THE NONLINEAR OPTIMAL CONTROLLER DESIGN OF HELICOPTER USING THE STATE DEPENDENT MATRIX EXPONENTIAL METHOD

Ji-Seung Hong, Min-Jae Kim, Sun-Goo Oh, Chang-Joo Kim, Aerospace Information System Engineering Department, Konkuk University, Korea

Abstract

This paper treats the State Dependent Matrix Exponential(SDME) technique to solve real time nonlinear optimal control problem. Nonlinear optimal control problem can be trasfored to the Nonlinear Programming (NLP) problem through Direct Multiple Shooting(DMS) technique. And the NLP problem can be resolved by using Quadratic Programming(QP) problem iteratively. Generally calculation of gradients and hessian matrix of the Karush-Kuhn-Tucker(KKT) system for the sequential quadratic programming(SQP) problem is most time-consuming routine in DMS framework. However the required function values to obtain matrices of KKT system can be expressed with integrals involving matrix exponentials without resorting to time integration. Therefore the SDME technique can reduce computational time, increase efficiency and accuracy of nonliear optimal control analysys for real time application of nonlinear optimal controller. The results of optimal control problem using the SDME method are compared with the other method in order to verify the computaional efficience and accuracy.

Nomenclature

A : state derivative matrix
B : control derivative matrix
D : control gradient of equality constraint
E : control gradient of inequality constraint
F : state gradient of equality constraint of matrix exponential
g : inequality constraint function
G : state gradient of inequality constraints
h : equality constraint function
H : Hessian matrix
J : cost function
K : control gain matrix
M : integral involving matrix exponential
N : number of shooting nodes
p : roll rate of search direction
q : pitch rate or control variable over each shooting interval
r : yaw rate
P : matrix solution of ARE
Q : integral involving matrix exponential
R : control weighting matrix
s : state variable at each shooting node
W : integral inveloving matrix exponential
x : state variable
x_0 : initial states or equilibrium states
X : solution of motion equation
$\tilde{x}$: perturbed state variable
u : control variable of longitudinal linear velocity
$\tilde{u}$: perturbed control variable
v,w : linear velocity in y-, z-direction
x : state vector or x-position
x_E,y_N,h : x-, y-positions and altitude
α : control parameter for line search
β : blade flap angle
δ : main rotor and tail rotor controls
ϕ : roll angle or terminal cost function
Φ : operating cost
θ : pitch angle and pitch control
λ : Lagrange multiplier for equality constraint
μ : Lagrange multipliers for inequality constraint
Δ : time variable $\left(=t-t_j\right)$
Δ_j : time step size $\left(=t_{j+1}-t_j\right)$

1. INTRODUCTION

The flight control system of rotorcraft has a short development history because of its instability and nonlinearlity of dynamics compared with fixed wing cases. And flight control system of helicopter which is represented with multi-variables has a limitation on design controller by using the classical control method which can not provide enough robustness of controller.

To improve the efficiency of rotorcraft controller, the design of controller basis of more accurate system model of rotorcraft is needed. Therefore the nonlinear optimal control method that offers enough robustness to controller and makes effective control possible is used to design of helicopter flight controller. Currently nonlinear optimal control techniques are extensively applied for the purpose of increasing efficiency of controller in various technical fields. Also the possibility of real time application of nonlinear optimal control techniques is on the increase gradually from the improvement of computer performance. But in case of rotorcraft which has complex nonlinear equation of motion, the capacity of computer is insufficient currently to use on-line optimal controller. To figure out this problem, many researches that develop the efficient numerical calculation method and apply this to the complex systems are performed at present.

There are various strategies to solve the nonlinear optimal control problem, including indirect multiple-shooting, direct multiple shooting and pseudospectral methods. The preceding direct multiple shooting method is solve the optimal control problem using the Nonlinear Programming method through parameterization and discretization of state and control variables. This paper apply the direct multiple shooting method to design of helicopter controller which is usually preferred for analyzing the nonlinear optimal control problem because of its vantage points for instance large convergence radius and easily handled constraints. The parameterized nonlinear programming problem obtains the solution by using the sequential quadratic problem which is repetition of quadratic problem algorithm. And when an optimal control problem is parameterized with piecewise constant control at DMS framework, the system optimality can be represented with structured KKT system. This type of control parameterization shows the best numerical performance among the DMS techniques, and the resultant KKT system can be effectively obtained for large scale systems. But the DMS technique applied to the complex system just like rotorcraft require a high weighting time even though use today's high-end computing environment. The time-consuming routines in the DMS method generally include the followings

- Building the KKT system,
- Solving the Quadratic Programming(QP) problem to find a search direction
- Conducting one dimensional search to update design variables for the next SQP iteration.

These evaluations require repeated time integration of complex motion equations to approximate gradient and Hessian matrix of cost and constraints function. Therefore, reducing the steps of calculation to estimate matrices can results in direct improvement of computational efficiency.

Accordingly, this paper treats the calculation method of gradients and Hessian matrix on nonlinear system stand on the linear system theory. This paper presents the adoption of the SDC factorization method for the nonlinear systems as a means to derive a linear system-like structure from nonlinear motion equations. In the case of a time invariant linear system, the exact solution under constant controls can be written in the form of a matrix exponential and its integrals. Therefore, the related quadratic cost function and continuity constraints in the DMS method can be represented with various integrals involving matrix exponential. Consequently, the KKT system for the SQP-based DMS method can be built without any time integration of motion equations, as opposed to the conventional method of using time integrators. Such an approach to a nonlinear system, if it works well, can improve the computational efficiency of the DMS method and enhance the accuracy of predictions of the KKT system matrices. In addition, the approach described above places no limitation on handling system constraints as opposed to the SDRE method, because its application is confined to the prediction of the KKT system matrices, specifically the matrices corresponding to the cost function and continuity condition given by system dynamic constraints. The preceding SDME method is applied to the trajectory tracking problem of rotorcraft. And compare the results that are attained through apply SDME and other techniques to helicopter maneuver trajectory tracking problem to verify the effect of accuracy and efficiency.

2. DIRECT MULTIPLE SHOOTING METHOD

2.1 DMS approach to general nonlinear optimal control problems

Nonlinear optimal control problems can be represent ed by the standard Bolza form:

(1) $$\min_{x,u,t_f} J(x,u,t_f) = \phi(x(t_f)) + \int_{t_0}^{t_f} \Phi(x,u)dt$$

s.t.

