

CLOSED-FORM SOLUTIONS FOR OPTIMUM ROTOR IN HOVER AND CLIMB

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ABSTRACT

Glauert's second approximation to momentum theory allows him to formulate a variational statement for the loading of an optimum rotor in hover or climb, Ref. [1], pp. 251-258. He then works out a numerical solution to the optimum rotor in hover, Ref. [1], pp. 310-314, which shows his optimum to be slightly better than the Betz approximate optimum loading, Ref. [2]. Reference [3] demonstrates that further development of Glauert's variational principle leads to a cubic equation for the optimum loading in hover. The cubic equation is shown to have a compact, closed-form solution for the optimum rotor in hover (as based on momentum theory). In Ref. [4], the Betz solution is utilized along with a third approximation to Glauert's equations in order to find wake contraction in hover due to a Betz distribution. In this paper, we show that Glauert's variational principle for optimum loading leads to a quartic equation in the case of climb. Although closed-form solutions exist for quadratics, in this case the solutions are awkward. As an alternative, a very accurate approximation to the quartic solution has been found in closed form for the optimum loading at all climb rates (including hover). The maximum error of this approximation (as compared to the exact solution of the quartic) is less than 0.5% over the entire range of allowed parameters. With this approximation, it is possible to compute rotor efficiency along with flow variables both at the rotor disk and in the far wake, including wake contraction.

NOTATION

C_T	thrust coefficient, $T/(\rho\pi R^2\Omega^2 R^2)$
C_P	power coefficient, $P/(\rho\pi R^2\Omega^3 R^3)$
$d(\)$	differential operator
DEN	denominator of expression for $\bar{\omega}$, nondimensional
$f(x)$	contraction function, (m^2)
K	contraction ratio = $[f/x^2]^{1/2}$
P	power ($N\cdot m/sec$)
q	normalized loading parameter, $q = v_0/(\eta + v_0)$
r	normalized radial coordinate, $r = x/[R(\eta + v_0)]$
R	rotor radius, (m)
T	thrust, (N)
U	climb rate, (m/sec)
u	induced flow at rotor, (m/sec)
\bar{u}	normalized induced flow, $u/[\Omega R(\eta + v_0)]$
x	radial coordinate, (m)
X	optimum velocity variable, $X = 2\Omega/\omega$
Γ	circulation on blade, (m^2/sec)
$\delta(\)$	variational operator
η	nondimensional climb rate, $U/(\Omega R)$

θ	angle in cubic equation, (rad)
Λ	Lagrange multiplier, (m/sec)
v_0	Glauert loading variable, nondimensional
ρ	density of air, (kg/m ³)
ϕ	angle of inflow and thrust vector, (rad)
ω	wake rotation rate just below disk, (rad/sec)
$\bar{\omega}$	nondimensional wake rotation rate just below disk, ω/Ω
$()_1$	quantity () evaluated in far downstream wake

MOTIVATION

Glauert's treatment of momentum theory has been a standard of rotor analysis for 80 years. The equations are elegant and offer deep insights into the energy and momentum relationships for a lifting rotor. Although Glauert developed some elegant, closed-form solutions for the Betz loading distribution, that distribution is only optimum in the limit of low-lift climb. Glauert offered a formulation of a more general optimum but was only able to solve it numerically for some simple hover solutions. Here, we wish to determine if further, closed-form results can be obtained for the general optimum with the hope of gaining insight into rotor behavior.

INTRODUCTION

The first order of business is to determine exactly what the optimality condition is for the general case of a lifting rotor in hover or climb. Glauert's second approximation to his momentum theory implies that the induced flow at the rotor disk is parallel to the local thrust vector.

$$(1) \quad \tan(\phi) = \frac{(\omega x/2)}{u} = \frac{(U+u)}{(\Omega x - \omega x/2)}$$

