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HAMILTONIAN MECHANICS AS A POSSIBLE ALTERNATIVE FOR DERIVING AERO-ELASTIC EQUATIONS

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SUMMARY

It is often a time consuming task to derive aero-elastic equations for complicated problems. In the case of non-linear problems, a large amount of analysis and "book keeping" can be avoided by introducing Hamilton's generalized momenta as variables. Doing so, the problem can be solved by direct numerical computation, without having to perform lengthy analytical differentiations as would be required by the procedure according to Langrange.

The method is explained, and illustrated with an example taken from an actual design analysis where the transients after an assumed pitch-link failure had to be analyzed.

INTRODUCTION

To avoid the tedious and time consuming task of deriving aero-elastic equations, one may use symbolic manipulation codes such as REDUCE, or one can replace the Langrangian derivation procedure by numerical differentiations (see Done, refs.1 and 2).

Each of these methods has it's pro's and cons. Using the REDUCE computercode one is still left with sometimes very large amounts of mathematical terms, with no way to discriminate between the physically most important terms and the large amounts of less relevant terms.

In the method described by Done in refs.1 and 2 this problem does not arise, because the mass-, stiffness and damping matrices are directly computed in numerical form. However, in the mentioned references, this method has been described for linearized treatments only.

By FDO (SPE) a third method was briefly explored, which appears to be suitable for strongly non-linear problems. This method too has disadvantages, because only solutions can be obtained in the time-domain, in the form of system simulations. Nevertheless, the method may have its own field of application, alongside the other methods mentioned.

The method, which is outlined in the following, uses some concepts taken from the classical Hamiltonian formulation of mechanics. It is not known to the author whether this approach is used by other institutes as well. The method is not mentioned in the most well-known textbooks on helicopter dynamics such as Bramwell (ref.3) or Johnson (ref.4). If only for that reason it seems useful to draw the attention to this particular application of the classical theory of Hamilton.

GENERAL THEORY

Let q_i be the Lagrangian generalized coordinates of the system, and let T and V denote the kinetic and potential energy respectively, expressed in terms of q_i and $\dot{q}_i$:

$$T = T(q_i, \dot{q}_i) \qquad (1)$$

$$\text{and } V = V(q_i) \qquad (2)$$

The usual Lagrangian equations of motion are given by:

$$\frac{d}{dt} (\partial T/\partial \dot{q}_i) - \partial T/\partial q_i + \partial V/\partial q_i = Q_i \qquad (3)$$

Where Q_i stands for the generalized forces acting in the system. Eq. (3) expresses n second order differential equations, where n is the number of Lagrangian coordinates.

In the Hamiltonian formulation of dynamics new coordinates are introduced, the socalled generalized momenta p_i . These are specific combinations of the velocities, defined by the transformation formulae:

$$p_i = \partial T/\partial \dot{q}_i \qquad (4)$$

In situations where the kinetic energy T of the system is a homogeneous quadratic function of the velocities, eq. (4) may be expanded in the form:

$$\begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} = \begin{bmatrix} & & \\ & A(q_i) & \\ & & \end{bmatrix} \begin{bmatrix} \mathring{q}_1 \\ \mathring{q}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathring{q}_n \end{bmatrix} \tag{5}$$

where the elements of the square matrix A are functions of q_i .

In Hamilton's theory, the equations of motions are transformed entirely so that all the dependent variables become functions of the new variables q_i and p_i . This process finally leads to Hamilton's canonical equations.

The purpose here is different, viz. it is sought to avoid the labour of writing out the equations, and replace it as much as possible by a purely computational task.
In that case it suffices to substitute eqs.(4) into (3), and use Hamilton's generalized momenta as purely auxiliary variables:

$$\mathring{p}_i = \partial T/\partial q_i - \partial V/\partial q_i + Q_i \tag{6}$$

Let us assume that at a certain instant of time t the values of q_i and $\mathring{q}_i$ are known. Through the set of equations (5) the initial values of p_i are then known as well.