(2) $$\dot{x}(t) = f(x(t),u(t),t), \quad t \in [0,t_f]$$
$$x(t_0) = x_0$$

(3) $$h(x(t_f),t_f) = 0$$

(4) $$g(x(t),u(t),t) \le 0$$

The DMS method transforms the above equations into a solvable nonlinear programming problem in finite dimension using suitable state and control parameterization methods. Namely, The above standard Bolza form formulation can be reduced to the NLP problem (5) through the DMS framework. The detail process of DMS framework is given in Ref. 1. The resulting NLP problem can be effectively resolved using the SQP method. The equality and inequality constraints of NLP problem shown in the equation (6).

(5)
$$\min J(s.q) = \phi(s_j) + \sum_{i=0}^{N-1} J_j(s_j, q_j)$$
$$J_j(s_j, q_j) = \int_j^{j+1} \Phi(x_j(t; s_j, q_j), q_j) dt$$

s.t

(6)
$$h_j(s_j, q_j) = X_j(t_{j+1}; s_j, q_j) - s_{j+1} = 0 \quad (j = 0, ..N-1)$$
$$h_N = (s_N, t_N) = 0$$
$$g_j(s_j, q_j, t_j) \le 0, \quad (j = 1, ..., N)$$

The SQP framework for solving above NLP problems is an iterative solution procedure which consists of a line search procedure and the the following QP:

(7)
$$\min J(\mathbf{p}) = \sum_{j=0}^{N} J_j(p_j^s, p_j^q)$$

s.t.

(8)
$$F_j p_j^s + D_j p_j^q + h_j - p_{j+1}^s = 0, \qquad j = 0, \ldots, N$$

(9)
$$G_j p_j^s + E_j p_j^q + g_j \le 0, \qquad j = 1, \ldots, N-1$$

where

$$\mathbf{p} = \left[p_0^q\ ,\ p_1^s\ ,p_1^q\ ,, \cdots ,\ p_{N-1}^s\ ,p_{N-1}^q, p_N^s\ \right]^T$$

$$J_0 = J_0^q p_0^q + \frac{1}{2}(p_0^q)^T H_0^{qq} p_0^q$$

$$J_j = \left(J_j^s \mid J_j^q\right)\begin{pmatrix} p_j^s \\ p_j^q \end{pmatrix} + \frac{1}{2}\left((p_j^s)^T \mid (p_j^q)^T\right)\begin{bmatrix} H_j^{ss} & (H_j^{qs})^T \\ H_j^{qs} & H_j^{qq} \end{bmatrix}\begin{pmatrix} p_j^s \\ p_j^q \end{pmatrix}$$
$$j = 1, \ldots, N-1$$

$$J_N = \left(\phi_N^s\right) p_N^s + \frac{1}{2}(p_N^s)^T\left(\phi_N^{ss}\right) p_N^s$$

$$J_j^s = \frac{\partial J_j}{\partial s_j}, J_j^q = \frac{\partial J_j}{\partial q_j}$$

$$H_j^{ss} = \frac{\partial^2 J_j}{\partial s_j^2}, H_j^{qs} = \frac{\partial^2 J_j}{\partial q_j \partial s_j}, H_j^{qq} = \frac{\partial^2 J_j}{\partial q_j^2}$$

$$\phi_N^s = \frac{\partial \phi}{\partial s_N}, \phi_N^{ss} = \frac{\partial^2 \phi}{\partial s_N^2}$$

$$F_j = \frac{\partial h}{\partial s_j}, D_j = \frac{\partial h}{\partial q_j}, h_j = h(s_j, q_j, t_j), \qquad j = 0, \ldots, N$$

$$F_0 = D_N = 0$$

$$G_j = \frac{\partial g}{\partial s_j}, E_j = \frac{\partial g}{\partial q_j}, g_j = g(s_j, q_j, t_j), \qquad j = 1, \ldots, N-1$$

The adjoined cost function can be defined using Lagrange multipliers λ_j and μ_j for the active constraints as follows:

(10)
$$L(p, \lambda, \mu) = \sum_{j=0}^{N} J_j\left(p_j^s, p_j^q\right) + \sum_{j=0}^{j=N} \lambda_j^T (h_j + F_j p_j^s + D_j p_j^q - p_{j+1}^s) + \sum_{j=1}^{j=N-1} \mu_j^T (g_j + G_j p_j^s + E_j p_j^q)$$

The KKT condition can be derived with the constraint equation shown in equations. (8) And (9), by setting the gradient of the above adjoined cost function to zero. The detailed derivation is given in Ref. 7, and the related KKT system for j=2,…,N-1 can be summarized as:

(11)
$$\begin{bmatrix} H_j^{ss} & (H_j^{qs})^T & (F_j)^T & (G_j)^T \\ H_j^{qs} & H_j^{qq} & (D_j)^T & (E_j)^T \\ F_j & D_j & & \\ G_j & E_j & & \end{bmatrix}\begin{bmatrix} p_j^s \\ p_j^q \\ \lambda_j \\ \mu_j \end{bmatrix} + \begin{bmatrix} (J_j^s)^T - \lambda_{j-1} \\ (J_j^q)^T \\ h_j - p_{j+1}^s \\ g_j \end{bmatrix} = 0$$

In Eq. (11), only active inequality constraints should be included with the positivity condition of $\mu_j \ge 0$. Therefore, the resultant KKT system is written as a linear system with a banded structure. Since the local cost function and continuity conditions include time integration terms, the gradient vectors of those functions requires repeated time integrations of motion equations, which are generally the most time-consuming elements in the SQP-based DMS method.

2.2 DMS approach to LQR problems with integrals involving matrix exponential

Consider the following local quadratic cost function in an LQR problem.

(12)
$$J_j(x, u) = \int_j^{j+1}\left(\frac{1}{2}x(\tau)^T Q_C x(\tau) + \frac{1}{2}u(\tau)^T R u(\tau)\right) d\tau$$

s.t

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$x(t_j) = s_j$$

The exact solution of a linear time invariant system under constant controls q_j , which is shown in Eq. (13), can be represented by Eq. (14).

