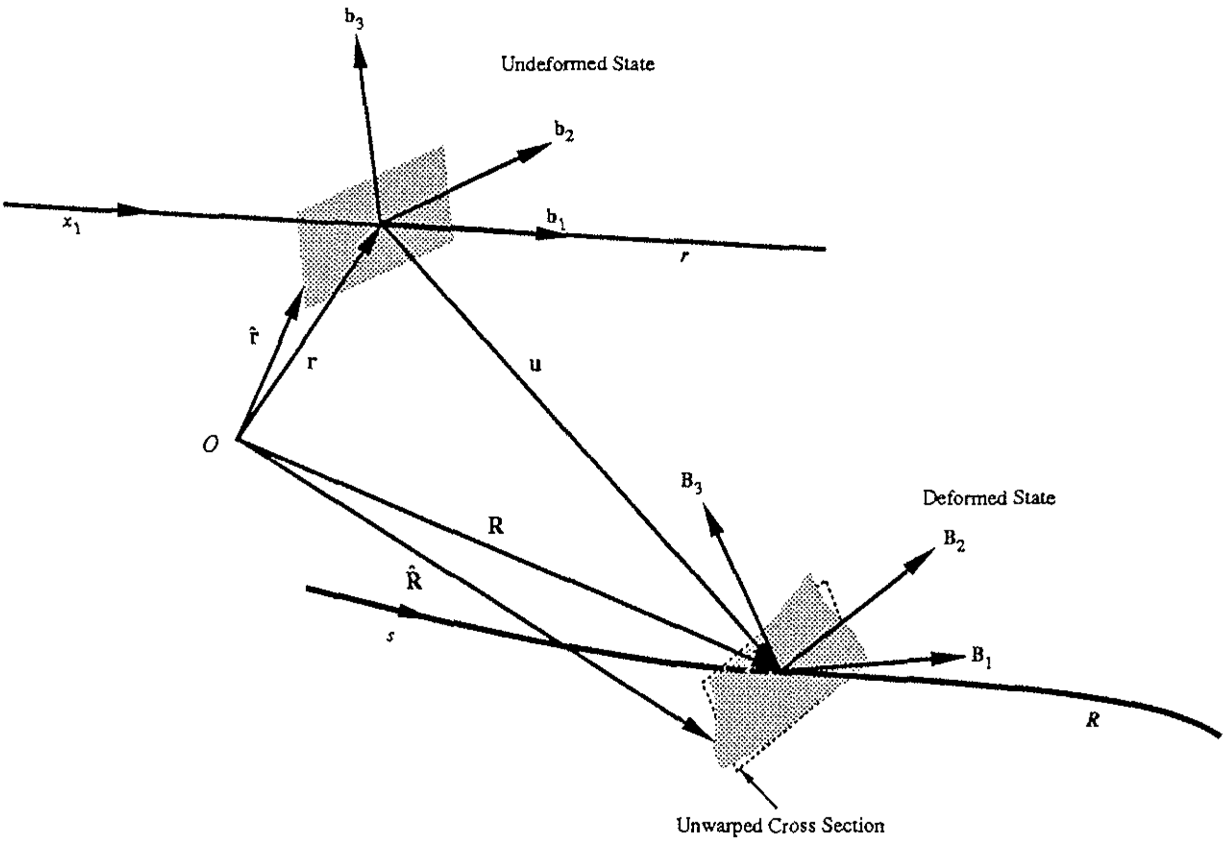

Fig. 1: Schematic of beam deformation

$$\mathbf{R}(x_1) = \mathbf{r}(x_1) + \mathbf{u}(x_1) \triangleq \frac{\langle \hat{\mathbf{R}}(x_1, x_2, x_3) \rangle}{\mathcal{A}} \tag{9}$$

where $\mathbf{u}(x_1)$ is a "displacement" vector, of sorts. This vector is properly understood as the position vector from a point at $x_1 = x_1^*$ on the reference line of the undeformed beam to a point $s(x_1^*)$ on the reference line of the deformed beam (the curved line of centroids for the deformed beam). Here s is the arc-length of the deformed beam reference line, which differs from x_1 by stretching.

The deformation can be described as a small warping displacement superimposed on possibly large rigid-body translation and rotation of the cross section. A schematic of this type of deformation is shown in Fig. 1. Thus, the position vector of any point in the deformed beam can be written as

$$\begin{aligned} \hat{\mathbf{R}}(x_1, \xi_2, \xi_3) =& \mathbf{R}(x_1) \\ &+ h\xi_\alpha \mathbf{B}_\alpha(x_1) \\ &+ hw_i(x_1, \xi_2, \xi_3)\mathbf{B}_i(x_1) \end{aligned} \tag{10}$$

Here we have introduced nondimensional cross-sectional coordinates ξ_α so that $x_\alpha = h\xi_\alpha$, and nondimensional warping displacement, w_i; h is the *maximum cross-sectional dimension.* This description is six times redundant; one can remove this indeterminancy by imposing constraints on the warping. By virtue of the definition of $\hat{\mathbf{R}}$ one can show that the warping must satisfy the following three constraints

$$\langle w_i(x_1, \xi_2, \xi_3) \rangle = 0 \tag{11}$$

With Eq. (11) applied, Eq. (10) is still three times indeterminate. Three more constraints will be introduced in the context of the reduction to one dimension. Note that the orientation of the vector $\mathbf{B}_1$ *is not necessarily tangent to the reference line of* the deformed beam. The orientation of $\mathbf{B}_\alpha$ will not be specified until the generalized strain measures are defined.

