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OPTIMAL DESIGN OF A HELICOPTER BLADE

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Abstract

The problem of minimum weight design of a helicopter rotor blade subject to a constraint on its coupled flap-lag-torsional natural frequency has been studied in this paper. Modern structural optimization technique based on optimality criteria approach has been applied for optimizing the weight of the blade. Optimum designs are presented for a typical soft-in-plane hingeless rotor configuration. The results indicate that the application of structural optimization techniques leads to benefits in rotor blade design not only through substantial reduction in weight but a considerable reduction in the vibratory hub shears and moments at the blade root due to proper placement of blade natural frequency.

Introduction

Many different considerations are necessary in designing a helicopter rotor blade. Some of these considerations include strength, damage tolerance, fatigue life, reliability and survivability. However, a very important design consideration is the requirement of separating the natural frequencies of the blade from the aerodynamic forcing frequencies to avoid resonance. This is done by a proper tailoring of the blade mass or the stiffness distribution to give a set of desired natural frequencies. However, this is not an easy task due to the presence of various coupling effects as discussed in Reference 1. One such reason is that the natural modes of the rotor blade are mostly coupled because of the pitch angle, blade twist, large aerodynamic damping and off-set between the elastic and inertia axes. The problem is further complicated by the fact that there are many forcing frequencies and they are separated by margins less than 20 percent in the range of importance. A failure to consider frequency placement at the stage of the preliminary design has the potential of significantly increasing the weight of the structure.[1] However, most of the present preliminary design practices are not to tailor the design and place the desired natural frequencies. After the design is completed, the designer checks for poorly placed natural frequencies and corrects for these poor placements by placing appropriate nonstructural masses at crucial locations.

In order to avoid such weight penalties, as explained by Peters,[2] it is now possible to design and fabricate a helicopter blade that includes appropriate prescribed variations in stiffnesses which permit placements of frequencies at the preliminary design stages. One of the reasons for this possibility is as follows. Rotor blades are being fabricated by the use of composite materials. The state of the art of structural dynamic system optimization techniques and parameter identification techniques have improved to a state such that it is possible to apply these techniques at the preliminary design stage of the rotor blade and obtain appropriate variations in stiffnesses that results in desired placement of natural frequencies.

BACKGROUND

Significant developments in the field of the application of optimization techniques to rotor blade designs can be traced to the works of Bielawa,[3] Bennett,[4,5] Friedmann,[6-8] Peters,[2] and Taylor.[9] Bielawa has developed an optimization procedure to reduce blade loads consistent with aeroelastic restraints. The method, however, is not completely automated. Taylor has considered the problem by the use of modal shaping. The objective of his work is to reduce vibration levels by modifying the mass and stiffness distributions in order to modify "modal shape parameters." These modal shape parameters have been sometimes interpreted as an 'ad-hoc' optimality criterion. A very brief summary of other works due to Bennett, Friedmann, and Peters can be explained as follows.

A discrete parameter form of the rotor blade equations is written as follows

$$[m]\{\ddot{w}\} + ([c] + [\hat{c}])\,\{w\} + ([k] + [A])\,\{w\} = \{f(t)\} \qquad (1)$$

In this equation [c] is the structural dynamic damping matrix, $[\hat{c}]$ is the aerodynamic damping matrix, [k] is the structural stiffness matrix and [A] is the aerodynamic stiffness matrix. Bennett has considered problems of minimizing hub shears and blade weight. However, he has considered the matrices $[c] = [\hat{c}] = [A] = 0$ in the equations in obtaining frequency constraints. It is equivalent to considering the system in vacuum. Friedmann has considered the problem of minimizing hub shears or hub vibratory rolling moments subject to aeroelastic and frequency constraints. His aeroelastic constraints are based on a fully coupled analysis of coupled flap-lag-torsional analysis of the blade. However, the frequency constraints of the problem are based on the uncoupled modes.

Peters has addressed the problem of the placement of frequencies alone. He has also considered $[c] = [\hat{c}] = [A] = 0$. In his work, the c.g. off-set and twist are not equal to zero. He has reduced the problem of multiple frequency constraints to a single objective function. This study, based on coupled analysis of frequencies, has been formulated with inequality constraints for frequency placements. This has been motivated by the difficulties associated with handling equality constraints by nonlinear mathematical programming techniques like CONMIN.