(13) $\dot{x} = Ax + Bq_j$ with $x(t_j) = s_j$

(14) $x(t) = F(t - t_j)s_j + H(t - t_j)q_j$

where

(15)
$$F(t - t_j) = e^{A(t - t_j)}$$
$$H(t - t_j) = \int_{t_j}^{t} e^{A(t-\tau)} B d\tau$$

The states at t_{j+1} and the local cost over $[t_j, t_{j+1}]$ as follows:

(16) $s_{j+1} = F(\Delta_j)s_j + H(\Delta_j)q_j$

$$J_j(s_j, q_j) = \int_0^h \left\{ \begin{array}{l} \frac{1}{2}\left(F(\Delta)s_j + H(\Delta)q_j\right)^T \\ Q_C\left(F(\Delta)s_j + H(\Delta)q_j\right) \\ + \frac{1}{2} q_j{}^T R q_j \end{array} \right\} d\tau$$

(17)
$$= \frac{1}{2} s_j{}^T Q s_j + s_j{}^T M q_j + \frac{1}{2} q_j{}^T W q_j + \frac{h}{2} q_j{}^T R q_j$$

where

$$h_j = t_{j+1} - t_j$$
$$\Delta = t - t_j$$
$$\Delta_j = t_{j+1} - t_j$$

(18) $F(\Delta) = e^{A(t - t_j)}$

(19) $H(\Delta) = \int_{t_j}^{t} e^{A(t-\tau)} B d\tau$

(20) $Q = \int_0^{\Delta_j} F(\tau)^T Q_C F(\tau) d\tau$

(21) $M = \int_0^{\Delta_j} F(\tau)^T Q_C H(\tau) d\tau$

(22) $W = \int_0^{\Delta_j} H(\tau)^T Q_C H(\tau) d\tau$

Therefore, the KKT system matrices related to the cost function and dynamic constraints can be expressed as a matrix exponential and its weighted integrals:

$$F_j = F(\Delta_j)$$
$$D_j = H(\Delta_j)$$
$$J_j^s = s_j{}^T Q + q_j{}^T M^T$$
$$J_j^q = s_j{}^T M + q_j{}^T W + h_j q_j{}^T R$$
$$H_j^{ss} = Q$$
$$H_j^{qs} = M^T$$
$$H_j^{qq} = W^T + h_j R$$

Since the derivation of the KKT system requires no time integration, the gradients and Hessian matrices can be evaluated with good accuracy so long as accurate numerical methods are used in estimating the related integrals. This approach is well suited to the direct multiple-shooting method parameterized with constant controls and with finely distributed shooting nodes.

3. FORMULATION OF NONLINEAR OPTIMAL CONTROL PROBLEMS

By assuming the SDC matrices $A(\tilde{x})$ and $B(\tilde{x})$ are locally constant, the SDC factorization method allows the definition of the SDME to be the same as in the linear system case. Thus, initial value problems to calculate the states and local cost over each interval can be defined as:

(23) $\dot{\tilde{x}}(t) = A_j(\tilde{x})\tilde{x}(t) + B_j(\tilde{x})\tilde{q}_j, \quad j = 0,1,\ldots,N-1$

where the initial condition and the control input can be written as:

$$\tilde{x}(t_j) = s_j - x_0(t_j)$$
$$\tilde{q}_j = q_j - u_0$$

At equilibrium, the control u_0 should be constant but the state x_0 can vary with a constant $\dot{x}_0$.

The solution of Eq. (23) can be approximated at the interval $t \in [t_j, t_{j+1}]$, as is done in the LQR problem:

(24) $\tilde{x}(t) = F(\Delta)\left(s_j - x_0(t_j)\right) + H(\Delta)\left(q_j - q_0\right)$

with $\Delta = t - t_j$

(25) or $x(t) = x_0(t) + F(\Delta)\tilde{s}_j(t) + H(\Delta)\tilde{q}_j$

with $\tilde{s}_j(t) = s_j - x_0(t_j)$

Then, the state solution and the equality constraints at (j+1)th shooting node can be written as:

(26) $X_j(t_{j+1}; s_j, q_j) = x_0(t_{j+1}) + F(\Delta_j)\tilde{s}_j(t_{j+1}) + H(\Delta_j)\tilde{q}_j$

with

(27) $\Delta_j = t_{j+1} - t_j$

(28) $h_j(s_j, q_j, t_j) = X_j(t_{j+1}; s_j, q_j) - s_{j+1}$

Therefore, the gradient of the equality constraints will have the same form as in the case of a linear system.

$$F_j = \frac{\partial X_j}{\partial s_j} = F(\Delta_j)$$

$$D_j = \frac{\partial X_j}{\partial q_j} = H(\Delta_j)$$

In order to derive the KKT system matrices for the cost function, consider the following local quadratic cost function for a trajectory tracking problem in the DMS framework:

(29) $$J_j(s_j, q_j) = \frac{1}{2}\int_{t_j}^{t_{j+1}} \Phi_j\left(x_j(t; s_j, q_j), q_j, t\right)dt$$

(30) $$\Phi_j(s_j, q_j, t) = \left(x(t) - x_R(t)\right)^T Q_C \left(x(t) - x_R(t)\right) + \left(q_j - u_0\right)^T R \left(q_j - u_0\right)$$

where

$x_R(t)$: prescribed reference trajectory

$t \in [t_j, t_{j+1})$

(31) $$x(t) - x_R(t) = \bar{x}(t) + F\tilde{s}_j(t) + H\tilde{q}_j$$

where

$\bar{x}(t) = x_0(t) - x_R(t)$

By substituting Eq. (30) into Eq. (29), the function $\Phi_j(s_j, q_j, t)$ can be written as:

(32) $$\begin{aligned}\Phi_j(s_j, q_j, t) &= \bar{x}(t)^T Q_C \bar{x}(t) + \tilde{s}_j^{\,T} F^T Q_C F\tilde{s}_j(t) \\ &+ \tilde{q}_j^{\,T} H^T Q_C H\tilde{q}_j + \tilde{q}_j^{\,T} R\tilde{q}_j \\ &+ 2\bar{x}(t)^T Q_C F\tilde{s}_j(t) \\ &+ 2\tilde{s}_j^{\,T}(t) F^T Q_C H\tilde{q}_j \\ &+ 2\bar{x}(t)^T Q_C H\tilde{q}_j\end{aligned}$$

The final formulation can be simplified by defining two matrices, $H_{B=I}$ and $M_{F=I}$, which can be obtained by substituting Eqs. (19) and (21) with $B = I$ and $F = I$, respectively.