From Eq. (1), u can be written in terms of ω or *vice versa*. The incremental thrust and power at a radial station on the blade from Glauert thus become:

$$(2) \quad dT = 2\pi\rho \left[\left(\Omega - \frac{\omega}{2} \right) \omega x^3 \right] dx$$

$$(3) \quad dP = 2\pi\rho \left[\left(\Omega - \frac{\omega}{2} \right) \Omega \omega x^4 \right] \tan(\phi) dx = 2\pi\rho \left[(U+u) \Omega \omega x^3 \right] dx$$

One can adjoin the thrust to the power with a Lagrange multiplier and obtain a variational statement for the minimum induced power P_i given a specified thrust.

$$(4) \quad P_i = P - UT; \quad J = P_i - T\Lambda; \quad \delta(J) = 0$$

This variational statement for power results in a quartic equation for the optimum angular velocity ω at any radial location, from which one can find the optimum u and the optimum loading, dT . The quartic equation is as follows:

$$(5) \quad \left[(1+3q-q^2)X - 2(2+2q-q^2) \right]^2 \times \left[(1-q)^2 X^2 + 4(X-1)r^2 \right] = \left[(1-q)^2 X^2 + 2r^2(3X-4) \right]^2$$

with

$$(6) \quad X = \frac{2\Omega}{\omega}; \quad q = \frac{v_0}{\eta+v_0}; \quad r = \frac{x}{R(\eta+v_0)}$$

where η is the nondimensional climb rate and v_0 is the nondimensional Lagrange multiplier—which then becomes the Glauert loading parameter.

Once Eq. (5) is solved for $X(r)$, Eq. (1) can be used to find ω , u , and the bound circulation of the optimum loading;

$$(7) \quad \bar{\omega} = \frac{\omega}{\Omega} = \frac{2}{X}; \quad \frac{\Gamma}{2\pi\Omega R^2(\eta+v_0)^2} = \bar{\omega}r^2$$

$$\bar{u} = \frac{u}{\Omega R(\eta+v_0)} = -\frac{(1-q)}{2} + \left[\frac{(1-q)^2}{4} + \left(1 - \frac{\bar{\omega}}{2} \right) \left(\frac{\bar{\omega}}{2} \right) r^2 \right]^{1/2}$$

Therefore, solution of the quartic Eq. (5) gives the entire solution for the optimum rotor in climb. The optimum thrust and power come directly from Eqs. (2) and (3) and are given by the following expressions:

$$(8) \quad \frac{dC_T}{dr} = (\eta + v_0)^4 (2\bar{\omega} - \bar{\omega}^2) r^3$$

$$(9) \quad \frac{dC_P}{dr} = (\eta + v_0)^5 (1 - q + \bar{u})(\bar{\omega}) r^3$$

Equations (5)-(9) are sufficient to describe the optimum rotor in climb under Glauert's second approximation to momentum theory. The next order of business is to determine useful solutions to these equations. We start with the special case of hover and then generalize.

SOLUTION

For hover ($\eta = 0$, $q = 1$), Eq. (4) reduces to a cubic equation in X which can be solved in a compact, closed form for the unknown X and, consequently for $\bar{\omega}$. As shown in Ref. [3], that cubic has a compact closed form solution.

$$(10) \quad \bar{\omega} = \frac{6}{5 + r^2 + 2(1 + r^2)\cos(\theta/3)}$$

where

$$(11) \quad \theta = \arccos\left(\frac{r^6 + 3r^4 + 3r^2 - 1}{r^6 + 4r^4 + 3r^2 + 1}\right)$$

When the rotor is in climb, there is a closed form to the quartic Eq. (5), but it is quite cumbersome. Investigation has found that an excellent approximation to Eq. (5) can be found (that is exact for $q = 1$ and exact for $q = 0$) by utilization of the form implied by Eqs. (10-11).

$$(12) \quad \bar{\omega} = \frac{2q(4 - q)}{DEN}$$

where

$$(13) \quad DEN = (4 + q) + (4q^2 - 7q + 4)r^2 + 2q(3 - 2q)(r^2 + 1)\cos(\theta/3) - (3/10)q(1 - q)(4 - q)r^2 + \left[(121/16)q^2(1 - q)^2 + 4q^2(3 - 2q)^2 r^2\right]^{1/2} - (11/4)q(1 - q) - 2q(3 - 2q)r$$

and θ is the same as in Eqs. (10 -11), $0 < \theta < \pi$. This function is designed to give the exact behavior at $r = 0$, at $q = 0$, and $q = 1$ with no more than 0.5% error for any value of r at any value of q . For the Betz, approximation, Eqs. (7-9) remain the same but with:

$$(14) \quad \bar{\omega} = \frac{2q}{1 + r^2}$$

For small q , the Glauert solution approaches the Betz solution.