Generalized Strains

The strain field can concisely be expressed in terms of so-called generalized strain measures [8]

$$\gamma \triangleq \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} \qquad K \triangleq \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix} \tag{12}$$

where

$$\begin{aligned} \gamma_{11} =& \mathbf{R}' \cdot \mathbf{B}_1 \\ 2\gamma_{1\alpha} =& \mathbf{R}' \cdot \mathbf{B}_\alpha \\ \mathbf{B}_i' =& K_j \mathbf{B}_j \times \mathbf{B}_i \end{aligned} \tag{13}$$

where $(\ )'$ denotes differentiation with respect to x_1. Here γ_{11} is the extensional strain, K_1 is the twist per unit length, K_α are the curvatures of the deformed beam, and $2\gamma_{1\alpha}$ are the transverse shear strain measures.

In addition to the three constraints of Eq. (11), if we choose the direction of $\mathbf{B}_1$ so that it is normal to the plane determined by $\langle \xi_\alpha \hat{\mathbf{R}} \rangle$, then two more constraints on the warping are found to be

$$\langle \xi_\alpha w_1(x_1, \xi_2, \xi_3) \rangle = 0 \tag{14}$$

Because of this, the shear strain measures $2\gamma_{1\alpha}$ from Eq. (13) are in general not zero. The vectors $\mathbf{B}_\alpha$ are determined within a rotation about $\mathbf{B}_1$; they can be fixed with one final constraint

$$\langle \hat{\mathbf{R}}_{,2} \cdot \mathbf{B}_3 - \hat{\mathbf{R}}_{,3} \cdot \mathbf{B}_2 \rangle = 0 \tag{15}$$

which is equivalent to the scalar condition

$$\langle w_{2,3} - w_{3,2} \rangle = 0 \tag{16}$$

The orientation of the kinematical deformed beam triad $\mathbf{B}_i$ relative to $\mathbf{b}_i$ is now specified uniquely; it can thus be represented by an arbitrarily large rotation in terms of orientation angles, Rodrigues parameters, or any suitable angular displacement parameters. For additional discussion of this matter, see [10].

It should be noted that $\mathbf{u}$ is *not* the displacement of a particular material point on the reference line of the undeformed beam, which would be given by

$$\begin{aligned} \left(\hat{\mathbf{R}} - \hat{\mathbf{r}}\right)\Big|_{\xi_2=\xi_3=0} =& \mathbf{u}(x_1) \\ &+ hw_i(x_1, 0, 0)\mathbf{B}_i(x_1) \end{aligned} \tag{17}$$

3. Dimensional Reduction

In constructing a one-dimensional beam theory from three-dimensional elasticity, we attempt to represent the energy stored in a three-dimensional body by finding the energy which would be stored in an imaginary one-dimensional body. This reduction from a three- to a one-dimensional model makes beam modeling more difficult than the analogous process for plates and shells, in which reduction from three to only two dimensions is necessary.

This modeling process *cannot* be performed in an *exact* manner. However, due to the interest of working with a simple one-dimensional theory, researchers have turned to asymptotical methods to reduce the dimension of the model for bodies which contain one or more small parameters. Beams are such bodies because the characteristic cross-sectional dimension of a beam is much smaller than its length.

Thus, in what follows we replace the three-dimensional beam problem by an *approximate* one-dimensional one in which the strain energy per unit length will be a function of only x_1. This will be done with the aid of the variational-asymptotical formulation originally developed by Berdichevsky [4].

In the following sections, we will apply this method for nonhomogeneous, anisotropic beams in order to obtain the asymptotically correct strain energy. Before doing so, however, it is appropriate to discuss the estimation procedure. To keep track of the orders of various terms in the strain field, we introduce a scalar parameter

$$\varepsilon \geq \sqrt{\gamma^T\gamma} + h\sqrt{K^TK} \tag{18}$$

Rather than write out complete expressions for the strain field, we will only write the needed terms of the appropriate order. As a first approximation, we will neglect all terms in the strain energy that are of the order $\mu\varepsilon^2\left(\frac{h}{\ell}\right)^2$, where μ is a constant which is of the order of the material elastic constants, and where ℓ is the lesser constant in the following inequalities

$$|\gamma'_{ij}| \leq \frac{\varepsilon}{\ell} \qquad |K'_i| \leq \frac{\varepsilon}{\ell} \tag{19}$$