$$H_{B=I} = \int_j^{j+1} F(\Delta)d\tau$$

$$M_{F=I} = \int_j^{j+1} Q_C H(\Delta)d\tau$$

Then, the local cost function can be rewritten as:

$$\begin{aligned}J_j(s_j, q_j) &= \frac{1}{2}h_j \bar{x}(\bar{t})^T Q_C \bar{x}(\bar{t}) + \frac{1}{2}\tilde{s}_j^{\,T}(\bar{t}) Q \tilde{s}_j(\bar{t}) + \frac{1}{2}\tilde{q}_j^{\,T} W\tilde{q}_j \\ &+ \frac{1}{2}h_j \tilde{q}_j^{\,T} R\tilde{q}_j + \bar{x}(\bar{t})^T Q_C \left(H_{B=I}\right)\tilde{s}_j(\bar{t}) \\ &+ \bar{x}(\bar{t})^T \left(M_{F=I}\right)\tilde{q}_j + \tilde{s}_j(\bar{t})^T M \tilde{q}_j\end{aligned}$$

Therefore, the gradients and Hessian matrices for the cost function have the following forms for the nonlinear quadratic regulator problem:

(33) $$J_j^s = \bar{x}(\bar{t})^T Q_C \left(H_{B=I}\right) + \tilde{s}_j(\bar{t})^T Q + \tilde{q}_j^{\,T} M^T$$

(34) $$J_j^q = \bar{x}(\bar{t})^T \left(M_{F=I}\right) + \tilde{s}_j(\bar{t})^T M + \tilde{q}_j^{\,T} W + h_j \tilde{q}_j^{\,T} R$$

$$H_j^{ss} = Q$$

$$H_j^{qs} = M^T$$

$$H_j^{qq} = W^T + h_j R$$

Since the initial value problems is completely solvable with initial conditions and control history, dynamic variation in the equilibrium states appears as dummy states originating from the superposition principle in the linear system theory. These states are generally related to slow dynamics, as in the position change in x-, y-, and z-directions. Variables to which no control action is applied can be excluded from the optimal control formulation. For example, if the objective of a turn maneuver is to track a prescribed heading angle while maintaining flight speed and altitude, there are no control actions on the x- and y-position. These non-active states can be predicted simply by integrating the related kinematical equations using the optimal control solution. Therefore, the system size can be reduced and the matrix condition of the resultant KKT system be improved because the continuity constraints for the slow dynamics generally have poor sensitivity to other states and controls. The corresponding relations for the cost function gradients have the following form:

(35) $$J_j^s = \left(x_0 - x_R(\bar{t})\right)^T Q_C \left(H_{B=I}\right) + \tilde{s}_j^{\,T} Q + \tilde{q}_j^{\,T} M^T$$

(36) $$J_j^q = \left(x_0 - x_R(\bar{t})\right)^T \left(M_{F=I}\right) + \tilde{s}_j^{\,T} M + \tilde{q}_j^{\,T} W + h_j \tilde{q}_j^{\,T} R$$

Since various numerical experiments based on the above gradient formulation present diverging solutions, the following modified gradient information is used, with which the trajectory tracking capability can be retained:

(37) $$J_j^s = \Delta\bar{s}(\bar{t}_j)^T Q_C \left(H_{B=I}\right) + \tilde{s}_j^{\,T} Q + \tilde{q}_j^{\,T} M^T$$

(38) $$J_j^q = \Delta\bar{s}(\bar{t}_j)^T \left(M_{F=I}\right) + \tilde{s}_j^{\,T} M + \tilde{q}_j^{\,T} W + h_j \tilde{q}_j^{\,T} R$$

where

$\Delta\bar{s}(\bar{t}_j) = \frac{1}{2}(s_j + s_{j+1}) - x_R(\bar{t}_j)$ with $\bar{t}_j = 0.5(t_j + t_{j+1})$

Since the resultant formulae for nonlinear systems are nearly the same as those for linear systems, the increase in computing time required to estimate the related gradients and Hessian matrices mainly depends on the SDC factorization of the nonlinear motion equations. Even though the SDC form of the motion equations should be repeatedly derived at each time interval for the nonlinear system, the number of function calls required for the numerical calculation of the gradients and Hessian matrices are generally much higher than the number required for the SDC factorization. Furthermore, this method places no limitation on the numerical solution of a nonlinear optimal control problem with state and control constraints, and it also provides a direct estimation of the Hessian matrices. The SQP algorithm generally adopts the BFGS (Broydon-Fletcher-Goldfrab-Shannon) formula to iteratively update the Hessian matrix. Since gradient information alone is required, the computation time can be greatly reduced by this technique. However, no general methods are capable of providing accurate initial guesses for the Hessian matrices. Since the cost function used in the SQP method interpolates the real cost function with the gradients and Hessian matrices, any prediction errors in the estimation of these matrices can directly affect numerical convergence and solution accuracy. Therefore, the proposed method can be used to provide good initial guesses of the Hessian matrices for the iterative BFGS method.

4. DYNAMIC MODELS AND NUMERICAL METHODS

The numerical methods outlined in the previous sections were applied to a nonlinear optimal control formulation of a rotorcraft slalom maneuver. Rotorcraft maneuver problems are extensively studied by the present authors[10, 14, 17-19]. Many contemporary applications are based on those previous research efforts that cover the rotorcraft flight dynamic modeling, optimal inverse simulation with the indirect method or the SDRE technique, etc. The related details are well documented in each cited reference. The major features of the rotorcraft models and numerical methods that are used in this paper are introduced briefly below.

4.1 Rotorcraft Model and Maneuver Trajectory

This paper presents the usage of a rotor dynamic model, proposed by R. T. N. Chen[20], where hub fixed flap states are used to derive the closed form expression for aerodynamic forces and moments based on quasi-linear aerodynamic theory. In this study, the aerodynamic forces and moments, generated by rotors, are calculated using the main and tail rotor trim solution. Since this study is focused on the possible use of linear system theory for the nonlinear optimal control problems using the SDME, the selected modeling retains enough nonlinear features for research purposes. The resultant computational burden can then be estimated based on the results from Ref [10]. For LQR problems, a linear time-invariant model is derived using a finite difference formula around trim flight conditions. A trajectory can be expressed as the sum of states at maneuver entry and its variation during the maneuver.