WAKE CONTRACTION

Glauert solved his equations after making an assumption that the rotation velocity in the far wake has little effect on the overall solution. This led to the result that the local induced velocity at the rotor disk is always in an opposite direction to the local lift. Reference [4] showed that one could make a far less stringent approximation that did not neglect rotational effect in the wake, and still obtain the useful result that local induced flow is parallel to local lift. The less stringent assumption then leads to a relationship for the contraction equation in the far wake.

As noted in Ref. [4], there is a differential equation for contraction ratio for a rotor with any loading under the third approximation to Glauert Momentum Theory.

$$(15) \quad \frac{df}{dx} = \frac{2x(U+u)}{U+2u+\omega x \left(\frac{x^2}{f} - 1 \right)^{1/2}}$$

$$(16) \quad K = \left(\frac{f}{x^2} \right)^{1/2}$$

With the equations developed here, the contraction ratio can be found for the Betz and Glauert loadings for any loading parameter, q . From that, streamlines can be drawn depicting how the wake contracts into the far field for various loading conditions.

The assumption for drawing the streamlines is that the pressure expands from the on-disk value to the far downstream value—the latter developed from Eq. (15)—by assuming the pressure functions in closed-form from potential flow theory in three-dimensions, Legendre functions of the second kind. With that pressure expansion, the Bernoulli equation then gives velocity along the streamlines; and then continuity gives the contraction behavior between the disk and far downstream.

NUMERICAL RESULTS

Results are plotted versus normalized radius, r , which allows them to be applicable to any loading condition. For a given climb rate η and loading condition v_0 , the tip of the blade is at $r = 1/(\eta + v_0)$. The parameter q is defined by Eq. (6) and is a normalized climb variable. One can see from that equation that $q = 0$ corresponds to a lightly-loaded rotor in climb, and $q = 1$ corresponds to hover. All results are plotted over the entire range of values of q .

Figure 1 shows the the percentage error between the exact solution—determined by numerical solution of the quartic equation in Eq. (5)—and our approximation Eq. (13). One can see that the relative accuracy of our closed-form approximation to the quartic equation for optimum loading is excellent. For hover, $q = 1$, the error is zero at all radial locations. That is to be expected since the approximation is

designed to give the exact solution to the cubic equation when $q = 1$. The error is also zero for lightly-loaded rotors, $q = 0$. Again, this is to be expected since we designed the approximation also to give the exact answer in that limit, which is the Betz solution. The maximum error at any point along the radius is only 0.5% and occurs at two different locations: 1.) normalized radius $r = 1.5$ for $q = 0.7$ and 2.) at $r = 0.5$ for $q = 0.3$. There is a residual errors as r approaches infinity. It is never any more than 0.4% with maximum error at $q = 0.5$.

Figure 2 shows the optimum values of wake rotation $\bar{\omega}$ versus normalized radial position for various loading conditions ranging from hover ($q = 1$) to lightly loaded ($q = 0$) and compares them to the Betz loading. From this, all other parameters can be found, including normalized circulation, which is equal to $\bar{\omega}r^2$. To the accuracy that can be seen in the plot, our result can be considered to be the exact value of the true optimum. Note that, in hover, the Betz circulation approaches 2.0 at the root whereas Glauert approaches 1.0. From Eq. (7) it follows that the axial velocity approaches a constant at the root for Galuert while it goes to zero for Betz.