This implies that ℓ is representative of the wavelength of the deformation pattern. For this sort of approximation it will turn out that we only need to keep the terms in the strain energy density functional that are of the order $\mu\varepsilon^2$. (Note that for some rotor blade problems it may be necessary to augment these terms with others of the order $\mu\varepsilon^3$ so that the nonlinear coupling between extension and torsion is properly accounted for [11].) This implies that the strains need only be written to $O(\varepsilon)$. To obtain them, one substitutes Eqs. (10) and (5) into Eq. (8). The strain components, which are a linear function of ϵ and $w = \lfloor w_1 \; w_2 \; w_3 \rfloor^T$, can be arranged as a 6×1 column matrix $\Gamma = \lfloor \Gamma_{11} \; 2\Gamma_{12} \; 2\Gamma_{13} \; \Gamma_{22} \; 2\Gamma_{23} \; \Gamma_{33} \rfloor^T$ given by

$$\Gamma = X\epsilon + \partial w \tag{20}$$

where

$$\epsilon \triangleq \begin{Bmatrix} \gamma \\ K \end{Bmatrix} \qquad X \triangleq \begin{bmatrix} I & -\tilde{\xi} \\ 0 & 0 \end{bmatrix} \tag{21}$$

and

$$\partial \triangleq \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix} \qquad \tilde{\xi} \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix} \tag{22}$$

Now the strain energy per unit length can be written as

$$2U = \langle \Gamma^T D \Gamma \rangle = \langle (X\epsilon + \partial w)^T D (X\epsilon + \partial w) \rangle \tag{23}$$

where D is the 6×6 matrix of three-dimensional material properties. This functional is to be minimized with the constraints found in Eqs. (11), (14), and (16). For general nonhomogeneous, anisotropic beams, analytical solutions do not exist. In what follows, a finite element solution of this two-dimensional variational problem will be developed.

Let us discretize the warping as

$$w = SW \tag{24}$$

where the matrix S contains the shape functions and W is the nodal displacement column matrix. Substituting this into the energy functional and taking the variations with respect to W and ϵ, one obtains

$$\delta U = \delta\epsilon^T A \epsilon + \delta W^T E W + \delta\epsilon^T R^T W + \delta W^T R \epsilon \tag{25}$$

where

$$
\begin{aligned}
A &= \langle X^T D\, X\rangle \\
E &= \langle (\partial S)^T\, D\, \partial S\rangle \\
R &= \langle (\partial S)^T D\, X\rangle
\end{aligned}
\tag{26}
$$

are matrices obtained in terms of the weighted integrals of the material properties and the geometry over the cross-sectional domain.

From Eq. (25) follows immediately a solution for the warping

$$\delta W^T (EW + R\epsilon) = 0 \tag{27}$$

The linear system of equations given by Eq. (27) can be solved with the aid of the discretized form of the constraints, which removes the indeterminancies (a total of six). In an equivalent sense, these indeterminancies can be thought of as six linearly dependent rows and columns and the isostatic constraint technique [6] is applicable. The solution is

$$W = \hat{E}^{-1} \hat{R} \epsilon \tag{28}$$

where $(\hat{\ })$ denotes ( ) after the six-fold indeterminacy has been removed.

Therefore, for the first approximation, the total strain energy per unit length is

$$2U = \epsilon^T \hat{S} \epsilon \tag{29}$$

where

$$\hat{S} = A - \hat{R}^T \hat{E}^{-1} \hat{R} \tag{30}$$

Note, however, that $\hat{S}$ can be reduced. This matrix is 6×6 because of the presence of shear deformation. There are also transverse shear related effects associated with slenderness, which are accounted for in higher asymptotical approximations. Thus, for slender beams one may not need to use the full 6×6 form of $\hat{S}$. Minimization of $\hat{S}$ with respect to the transverse shear measures $2\gamma_{1\alpha}$ produces a 4×4 stiffness matrix denoted by $\mathcal{S}$. This minimization is equivalent to undertaking the following operations on the stiffness matrix: (1) invert the 6×6 matrix; (2) ignore the rows and columns associated with transverse shear, leaving a 4×4 matrix; (3) invert this resulting 4×4 matrix yielding the "reduced" stiffness matrix associated with extension, torsion, and two bending measures. The result is an approximate strain energy per unit length of the form

$$2U^* = \begin{Bmatrix} \gamma_{11} \\ K \end{Bmatrix}^T [\mathcal{S}] \begin{Bmatrix} \gamma_{11} \\ K \end{Bmatrix} \tag{31}$$

Thus, the strain energy is in the same *form* as in classical theory (*i.e.*, no shear deformation in the one-dimensional energy). However, the complex coupling effects involving transverse shear are present in the energy, and the numerical values of these elastic constants can differ considerably from those of classical theory, in which shear deformation is set equal to zero at the outset. Note that this reduced form of the one-dimensional strain energy allows for simple modification of existing blade analyses, such as GRASP [12], to treat composite beams.

Cross-Sectional Analysis Code

A cross-sectional analysis code called VABS (*V*ariational-*A*symptotical *B*eam *S*ectional Analysis) has been developed based upon the theoretical formulation presented herein. From it one gets a reduced, asymptotically correct stiffness matrix and warping displacements for a general, nonhomogeneous, anisotropic beam cross section. The discretization of the cross-sectional domain is made with the finite element technique. The element which has been developed is four-noded, planar, and rectangular, with three degrees of freedom per node. The algebraic operations at the element level, including element quadrature, were carried out via symbolic manipulation by using *Mathematica* [13]. This has the main advantage of allowing any kind of element dimensions without loss of accuracy, as can happen when element quadrature is performed numerically [14].