Futhermore, let us assume that *expressions* for T and V as functions of q_i and $\mathring{q}_i$ have been set up. The differentials $\partial T/\partial q_i$ and $\partial V/\partial q_i$ required in (6) may be computed numerically by replacing them by differences.
With $\mathring{p}_i$ known from eq.(6) and $\mathring{q}_i$ from the initial conditions, one can by any convenient numerical integrationscheme predict the values of p_i and q_i at the next required instant of time.
The corresponding values of $\mathring{q}_i$ at the new instant of time are finally determined by substituting p_i and q_i into eq. (5), and solving this set of simultaneous algebraic equations for $\mathring{q}_i$.

Note that in this way the most laborious part of setting up the equations of motion has been bypassed, viz. the analytical differentiations $d/dt\ (\partial T/\partial \mathring{q}_i)$ and $\partial T/\partial q_i$.

The analytical preparation which is still needed is the determination of the functions $T(q_i,\mathring{q}_i)$, $V(q_i)$ and $p_i = \partial T/\partial \mathring{q}_i$. Since these functions in general do not yet involve excessive amounts of terms, this operation may conveniently be performed by a symbolic manipulation code like REDUCE.
An alternative which has not yet been attempted by the author in actual practice but which would deserve some study is, to set up only the series of transformation matrices describing the geometrical configuration under consideration. One may express T, p_i, and the elements of the matrix A in terms of these transformation matrices, which would be a sufficient specification for performing the numerical operations required by the above computational scheme.

EXAMPLE PROBLEM

The above described method was used by FDO to perform a design analysis of an assumed failure condition of a large MW-windturbine.
The turbine in question has blades with relatively large aft chordwise position of the centre of gravity, and a not too large flapwise stiffness. Flutter is normally prevented by the rather large torsional stiffness both of the pitchcontrol mechanism and the bladestructure.

In case of pitch linkage failure however, flutter may be expected to develop almost certainly. Such a failure is detected automatically, with the result that a mechanical brake at the shaft is immediately activated. The usual braking method is by feathering the blades, but this of course would not be functional in case of pitch linkage failure. The design aim was, to prevent the flutter from becoming so severe that the blades would fail before the rotor is sufficiently retarded.

Something to take into account was, that the blade aerofoilsections have rather large negative pitching moments, so that a natural tendency towards positive feathering exists. It may nevertheless not be expected that the blades under the action of aerodynamic pitchmoments alone would rotate through the negative stall towards large feathering angles.

It was decided to simulate the "manoeuvre" and thereby to determine a safe choice of design parameters. The most important parameter appeared to be the layout of the pitchmechanism. Actually this mechanism does not consist of a real pitch link, but is built up from several gear stages between a hydraulic actuator motor and blade. By rearranging the layout of these gears, the likely failure modes were such that the blades would still be coupled mutually together through several gears. The effective inertia of these latter gearwheels appeared a strong remedy against "explosive" growth of a fluttercondition in the short time before the rotor can be decelerated to a safe, low speed.

ANALYSIS OF THE EXAMPLE PROBLEM

It will be clear that the situation described above cannot be treated by a linearized analysis. In fact, the angular velocity of the rotor drops from its normal operating speed to a much lower level, whereas at the same time the blade pitch is driven from its normal operating value into the feathering position by aerodynamic, centrifugal and inertia forces. During the motion both positive as well as negative stall conditions may be met.

For this reason the above described numerical simulation approach was considered. The blade flapping degrees of freedom were modelled by an offset-hinge model with appropriate springs. The formulae given in the appendix however, represent the somewhat simpler case of acentral-hinge model with springs, which is sufficient and more elucidating for the purpose of illustrating the method.
The expressions in the appendix show the actual form of eqs.(1), (2) and (5) in this particular example problem.

The expressions for the aerodynamic generalized forces are not shown, since they are not relevant for the subject of the present paper. Features of the aerodynamic model were:

- At ten different blade stations the relative flow velocities and angles of attack were determined from the instantaneous dynamic conditions.

- The aerofoil charcteristics used were quasi-steady and covered the whole range between -90 and +90 degrees angle of attack, according to the method suggested by Viterna and Corrigan (ref.5). This method corrects for three-dimensional effects within the stall-region.