Figure 3 presents the contraction ratios for both the true Glauert optimum and Betz distributions as determined from the solution of the differential equation in Eq. (15). As noted in Ref. [3], the Betz distribution has a strong singularity at $r = 0$ for $q = 1$ which creates a rapid change in K near the root approaching $K = 0$ at the rotor center. All curves at all climb rates begin at $K = 0$ for the Betz solution (except for $q = 1$), with K for values of q close to 1.0 rapidly moving towards the $q = 1$ curve. For Betz, the fact that $K = 0$ at the root implies an infinite contraction, whereas for Glauert K approaches 0.3827 at $q = 1$, which indicates no concentrated singularity—but only a fairly weak singularity.

Figure 4 shows notional streamlines for the Glauert and Betz distributions in hover to give an idea of how the different loadings affect contraction. It can be seen that the streamlines are not much different for the two distributions.

Figure 5 gives the downstream values of wake rotation for the Betz distribution, and Fig. 6 gives them for the Glauert optimum distribution. Because of the strong singularity in contraction for Betz, $\bar{\omega}$ in the far wake approaches infinity at the root; and u approaches a constant. Glauert has a weaker behavior near the root. Thus, $\bar{\omega}$ in the far wake for Glauert approaches a large value, $1/(\cdot 3827)^2 = 6.827$ but not infinity. For either case, the axial flow far downstream approaches 2.0 (twice the velocity at the disk) at large r .

CONCLUSIONS

- 1) Application of a variational principle to Glauert's momentum theory gives a quartic equation for the optimum loading for a powered rotor in hover and climb. It is shown that a closed form approximate solution to this quadratic equation gives accurate results as compared to the exact solution.
- 2) Comparison of the optimum values of the wake rotation obtained from the approximate solution and the Betz loading shows that for the rotor in hover, Betz circulation approaches the value of 2.0 at the root and the axial velocity reaches a constant value. On the contrary, for Glauert, circulation approaches the value of 1.0 at the root and the axial velocity goes to zero.
- 3) Betz shows strong singularity in contraction ratio at the root which results in the infinite value for $\bar{\omega}$ in far wake. However, due to a weaker singularity in the root for Glauert, $\bar{\omega}$ gets a large value but not infinity in the far wake.