Constraints can be imposed in two equivalent ways: (a) by using Eqs. (11), (14), and (16); and (b) by eliminating rows and columns [6]. Method (b) is better since it requires neither extra memory allocation nor additional computational time to handle extra matrices. The stiffnesses that result from using (a) and (b) are numerically the same, but the warping from (b) must be transformed in order to ensure that it satisfies Eqs. (11), (14), and (16).

4. Nonlinear Beam Analysis

The asymptotically correct expression for the reduced strain energy per unit length of an anisotropic beam is now available from Eq. (31). The expression for the energy is quite simple and the constants of the constitutive law coincide with those of linear theory, *although the theory is valid for arbitrarily large displacements and rotations* (which enter through nonlinear expressions for the generalized strains) as long as the strains remain small.

The one-dimensional elastic law then follows as

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = [S] \begin{Bmatrix} \gamma_{11} \\ K_1 \\ K_2 \\ K_3 \end{Bmatrix} \tag{32}$$

Now, let us recapitulate the ingredients of the theory as it now stands. The beam boundary value problem is based on six nonlinear intrinsic equilibrium equations [9] which contain the six stress resultants (F_1, F_2, F_3, M_1, M_2, and M_3) and the six generalized strain measures (γ_{11}, $2\gamma_{12}$, $2\gamma_{13}$, K_1, K_2, and K_3). Four of the stress resultants and four of the generalized strains are related through the four scalar equations in the elastic law in Eq. (32). The shear forces F_2 and F_3 are not available from the constitutive law, but rather must be determined from the equilibrium equations. The shear strain measures can be calculated by setting

$$\frac{\partial U}{\partial 2\gamma_{1\alpha}} = 0 \tag{33}$$

where U is given by Eq. (29). Recalling the kinematical development above and following the procedure in [9], one can find relations between the six generalized strain measures (γ_{11}, $2\gamma_{12}$, $2\gamma_{13}$, K_1, K_2, and K_3) and the three displacement measures ($\mathbf{u}\cdot\mathbf{b}_i$) and three suitable orientation parameters. The resulting system of 18 equations has 18 unknowns.

One can alternatively use Eq. (29) instead of Eq. (31); for the cases studied below, this choice makes a negligible difference in the results. Both ways possess equivalent energy. If, however, one sets $2\gamma_{1\alpha}$ equal to zero at the outset, one obtains classical theory. As will be seen below, this latter approximation yields incorrect results in some cases.

5. Applications

In this section, numerical results obtained for the stiffness constants and for the global deformation parameters are presented and, where possible, compared with experimental data. Three cantilevered composite beams are considered. We first present results obtained for the stiffness constants of these beam cross sections, based on the approaches outlined above and making use of the programs VABS and NABSA. We then present nonlinear static deflection results for these beams under various loadings. The intent here is to validate that knowledge of the reduced 4×4 stiffness matrix is sufficient to predict static deflections of slender composite beams. This is accomplished by a comparison with previously published experimental results and an examination of the influence of the stiffness calculation on the global deformation.

Comparison of Results for Stiffness Constants

We first verified that VABS gives, for the same elements and mesh, the same stiffnesses as NABSA. We compare all of our results with those from NABSA, which has been shown to yield asymptotically correct extension, bending, torsion, and all possible coupling stiffnesses [2]. The values of the sectional stiffness constants reported herein were obtained from NABSA using a sufficiently large number of 8-noded planar quadrilateral elements to obtain a converged result.

Two of the beams were studied both experimentally and theoretically by Minguet [15]. These have thin rectangular cross sections of width 1.182 in. The two layups are $[45°/0°]_{3s}$ (L1) and $[20°/-70°/-70°/20°]_{2a}$ (L2). The third beam was studied in [16]. It is a rectangular box beam which has layup $[15°]_6$ on all four sides. The exterior of this cross section had a width of 0.953 in. and a depth of 0.53 in., with a total wall thickness of 0.030 in. For all three beams, the material is $AS4/3501-6$ Graphite/Epoxy, the properties of which are given in Table 1.

Table 1: Properties of AS4/3501-6 Graphite/Epoxy [17] (note that the "1" direction is along the fibers and "3" is normal to the laminate)

$$E_{11} = 20.6 \times 10^6 \text{ psi}$$
$$E_{22} = E_{33} = 1.42 \times 10^6 \text{ psi}$$
$$G_{12} = G_{13} = 0.87 \times 10^6 \text{ psi}$$
$$G_{23} = 0.696 \times 10^6 \text{ psi}$$
$$\nu_{12} = \nu_{13} = 0.3; \quad \nu_{23} = 0.34$$

Stiffness results (for S and $\hat{S}$, both denoted generically by S) for these three beam cross sections are shown in Tables 2 – 4. Different choices for the laminate thicknesses produce different stiffness results for the strips. Stiffnesses for the strips were determined from VABS and NABSA based on the so-called "effective thickness" as suggested in Minguet [15]. Specifically, the lamina thicknesses were taken to be 0.05792 in. for (L1) and 0.07565 in. for (L2). The resulting stiffnesses are given in Tables 2 - 3, while those for the box beam are given in Table 4. The heading NABSA$_R$ refers to the reduced form of the NABSA stiffness model. For the (L1) layup, NABSA results were obtained by using 8-noded elements in a 12×20 element mesh while VABS results were based on 4-noded elements in a 12×50 mesh. For the (L2) layup, a 16×10 8-noded element mesh for NABSA and a 16×44 4-noded element mesh for VABS were used. For the box beam case, NABSA used a 216-element proportional mesh with 8 nodes per element. The small differences between the corresponding results from NABSA$_R$ and VABS are basically due to the superior convergence property of the NABSA elements; the influence of these small differences on the static behavior is considered below.