In this paper, the optimum design problem of placement of natural frequencies has been formulated with equality constraints on frequencies and a minimum weight objective. The intended procedure is to consider the minimization problem with one frequency constraint at each time. Later, an approach based on the game theory[10] can be used to select an optimum design for multiple frequency constraints. This part has not been discussed in this paper. Also, the matrix [A] has not been equated to zero, although the structural damping matrix [c] has been assumed to be zero. A c.g. off-set has been considered. The optimization is based on an optimality criterion approach that can consider equality constraints with little difficulty. A purpose of selecting this reduced problem is to understand the problem of frequency placement first and then consider the problem of combined frequency and aeroelastic constraints by the use of optimality criterion approach.

Background on Optimum Design with Frequency Constraints

The problem considered in this paper falls in the category of optimum design of beam problems with frequency constraints.

A first investigation of the optimal beam vibration problem is attributed to Niordson.[11] He considered the problem of finding the best taper that yields the highest possible natural frequency. Following the initial work of Niordson, many different investigators have considered different problems in the field of optimal vibration of beams. Reference 12-17 deal with the problem of maximization of fundamental frequencies. The problem of maximizing higher order frequencies and rotating beams was addressed by Olhoff.[18-20] The problem of minimizing weight for a specified frequency constraint has been addressed in References 21-27. Multiple frequency constraints have been addressed in References 28-30. An optimality criteria approach has been discussed in References 26 and 27.

The work of this paper is also based on the optimality criterion approach. However, all the aforementioned works have considered reciprocal relationships and symmetry of matrices. The problem considered here does not always have symmetric matrices because of the aerodynamic contributions. The optimality criterion has been derived by using biorthogonal eigenvectors. Also, the effect of aerodynamic dissipative terms [$\hat{c}$] is important in the analysis. In earlier works, structural or aerodynamic damping terms have not been included. In order to approach the problem step by step, initial numerical work has been done with [c] and [$\hat{c}$] being equal to zero.

Problem Formulation

As discussed in the section "Introduction," an important design criterion that is desired in the design of helicopter rotor blades is the placement of the natural frequencies of the blade away from the rotor frequencies to avoid resonance. This is done by a proper tailoring of the blade mass or the area distribution to give a set of desired natural frequencies. However, this is not an easy task due to the presence, of various coupling effects as discussed in Reference 24. One such reason is that the natural modes of the rotor blades are mostly coupled because of the pitch angle, blade twist and an off-set between the elastic and inertia axes. The coupling effect, due to this off-set is considered here. The scope of the present work is to find a suitable mass distribution of the blade which minimizes the weight while holding the selected natural frequency at a specified value. Minimum gauge constraints are imposed on the selected design variables to prevent them from reaching impractical values or limiting values during the design optimization process. This constraint also accounts for the autorotational constraint.

Fig. 1 depicts a typical rotor blade with a thin-walled box beam running along the span and leading edge tuning weights distributed along the span. Following are the assumptions that have been made to simplify the analysis:

(1) The stiffness of the blade is contributed by the unsymmetric box section with nonuniform wall thickness and tip mass.

(2) Stiffness contributed by skin, etc., is negligible.

(3) The material density is uniform throughout.

(4) Thin-wall approximations are used.

(5) The effect of warping has been included.

Simplified Problem

Objective function: The weight of the blade which is assumed to be the sum of the weights of the box beam and the distributed tip turning mass.

Design variables: Dimensions of the box beam, e.g., b, h, t_1, t_2, t_3, etc. (See Fig. 2).

Pre-assigned parameters:

Rotor blade radius	= 193.26"
Blade semi-chord (non-dimensionalized with respect to blade radius)	= 0.0275
Number of blades	= 4
Off-set between aerodynamic center and elastic axis	= 0
Blade root offset	= 0
Tip loss factor	= 1.0
Speed of rotation	= 425 rpm
Lock number	= 5.5
Blade solidity	= 0.07
Flight path angle measured from horizontal	= 0
2-D lift curve slope	= 2 π
Weight coefficient	= 0.005

The problem of weight minimization subject to a constraint on a natural frequency is often referred to as the dual problem. The primal problem is the one where the natural frequency is maximized holding the weight to a specified value. Both of these problems as applied to optimum design of nonrotating beams with thin-walled cross sections undergoing coupled flexural-torsional vibrations have been addressed by the first two of the authors in the recent past [31, 32]. It has been observed that the optimum distributions differ largely with and without the coupling effects.