ACKNOWLEDGMENT

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Figures

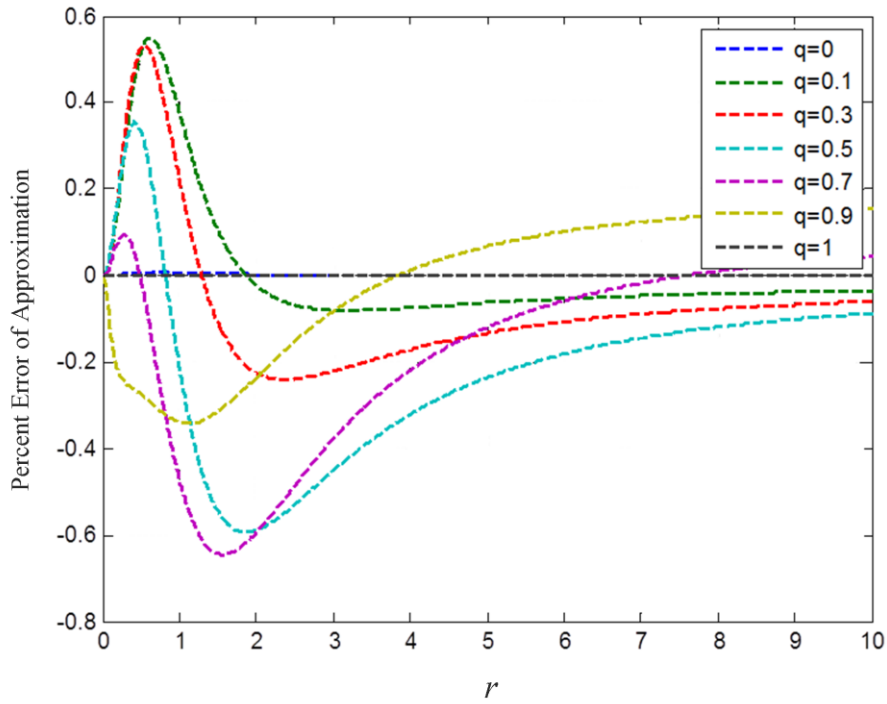


Figure 1. Percent Error of Approximation over Parameter Range

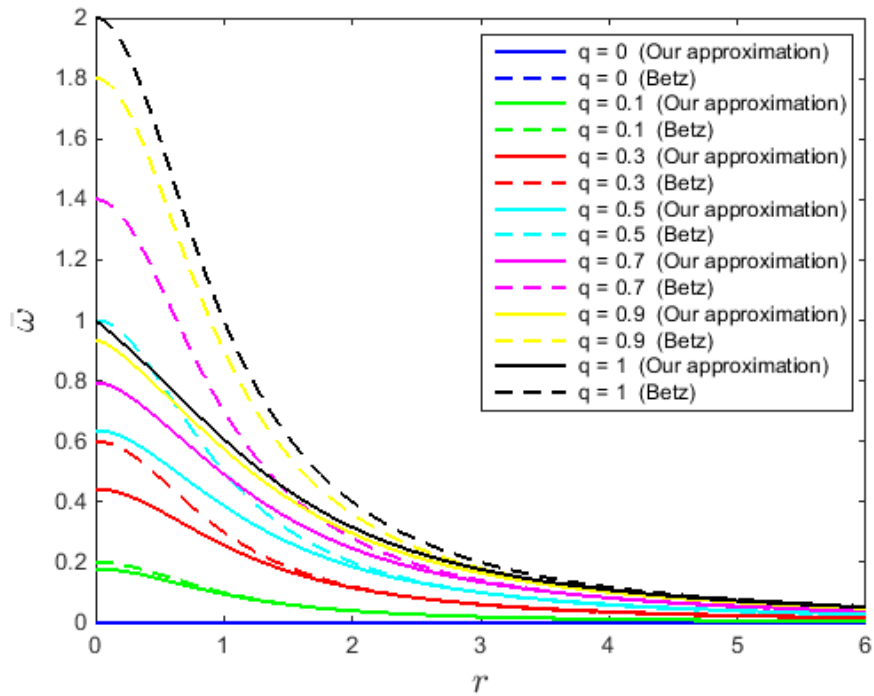


Figure 2. Rotation of Optimum Loading Distribution $\bar{\omega}$ at Various Loading Parameters as Compared with Betz Loading. (Nondimensional circulation = $\bar{\omega}r^2$)