Table 2: Stiffness results (lb., lb.-in., and lb.-in.^{2}) for (L1) (1 extension; 2, 3 shear; 4 torsion; 5, 6 bending)

S	NABSA	NABSA$_R$	VABS
S_{11}	0.8115×10^6	0.7884×10^6	0.7884×10^6
S_{12}	-0.4655×10^5	–	–
S_{22}	0.9368×10^5	–	–
S_{33}	0.6882×10^4	–	–
S_{44}	0.1251×10^3	0.1251×10^3	0.1290×10^3
S_{45}	0.3455×10^2	0.3455×10^2	0.3653×10^2
S_{55}	0.1852×10^3	0.1852×10^3	0.1864×10^3
S_{66}	0.9178×10^5	0.9178×10^5	0.9179×10^5

Table 3: Stiffness results (lb., lb.-in., and lb.-in.^{2}) for (L2) (1 extension; 2, 3 shear; 4 torsion; 5, 6 bending)

S	NABSA	NABSA$_R$	VABS
S_{11}	0.7585×10^6	0.7585×10^6	0.7594×10^6
S_{14}	-0.8587×10^4	-0.8587×10^4	-0.8603×10^4
S_{22}	0.1324×10^6	–	–
S_{25}	0.3636×10^4	–	–
S_{33}	0.9946×10^4	–	–
S_{36}	0.6205×10^2	–	–
S_{44}	0.3675×10^3	0.3675×10^3	0.3683×10^3
S_{55}	0.3762×10^3	0.2763×10^3	0.2778×10^3
S_{66}	0.8460×10^5	0.8460×10^5	0.8491×10^5

Table 4: Stiffness results (lb., lb.-in., and lb.-in.^{2}) for box beam (1 extension; 2, 3 shear; 4 torsion; 5, 6 bending)

S	NABSA	NABSA$_R$
S_{11}	0.1438×10^7	0.1438×10^7
S_{14}	-0.1075×10^6	-0.1075×10^6
S_{22}	0.9018×10^5	–
S_{25}	0.5204×10^5	–
S_{33}	0.3932×10^5	–
S_{36}	0.5637×10^5	–
S_{44}	0.1678×10^5	0.1678×10^5
S_{55}	0.6622×10^5	0.3619×10^5
S_{66}	0.1726×10^6	0.9179×10^5

The reduced 4×4 stiffness matrix either for NABSA$_R$ or for VABS were obtained by the minimization process described above. Note that due to extension-shear coupling there is a certain reduction (2.85%) in the extension stiffness for the (L1) layup. On the other hand, for the (L2) layup, the bending stiffness is reduced by 26.6% due to the bending-shear coupling. More severe still is the case of the box beam problem, in which the bending-shear couplings reduce the bending stiffnesses by about 46%! Changes in the predicted static behavior of a beam which stem from neglecting these effects (*i.e.*, adopting classical beam theory) are considered below.

Sensitivity of Global Behavior to Stiffnesses

The different stiffness modeling approaches yield different stiffnesses, as shown above. Here we consider the predictive capability of the different stiffness models by using these different stiffness constants in the same beam formulation.

The one-dimensional beam formulation adopted here is the mixed, weak formulation derived in [9]. The equilibrium and kinematical equations therein are exact because all terms have been retained; that is, no ordering scheme has been used to create approximate equations. This formulation has been applied to the nonlinear statics [1], linear dynamics [18], and linearized dynamics about nonlinear equilibrium [2]. Here we consider nonlinear static behavior once more, analyzing different laminates and focusing on the effects of the reduced stiffness model.

In Figs. 2 and 3, deflection results from our calculations versus load are compared with experiment for laminated beams (L1) and (L2). Note that for both figures, the deflection components were measured 19.70 in. from the root and the load was applied at the 21.67 in. station. In addition, the beam's deflection due only to its own weight was subtracted from the results such that the deflection curves pass through the origin. The beams are essentially flat strips, both oriented in the horizontal plane, and loaded with vertical transverse loads.

In Fig. 2 the displacements of the symmetric laminate (L1) are shown as a function of the magnitude of the vertical load. The mass per unit length used in the calculations was 1.07×10^{-5} lb. sec.^{2}/in.^{2} [15]. The theoretical results from all the stiffness models, including the full NABSA 6×6, the reduced NABSA 4×4, NABSA with transverse shear deformation set equal to zero (classical theory), and the present result from VABS, all show as one curve to within plotting accuracy, and agree with the experimental data very well. This is not too surprising since in this case the reduction operation only slightly changes the axial stiffness (because of extension-shear coupling). Studying only these results, one could (falsely, as shown below) conclude that transverse shear deformation could be set equal to zero at the outset and not hamper the predictive capability of the model.