(39) $$x(t) = x(t_{entry}) + \Delta x(t)$$

(40) or $$x(t) = x(t_{entry}) + \int_{t_{entry}} \Delta\dot{x}(\tau)d\tau$$

The lateral-position change during a slalom maneuver is initially described with the following formula:

(41) $$\Delta x(\bar{t}) = \frac{(\Delta x)_{\max}}{46.8}\begin{bmatrix} 32 + \sin(2\pi\bar{t}) - 20\sin(4\pi\bar{t}) \\ + 2\sin(8\pi\bar{t}) \end{bmatrix}$$

where

$\bar{t} = (t - t_{entry})/(t_{finish} - t_{entry}) \quad 0 \le \bar{t} \le 1$

The times t_{entry} and t_{finish} designate maneuver entry and finish times, respectively, and $(\Delta x)_{\max}$ is the maximum amplitude of the Y-position, which determines the maneuver aggressiveness for a given duration. These trajectory deviations are considered when we define a cost function for optimal control problems. The following form of the quadratic cost function with no terminal cost is implemented in this study[14]:

(42) $$f_{CO}(\bar{x}_R(t), u(t), t) = 0.5(\bar{x}_R - \bar{x}_{target})^T Q_C(\bar{x}_R - \bar{x}_{target}) + 0.5(u - u_{trim})^T R(u - u_{trim})$$

where

$\bar{x}_R$: reduced rigid body states

$\bar{x}_{target}$: target states

$$\bar{x}_R(t) = [u, v, w, p, q, \dot{\psi}, \phi, \theta, \psi, x_E, y_N, h]^T$$
$$R = diag(r_{\delta_0},\ r_{\delta_{1C}},\ r_{\delta_{1S}},\ r_{\delta_{TR}})$$
$$Q_C = diag(\ q_u,\ q_v,\ q_w,\ q_p,\ q_q,\ q_{\dot{\psi}},\ q_\phi,\ q_\theta,\ q_\psi,\ q_{xE},\ q_{yN},\ q_H)$$

For simplicity of analysis, no system constraints are imposed. The target states $\bar{x}_{target}(t)$ are set to be the trim states $(x_R)_{trim}$, except in cases where they require a description of their time variation for a specific maneuver.

4.2 Numerical Methods for the SQP-based DMS

Gradients and Hessian matrices related to the continuity condition and cost function can be estimated using the SDME technique, as described in the previous section. The Hessian matrix can also be updated using the BFGS method. The BFGS method iteratively update the Hessian matrix, H , for the following general NLP problem with the design variable x :

(43) $\min J(x)$ subject to $h(x) = 0$

The BFGS method is the most popular technique for the iterative Hessian update in the SQP method because it is fast and widely applicable despite its simplicity. The line search procedure based on the Powell's method[24] is applied in this study, using the L1-penality function (or the L1-merit function) P_P , which can be used to represent a system having equality constraints $h_i(y)$, $i = 1,\ldots,m$ and active inequality constraints $g_j(y), j = 1,\ldots,m$, as follows:

(44) $$P_P(x,\sigma,\tau) = J(x) + \sum_{i=1}^{m} \sigma_i \left|h_i(x)\right| + \sum_{j=1}^{l} \tau_j \left|\max(0, g_j(x))\right|$$

Next, the one-dimensional line search and the update of design variables are performed using the following formulae:

(45) $$x^{k+1} = x^k + \alpha_k d_k$$

(46) $$\alpha_k = \arg\{\min T(\alpha)\}$$

(47) $$T(\alpha) = P_P(x^k + \alpha d_k, \sigma_k, \tau_k)$$

One-dimensional optimization, shown in Eq. (46), is performed using the Powell's algorithm in Ref [20]. The above steps generally work well in the DMS method with a linear system but sometimes the initial step size can becomes too large to reach a converged solution when applied to a DMS with nonlinear dynamics. In this case, the initial step size is limited in order to guarantee numerical convergence by using this formula:

(48) $\alpha_k^{(0)} = \min(1.0, \kappa_1 \|x_{k-1}\| / \|d_{k-1}\|)$, and $\kappa_1 = 0.8$

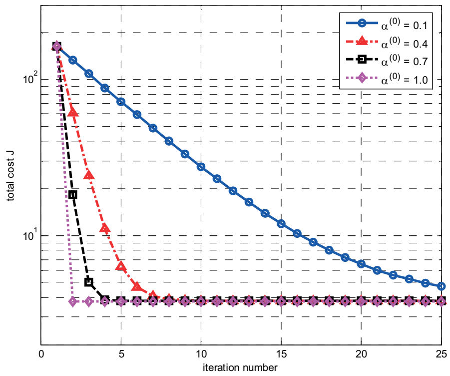
There exist various other strategies to improve the numerical efficiency of the one-dimensional search, such as higher-order correction methods to prevent the Maratos effect, the watchdog technique to cope with the cycling effect, etc. However, the appropriate selection from such methods depends on the problem at hand and, in most cases, is heuristically determined resulting in increased computational burden. For this reason, the convergence characteristics of the one-dimensional search algorithm in question were investigated during the code development stage.

5. APPLICATIONS

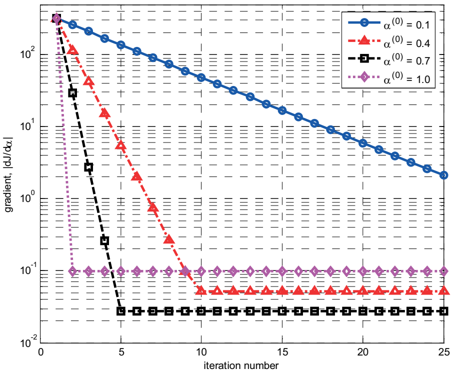
The proposed SDME technique was applied to the slalom maneuver problems of the Bo-105 helicopter, where the slalom trajectory was defined as follows:

$(\Delta y)_{\max}$	5.0m
t_{entry}	1.0s
t_{finish}	9.0s

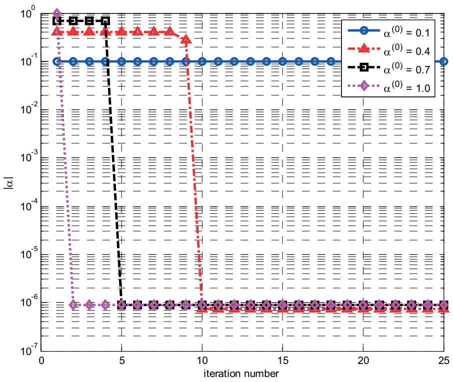
In this setup, the slalom maneuver begins in a steady level flight at a forward speed of 60 knots. The 4-stage Runge-Kutta time integrator was used for the forward simulations. the time interval between two adjacent shooting nodes was divided into 32 integration stages. In case of the LQR problem, KKT system matrices corresponding to the cost function and continuity conditions could be expressed exactly, with the SDME under the DMS framework. Therefore, the pure SDME method (L-F1-G1-H1: the code is defined below) can provide a baseline solution to compare the pros and cons of various approaches. Before analyzing the results of the present SDME application, the details regarding various possible approaches to building up the KKT system matrices will help identify each method with a different code. Table 1 presents a summary of the available methods for each component of the KKT system, along with the corresponding identification codes.