- Dynamic inflow was taken into account by a simple first order time lag model.

- A dynamic stall model was not used yet. It is intended to incorporate such a model later.

The actual numerical simulation appeared to be very simple to perform. It could even be accomodated easily on a small desktop computer.

As an example, one of the simulation results is shown in fig.1.
It may be seen that the blade is in principle unstable: both the flapping and pitching angles vehemently deviate from their initial values and a large amplitude, coupled oscillation immediately starts to grow. After two cycles the amplitudes do not grow further because the blade starts to periodically enter into a condition of negative stall, which is associated with strong restoring pitching moments.
The angular speed oscillates too, but on the average decreases. After some time this leads to smaller blade motions. Whilst the rotorspeed drops, the average pitch-angle increases towards feathering.
Different simulations showed that it is possible to influence effectively the dynamic behaviour by the choice of design parameters. The most strongly controlling factor appeared to be the equivalent inertia of the gears that remain inside the pitch-circuit after the linkage failure.

CONCLUSIONS

In the case of complicated, non-linear problems where simulations in the time-domain are sufficient to get insight, there is no need to write down the complete equations of motion explicity. Just the first few analytical steps required by Lagrange's theory are needed. After these preliminaries, a coordinate transformation towards Hamilton's generalized momenta renders the problem conditioned for a further, purely numerical solution. This may save much tedious "book keeping" as well as precious time of the analyst, at the cost of a slight increase in computing time.

The procedure has been illustrated by an example taken from a practical design situation.

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APPENDIX: Dynamic equations for the example problem.

Notations:

I_x = blade moment of inertia w.r. to flapping axis.

I_y = blade moment of inertia w.r. quarter-chord line.

C_{xy} = inertia product w.r. to flapping axis and quarter-chord line.

J_{tr} = equivalent polar moment of inertia by drive train and generator.

J_{pitch} = equivalent polar moment by inertia of gearwheels in pitch mechanism.

β = flapping angle.

θ = pitch angle.

ψ = azimuth angle.

K_β = stiffness of equivalent flap spring.

Kinetic energy

$$
\begin{aligned}
T = {} & \tfrac{1}{2}.I_x . [\dot{\psi}.\sin\theta + \dot{\beta}]^2 + \\
& + \tfrac{1}{2}.I_y . [\dot{\theta}.\cos\beta - \dot{\psi}.\cos\theta.\sin\beta]^2 + \\
& + \tfrac{1}{2}.[I_x + I_y] . [\dot{\theta}.\sin\beta + \dot{\psi} \cos\theta.\cos\beta]^2 + \\
& + C_{xy} . [\dot{\psi}.\sin\theta + \dot{\beta}] . [\dot{\theta}.\cos\beta - \dot{\psi}.\cos\theta.\sin\beta] + \\
& + \tfrac{1}{2}.J_{tr} . \dot{\psi}^2 + \tfrac{1}{2}.J_{pitch} . \dot{\theta}^2
\end{aligned}
$$

Potential energy

$$V = \tfrac{1}{2} . K_\beta . \beta^2$$

Transformation equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \mathring{\psi} \\ \mathring{\theta} \\ \mathring{\beta} \end{bmatrix} = \begin{bmatrix} p_\psi \\ p_\theta \\ p_\beta \end{bmatrix}$$

with

$$a_{11} = I_x \cdot \sin^2\theta + I_y \cdot \cos^2\theta + I_x \cdot \cos^2\theta \cdot \cos^2\beta +$$
$$- 2 \cdot C_{xy} \cdot \sin\theta \cdot \cos\theta \cdot \sin\beta + J_{tr}$$

$$a_{12} = I_x \cdot \cos\theta \cdot \sin\beta \cdot \cos\beta + C_{xy} \cdot \sin\theta \cdot \cos\beta$$