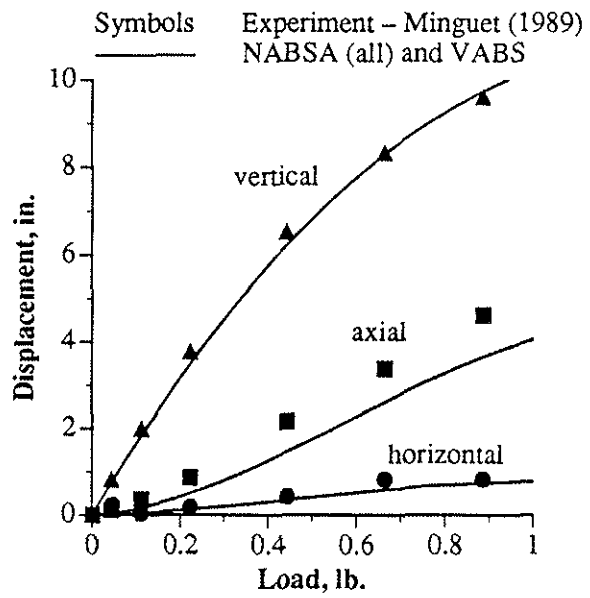

Fig. 2: Displacements of symmetric laminated beam (L1) – for root angle of 0°

In Fig. 3 the displacements of the beam with the antisymmetric laminate (L2) are shown. The mass per unit length was 1.27×10^{-5} lb. sec.^{2}/in.^{2} [15]. The dashed line is the "classical" result obtained by setting shear deformation equal to zero in the strain energy based on the full 6×6 stiffness matrix. These results are clearly inferior because the model is considerably stiffer than it should be. However, the theoretical results from the other three stiffness models, including the full NABSA 6×6, the reduced NABSA 4×4, and the present result from VABS, all show as one curve to within plotting accuracy. This shows that for this case the 4×4 stiffness model is sufficient for predicting the same behavior as the 6×6 full model.

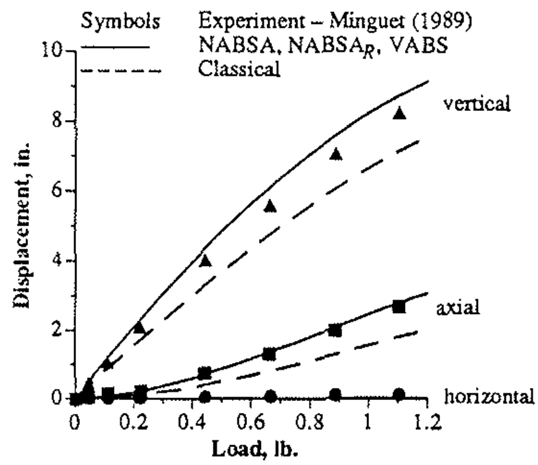

Fig. 3: Displacements of antisymmetric laminated beam (L2) – for root angle of 0°

We now turn to the box beam. In Figs. 4 and 5 the twist of the box beam versus axial coordinate is shown, due to a tip twisting moment and a tip axial force, respectively. The beam axis was parallel to the gravity vector with the tip above the root, and the weight of the beam produces negligible deflections compared to those created by the tip loads. The nonlinearity of the experimental data is due to a very slight restrained warping effect [16] which is not treated in the theory. The theoretical results from three NABSA stiffness models (full, reduced, and with shear deformation set equal to zero) again show as one curve to within plotting accuracy. One might (again falsely) conclude that all of these models are of equal predictive capability. To see that this is not true, Fig. 6 shows the displacements due to a transverse load applied at the tip. As with the (L2) laminated strip, the presence of bending-shear coupling in the full 6×6 stiffness model from

NABSA greatly reduces the effective bending stiffness, as seen in the reduced 4×4 model (see Table 4). The model obtained from setting shear deformation equal to zero (the classical result) is much too stiff as shown in Fig. 6.