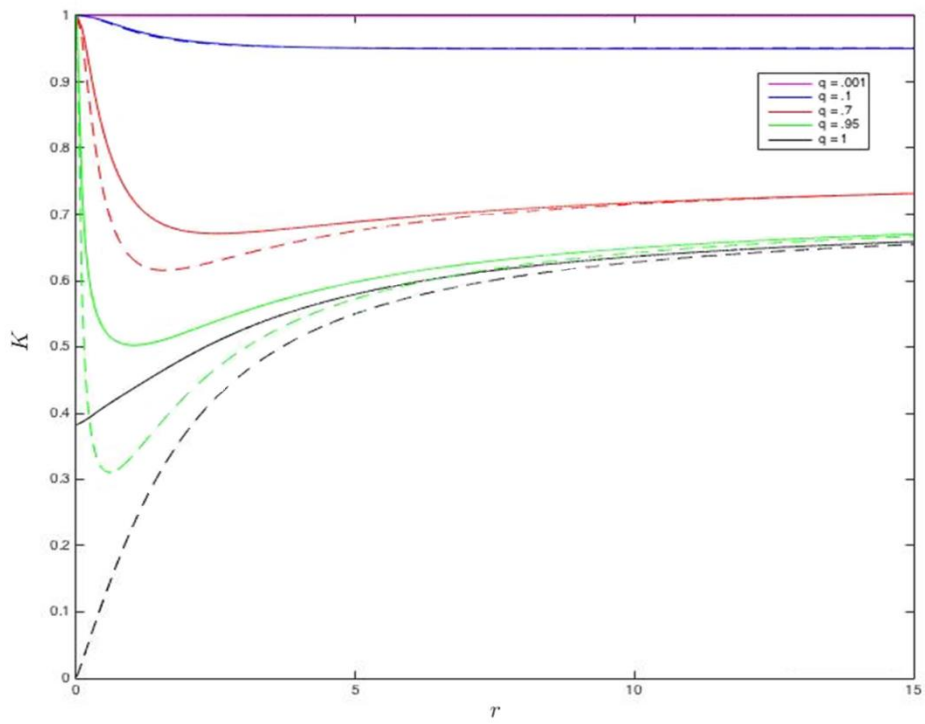


Figure 3. Contraction Ratio of Glauert Optimum as Compared to Betz Loading at Various Loading Parameters.

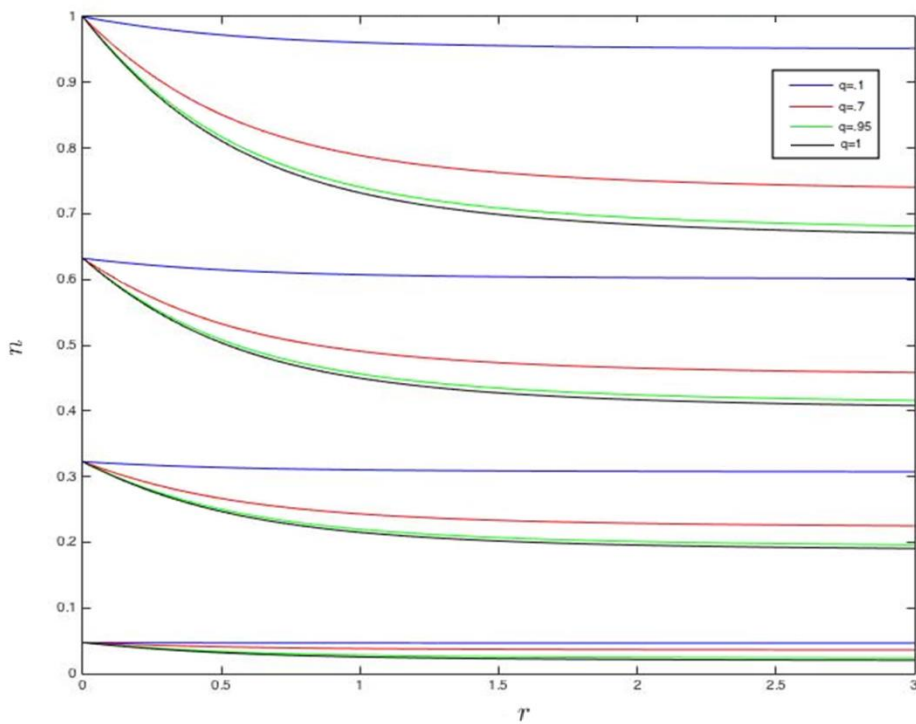


Figure 4. Notional Streamlines for Glauert Optimum Distribution in Hover and Climb.