$$a_{13} = I_x \cdot \sin\theta - C_{xy} \cdot \cos\theta \cdot \sin\beta$$

$$a_{21} = a_{12}$$

$$a_{22} = I_y + I_x \cdot \sin^2\beta + J_{pitch}$$

$$a_{23} = C_{xy} \cdot \cos\beta$$

$$a_{31} = a_{13}$$

$$a_{32} = a_{23}$$

$$a_{33} = I_x$$

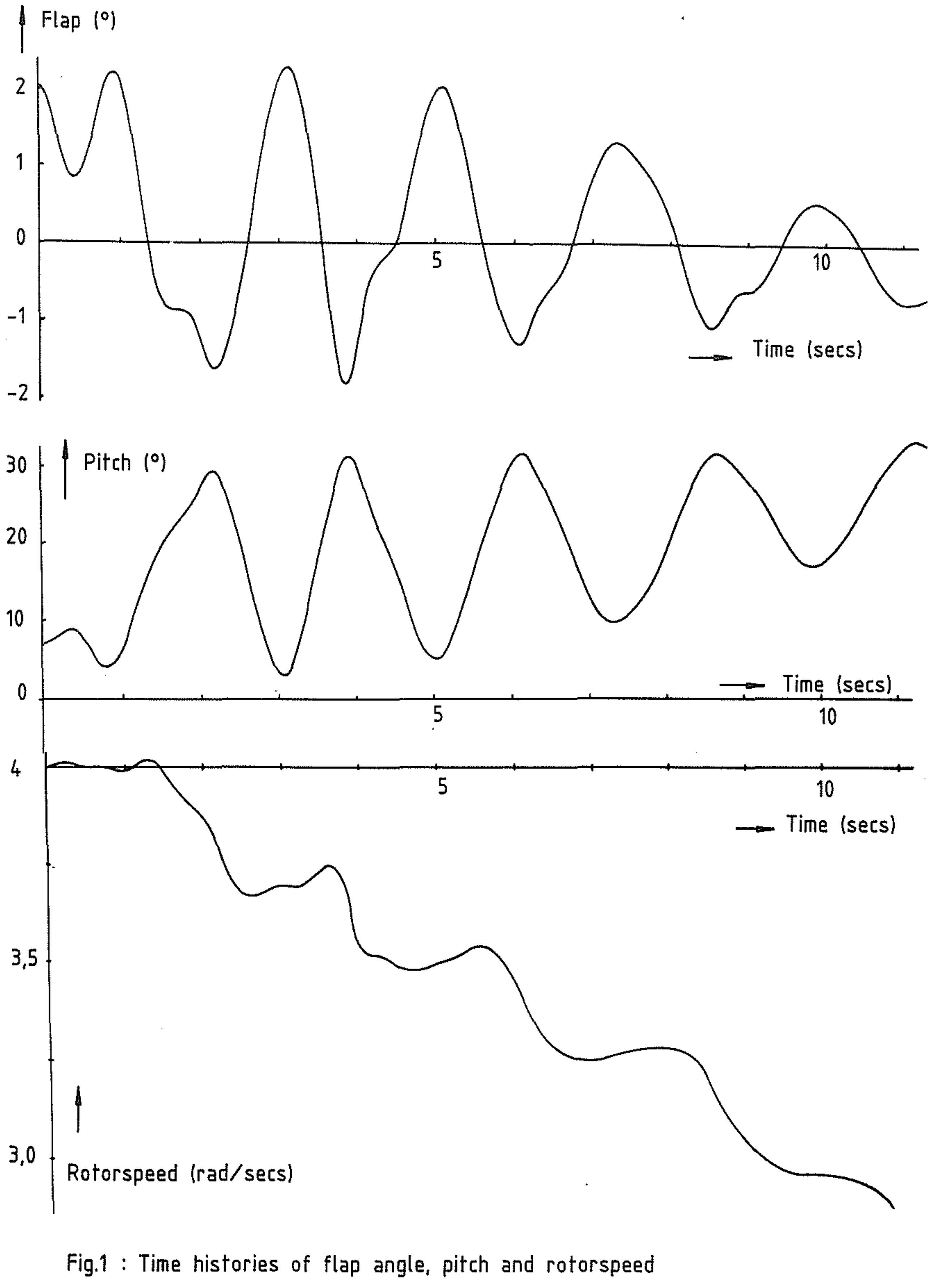

Fig.1 : Time histories of flap angle, pitch and rotorspeed