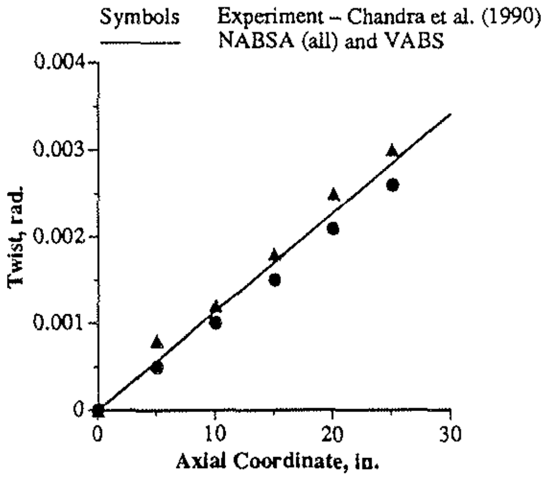

Fig. 4: Twist for box beam for a 1 in-lb twisting moment applied at the tip

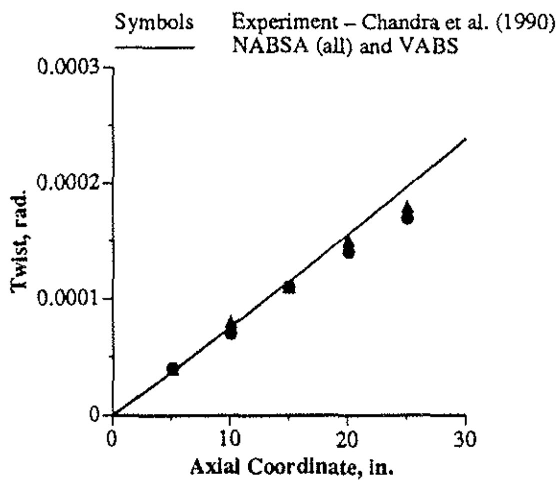

Fig. 5: Twist for box beam for a 1 lb axial force applied at the tip

As noted above, the essential difference in predictive capability between the full 6×6 and the reduced 4×4 stiffness models is related to the slenderness of the beam. If the beam is sufficiently slender for given sectional characteristics, then the reduced model is adequate. A meaningful question, then, is how slender a beam of given sectional stiffness characteristics must be. Rehfield *et al.* [7] treats a circular tube with extension-twist coupling. Results presented therein imply that, for slenderness ratio $\frac{L}{D} \geq 8$ where L is the length and D is the diameter, the reduced model is sufficient.

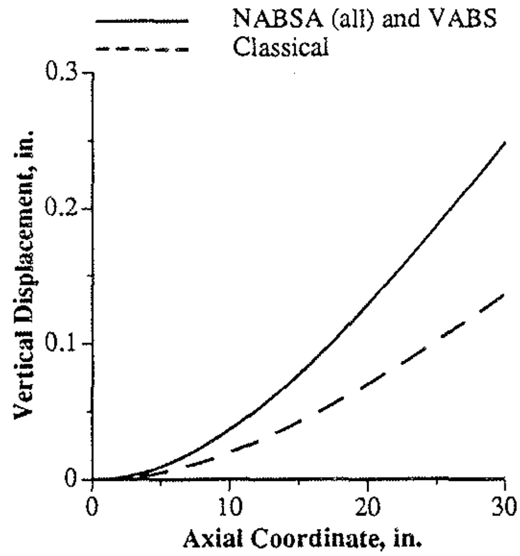

Fig. 6: Vertical displacement for box beam for a 1 lb vertical force applied at the tip

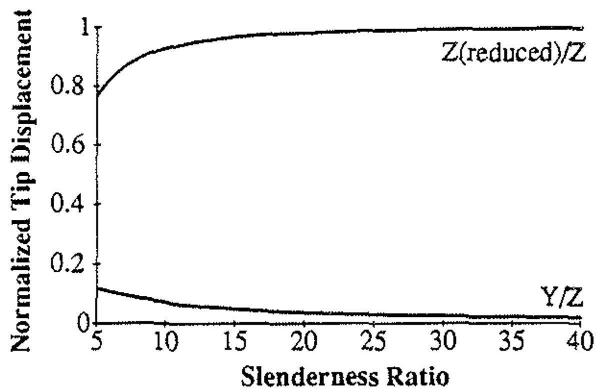

Fig. 7: Normalized tip displacements of box beam for distributed transverse loading versus slenderness ratio $\frac{L}{b}$

The linear solution for the box beam with a uniformly distributed transverse load can be obtained analytically. Consider the tip displacements $Y = u_2(L)$ and $Z = u_3(L)$ from the full 6×6 model, and also corresponding displacements from the reduced model; note that Y(reduced)=0. In Fig. 7, Y

and Z(reduced), both normalized by Z, are shown plotted versus slenderness ratio $\frac{L}{b}$ where b is the width of the box beam. Similar to the results of [7], for beams of modest slenderness, say $\frac{L}{b} \geq 8$, the difference between the full and reduced models is quite small. Also, the horizontal deflection is small for slender beams, indicating that the reduced model is adequate for beams with these sectional characteristics. Dynamic behavior with the reduced model has not yet been investigated.

6. Concluding Remarks

We have presented an asymptotically correct first approximation of composite beam stiffnesses for use in nonlinear deformation theory. The present development is based on the variational-asymptotical method, which allows consistent determination of the governing equations for the complete beam problem, including the three-dimensional relations necessary to predict the displacement, stress, and strain throughout the beam. An asymptotically correct strain energy function was obtained for the case of a generally anisotropic, prismatic, slender beam. The beam deformation is governed by the geometrically exact equations presented in [9].