a. Total cost



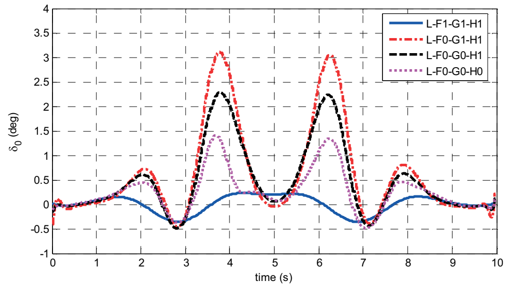
b. Gradient of total cost



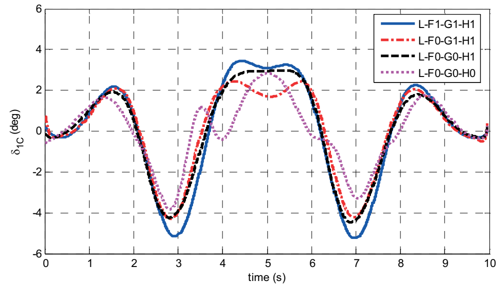
c. Line search parameter

<Fig 1. Convergence characteristics of LQR problem with varying initial line search parameter (L-F1-G1-H1)>

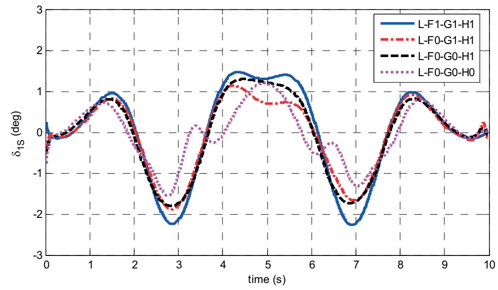
Fig. 1 shows the convergence characteristics with varying initial values of the line search parameter $\alpha_k^{(0)}$. The analysis with $\alpha_k^{(0)} = 1.0$ requires only one or two iterations to achieve a fully converged value of the cost function.



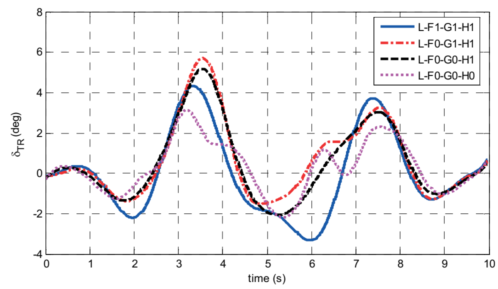
a. Main rotor collective pitch



b. Lateral cyclic pitch



c. Longitudinal cyclic pitch

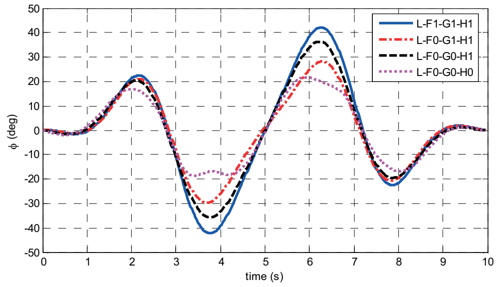


d. Tail rotor collective pitch

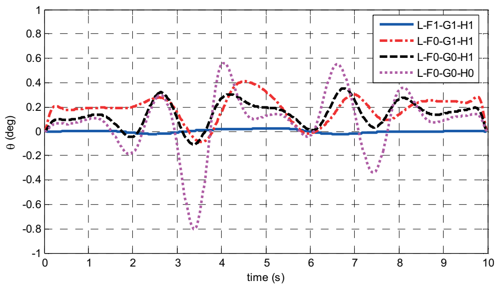
<Fig. 2 Control history for LQR problem>

Table 1. Codes for the identification of calculation methods

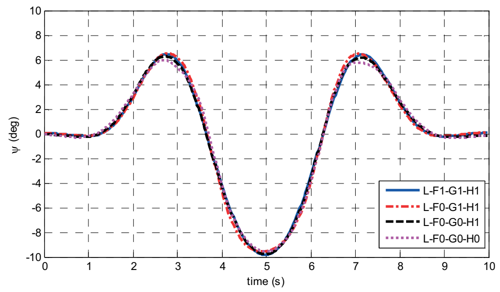
Items in KKT system	Estimation methods		Code
Functions of the cost function and continuity conditions	Time integration		F0
	SDME		F1
Gradients of the cost function and continuity conditions	Finite difference formula		G0
	SDME		G1
Hessian matrices for the cost function	Finite difference formula		H0
	SDME		H1
	BFGS with initial Hessian matrix by using	Identity matrix	H3
		Finite difference	H4
		SDME	H5



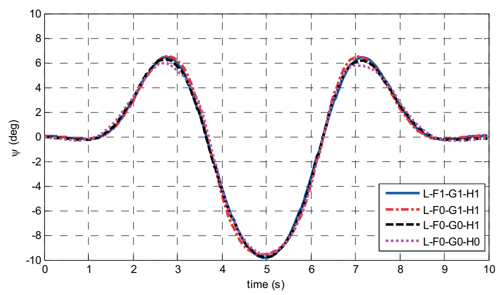
a. roll attitude angle



b. Pitch attitude angle



c. Heading angle

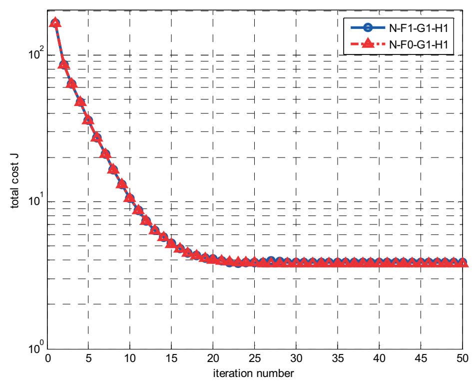


d. Y-positioni

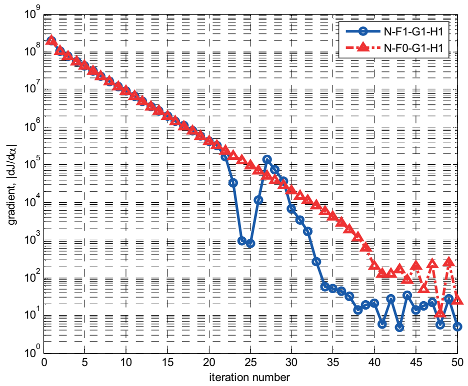
<Fig 3. Body state changes for LQR problem>