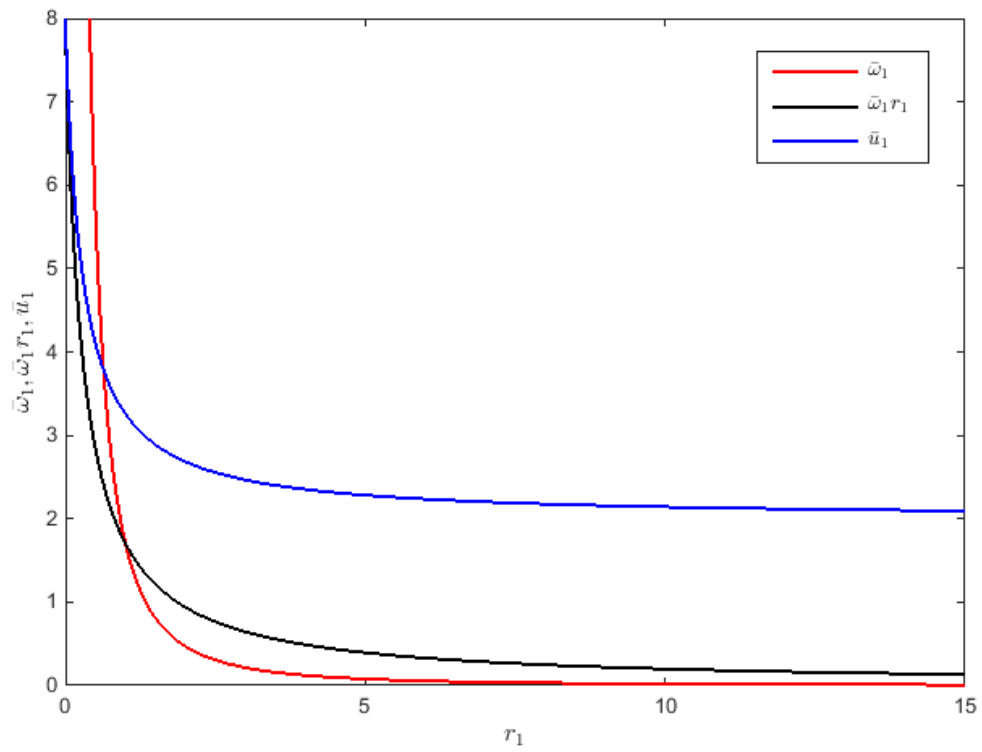


Figure 5. Downstream Variables for Betz distribution.

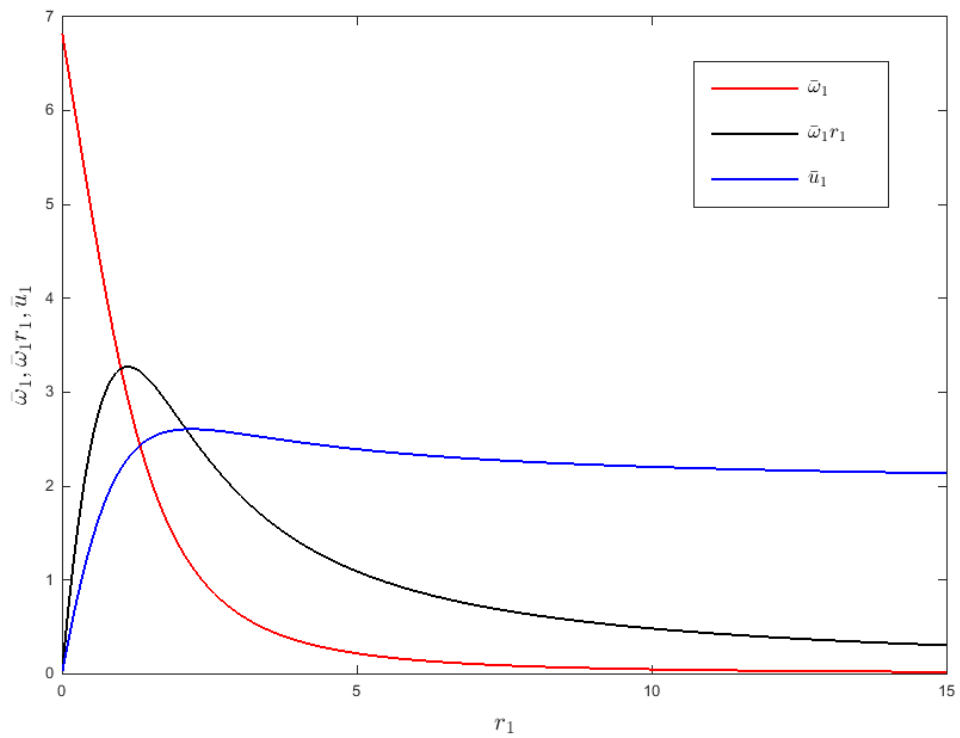


Figure 6. Downstream Variables for Glauert optimum distribution.