The splitting of the problem into linear two-dimensional and nonlinear one-dimensional analyses, which is a natural outcome of applying the variational-asymptotical analysis, has been confirmed experimentally for slender composite beams. Recalling that NABSA, which is based on [6], produces a 6×6 matrix of elastic constants, we hypothesized that a reduced 4×4 form of this matrix, obtained from minimizing the energy with respect to the transverse shear strain measures, is sufficient for modeling slender composite beams. The reduced model is of the classical form, but the stiffness constants may be quite different from those of classical theory. The extensional, bending, and torsional constants of the reduced NABSA stiffness matrix are in agreement with our results. The agreement of the predicted nonlinear deformation with experimental data, based on the reduced stiffnesses, appears to confirm our hypothesis. Further work, however, will need to be done in order to investigate dynamic effects.

The form of the one-dimensional strain energy obtained allows for simple modification of existing blade analyses, such as GRASP [12], to treat composite beams. Since real beams may be initially twisted and curved, it is important to extend the work in that direction. Initial twist and curvature not only appear in the equilibrium and kinematical equations, but they also influence the section modeling. Such a refined theory has now been developed by the first and third authors and will be presented in a later paper.

Acknowledgements

Technical discussions with Prof. Victor L. Berdichevsky of Georgia Institute of Technology are gratefully acknowledged. The authors also appreciate the supplying of certain numerical data by Dr. Ramesh Chandra of the University of Maryland and students in the TELAC group at Massachusetts Insitute of Technology. This work was supported by the U.S. Army Research Office under contracts DAAL03-88-C-0003 (the Center of Excellence for Rotary Wing Aircraft Technology) and DAAL03-89-K-0007 of which Dr. Gary L. Anderson is the technical monitor.

References

1. Atılgan, A. R., and Hodges, D. H., "A Unified Nonlinear Analysis for Nonhomogeneous, Anisotropic Beams with Closed Cross Sections," *AIAA Journal*, to appear, 1991.

2. Atılgan, A. R., Hodges, D. H., and Fulton, M. V., "Nonlinear Deformation of Composite Beams: Unification of Cross-Sectional and Elastica Analyses," *Applied Mechanics Reviews*, to appear, 1991.

3. Hodges, D. H., "Review of Composite Rotor Blade Modeling," *AIAA Journal*, Vol. 28, No. 3, 1990, pp. 561 – 565.

4. Berdichevsky V. L., "On the Energy of an Elastic Rod," *PMM*, Vol. 45, No. 4, pp. 518 – 529, 1981.

5. Borri, M., and Mantegazza, P., "Some Contributions on Structural and Dynamic Modeling of Rotor Blades," *l'Aerotecnica Missili e Spazio*, Vol. 64, No. 9, 1985, pp. 143 – 154.

6. Giavotto, V., Borri, M., Mantegazza, P., Ghiringhelli, G., Carmashi, V., Maffioli, G. C., and Massi, F., "Anisotropic Beam Theory and Applications," *Computers and Structures*, Vol. 16, Nos. 1 – 4, 1983, pp. 403 – 413.

7. Rehfield, L. W., Atılgan, A. R., and Hodges, D. H., "Nonclassical Behavior of Thin-Walled Composite Beams with Closed Cross Sections," *Journal of the American Helicopter Society*, Vol. 35, No. 2, Apr. 1990, pp. 42 – 50.

8. Danielson, D. A., and Hodges, D. H., "Nonlinear Beam Kinematics by Decomposition of the Rotation Tensor," *Journal of Applied Mechanics*, Vol. 109, pp. 258 – 262, 1987.

9. Hodges, D. H., "A Mixed Variational Formulation Based On Exact Intrinsic Equations for Dynamics of Moving Beams," *International Journal of Solids and Structures*, Vol. 26, No. 11, 1990, pp. 1253 – 1273.

10. Hodges, Dewey H., "Finite Rotation and Nonlinear Beam Kinematics," *Vertica*, Vol. 11, No. 1/2, pp. 297 – 307, 1987.

11. Danielson, D. A., and Hodges, D. H., "A Beam Theory for Large Global Rotation, Moderate Local Rotation, and Small Strain," *Journal of Applied Mechanics*, Vol. 55, No. 1, 1988, pp. 179 – 184.

12. Hodges, D. H., Hopkins, A. S., and Kunz, D. L., "Analysis of Structures with Rotating, Flexible Substructures Applied to Rotorcraft Aeroelasticity," *AIAA Journal*, Vol. 27, No. 2, 1989, pp. 192 – 200.

13. Wolfram, S., *Mathematica*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1988.

14. Kikuchi, M., "Application of the Symbolic Mathematics System to the Finite Element Program," *Computational Mechanics*, Vol. 5, 1989, pp. 41 – 47.

15. Minguet, P., "Static and Dynamic Behavior of Composite Helicopter Rotor Blades Under Large Deflection," Ph. D. Dissertation, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, (TELAC Report 89-7A), May, 1989.

16. Chandra, R., Stemple, A., and Chopra, I., "Thin-Walled Composite Beams Under Bending, Torsional, and Extensional Loads," *Journal of Aircraft*, Vol. 27, No. 7, 1990, pp. 619 – 625.

17. Students at the MIT Technology Laboratory for Advanced Composites (TELAC), Private Communication, July 1991.

18. Hodges, D. H., Atılgan, A. R., Fulton, M. V., and Rehfield, L. W., "Free-Vibration Analysis of Composite Beams," *Journal of the American Helicopter Society*, to appear, 1991.