Figs. 2 and Fig. 3 show the time history of the controls and body states with different ways of building the KKT system matrices. The converged solutions using the SQP-based DMS depend significantly on the estimation of the KKT system matrices. The matrix exponential approach in the DMS framework provides an exact formula for function values, gradients, and Hessian matrices. Therefore, the solution accuracy using the L-F1-G1-H1 method depends on the computational accuracy of the matrix exponential and its integrals.

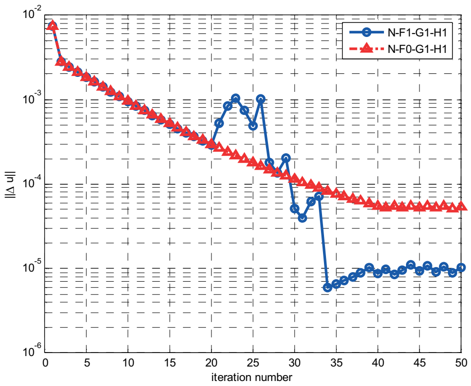
The nonlinear optimal control analyses using the SDME method involve several difficulties not found in their linear counterparts. In the case of the LQR problem, where an evenly distributed node system was used, the KKT system matrices corresponding to the cost function and continuity constraints were constant over the SQP iteration. Therefore, the calculation of residual vectors in the KKT system alone was enough to continue the SQP iteration. Furthermore a full one-dimensional search step with $\alpha = 1.0$ allowed fast convergence in the SQP procedure, as shown in Fig. 1. On the other hand, the SDC form of the general nonlinear system has state-dependent system matrices, which resulted in local variation in the related KKT system matrices. The SDME method also inherited numerical characteristics of the SQP approach for the general NLP problems.



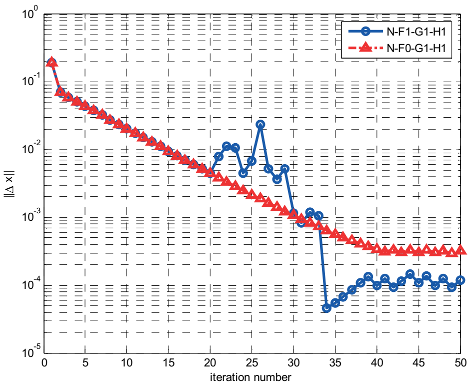
a. Total cost



b. gradient of total cost



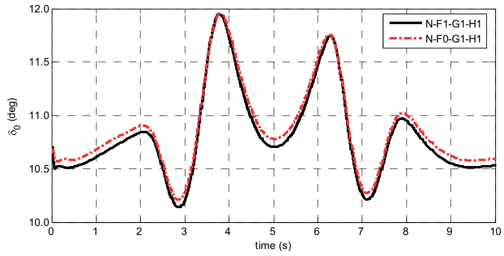
c. Control correction



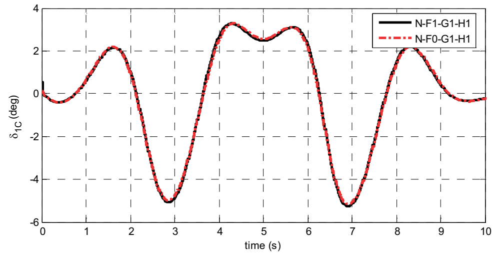
d. State correction

<Fig 4. Convergence characteristics of SDME method>

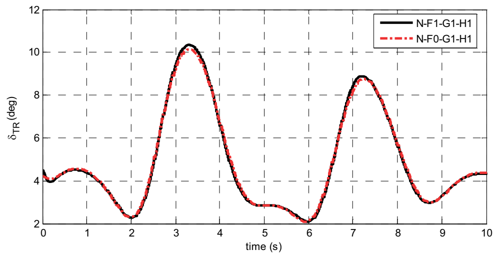
Fig. 4 shows the convergence characteristics with variations in the SDME method. The history of the cost function and error corrections for the states and controls could be used to measure the solution convergence. Various combinations of methods, as defined in Table. 1, have been tried in order to obtain a converged solution, but only the analyses with the N-F1-G1-H1 and N-F0-G1-H1 methods converged. Moreover, any combined analyses with the BFGS update formula failed to achieve a successful solution with the present SQP strategy.



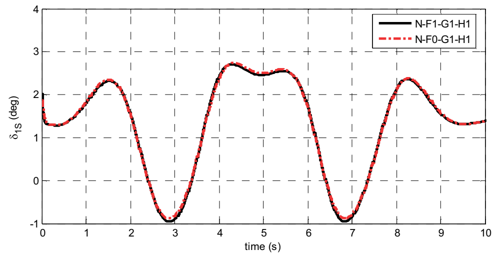
a. Main rotor collective pitch



b. Lateral cyclic pitch

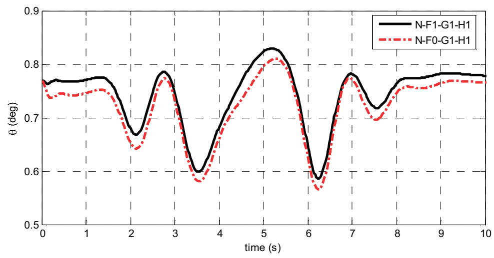


c. Longitudinal cyclic pitch

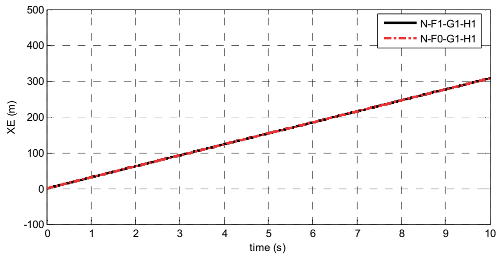


d. Tail rotor collective pitch

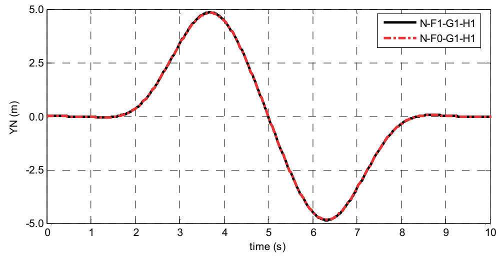
<Fig 5. Effect of state calculation method on controls>



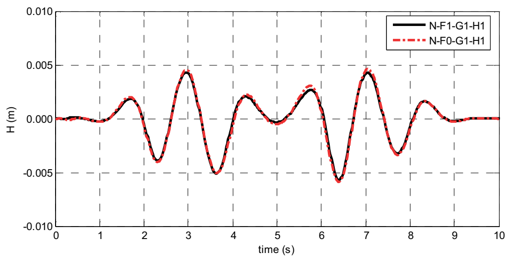
a. Pitch attitude angle



b. X-position



c. Y-position

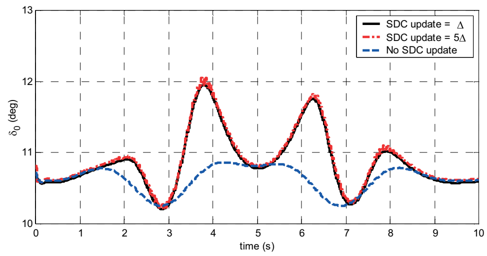


d. Altitude

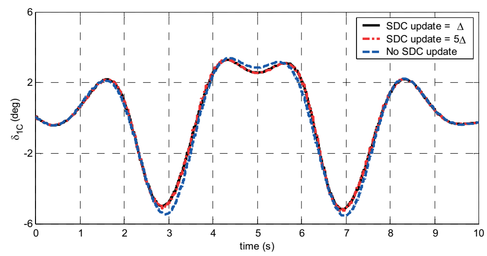
<Fig 6. Effect of state calculation method on pitch attitude angle and positions>

Fig. 5 and Fig. 6 compare the converged solutions with the N-F1-G1-H1 and N-F0-G1-H1 methods. Minor deviations in the main rotor collective pitch and pitch attitude angle differentiates the effect of the time integration methods on the solution of the nonlinear system equations. The SDME approach provided comparable performance in integrating the nonlinear motion equation as the Runge-Kutta method. This result can prove the usefulness of the SDME method in accurately predicting the KKT system matrices.

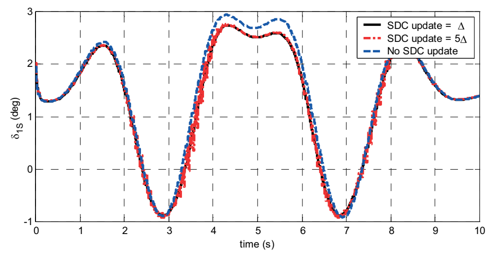
In principle, the SDC form of the motion equations should be re-calculated at each time interval in order to implement the SDME method. This could be a big drawback in its application to the MPC because of the large computational burden. If the helicopter maintains a nearly steady flight condition, the SDC matrices calculated in the previous time step could be used, since they generally involve negligible change. This could provide the update logic for the SDC matrices to increase computational efficiency. The final application in this paper is related to this kind of update logic. The standard SDME method updates the KKT system matrices at every time interval, which corresponds to the update time of $\Delta_j = t_{j+1} - t_j,\ j = 0,\ldots,N-1$. The matrices calculated at $\Delta_0 = t_1 - t_0$ could be used without any update, if minor variation exists in system states. In such an extreme case, a one-time forward simulation is enough to build the KKT system over whole time horizon, which could provide various opportunities to enhance the overall computational efficiency in the MPC application. For this purpose, Fig. 7 compares the effect of the update frequency of the SDC matrices on the optimal control solution when the shooting nodes are evenly distributed. Three different update logics were selected with update frequencies of Δ , 5Δ , and $N\Delta$, respectively.



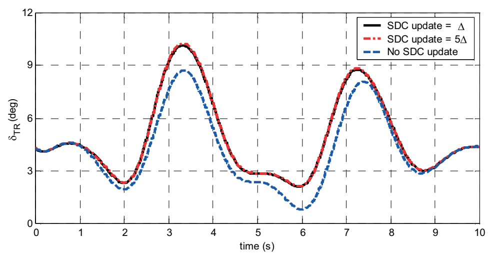
a. Main rotor collective pitch



b. Lateral cyclic pitch



c. Longitudinal cyclic pitch



d. Tail rotor collective pitch

<Fig 7. Effect of update frequency of SDC Matrices on controls>

Even though the results show some discrepancy, especially in the main rotor collective pitch and pitch attitude angle, the trajectory comparison in the Y-position shows that the SDME method maintained its excellent trajectory tracking performance without any update to the KKT system matrices over the time horizon of interest. Where the update frequency was five times that of the standard application, the longitudinal cyclic pitch shows oscillatory behaviors. However, the calculated trajectory and controls were nearly the same as those calculated with the standard application. Therefore, these analyses provide a good incentive to use the SDME technique in real-time MPC applications.

6. CONCLUSION

A new approach to the estimation of the KKT system matrices using integrals involving matrix exponential has been proposed. Applications to linear quadratic regulator problems showed that the matrix exponential approach yields better numerical accuracy and convergence with the direct multiple-shooting method than the conventional estimations of the KKT system matrices. More importantly, this approach simultaneously calculates system states at the end of each shooting node as well as gradients and Hessian matrices for the cost function and continuity constraints. The derivations for nonlinear regulator problems resulted in nearly the same formula as for linear regulator problems. Here, repeated computations for the state-dependent factorization and integrals weighted by matrix exponential are the major contributors to long computing times for nonlinear optimal control analyses. However, the related computational burden is generally much less than that associated with the finite difference methods to estimate gradients of the cost function and continuity constraints. The proposed method calculates the converged solutions for the nonlinear trajectory tracking problem, even though the solutions using conventional approaches are mostly divergent. In addition, the state-dependent matrix exponential approach can be used to integrate nonlinear motion equations. However, the Runge-Kutta time integrator showed better convergence characteristics when combined with the present matrix exponential approach in the direct multiple shooting method. The update frequency of state-dependent coefficient matrices had a minor effect on the accuracy of trajectory tracking over the present time horizon. Therefore, the compared results could be utilized to design an efficient MPC framework using the present method.

Acknoledgements

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