Brief Description of the Equations of Motion

Since the natural frequency of the rotor blade is the major behavioral constraint of the optimum design problem its calculation becomes necessary at each step of the optimization procedure. The coupled flap-lag-torsional equations of motion as presented in Reference 33 has been used in this analysis. The basic assumption behind the derivation of these equations is that the blade undergoes small strains and finite slopes. An ordering scheme is used and terms involving the squares of slopes and the products of slopes are assumed to be negligible when compared to the terms of the order of unity.

These equations of motion are capable of simulating general coupled flap-lag-torsional dynamics of a hingeless rotor blade for both forward flight and hover conditions with arbitrary mass and stiffness distributions. Cross-sectional mass center and aerodynamic center offsets from the elastic axis are also included. Quasisteady aerodynamics has been used in developing the equations neglecting compressibility and stall effects. The analysis also includes reverse flow and cyclic pitch changes for forward flight. Since the equations represent the isolated blade dynamics, shaft motions are not included.

Discrete Parameter Form of the Equations

Fully coupled flap-lag-torsion equations form a set of partial differential equations in x and t. These partial differential equations have been reduced to a discrete parameter form[33] by using the method of weighted residuals. Thismethod is also often referred in the literature as Galerkin finite element method. The differential equation is of the form:

$$L\{u\} = f \tag{2}$$

where

$$\{u^T\} = \{v, w, \phi\} \tag{3}$$

In extended Galerkin approximation, an admissible form of {u} that satisfies only the geometric boundary conditions are assumed

$$\{u\} = [\phi]\{b\} = \sum_{m=1+1}^{n} \phi_m b_m \tag{4}$$

where [ϕ] denotes the set of admissible functions and {b} the nodal displacement vector. The quantity n denotes the degree of discretization. Then, a sum of the integral of the weighted errors of the differential equations over the domain Ω and the integral of the weighted error of the natural boundary conditions over the surface S are equated to zero.

$$\int_{\Omega} W_m e \, d\Omega + \int_{S} W_m e_S \, ds = 0 \tag{5}$$

In general, the weighting functions or the test function need not be the same as ϕ_m. However, in structural dynamics problems these weighting functions are selected to be the same as the assumed functions to preserve the symmetry of the resulting system matrices. In some problems, selection of a weighting function other than the assumed function has resulted in improved but unsymmetric models. In the present case the weighting functions are the same as the assumed functions. However, system matrices are unsymmetric due to reasons such as the aerodynamic terms. Furthermore, the integrals of equation (5) have been integrated by parts to reduce the order of differentiation. This has effect of lowering the interelement continuity requirements. Then, the resulting discrete parameter model has the form of equation (1) or simply

$$Mb + Cb + Kb = F(t) \tag{6}$$

$$C = \hat{C}+C^{s} \tag{7}$$

$$K = \hat{K}+A \tag{8}$$

where C^{s} is the structural damping matrix, $\hat{K}$ is the structural stiffness matrix. $\hat{C}$ is the aerodynamic damping matrix and A the aerodynamic contribution to stiffness. For an eigenvalue analysis and response study, a state vector is defined as follows

$$\{x\}^{T} = \{b, b\} \tag{9a}$$

then equation (6) becomes

$$[M]\{x\} + [K]\{x\} = \{f\} \tag{9b}$$

where

$$[\tilde{M}] = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \tag{10}$$

and

$$[\tilde{K}] = \begin{bmatrix} C & K \\ -I & 0 \end{bmatrix} \tag{11}$$

In these equations [M] and $[\tilde{K}]$ are the real unsymmetric matrices. The eigenvalue problem now reduces to the following form

$$\lambda[\tilde{M}][Q] + [\tilde{K}][Q] = o \tag{12}$$

and

$$\lambda[\tilde{M}^{T}[S] + [\hat{K}]^{T}[S] = o \tag{13}$$

where {q} and {s} are the right and left eigenvectors. The solution consists of 2N eigenvalues or N pairs of complex conjugate eigenvalues two sets of eigenvectors {q} and {s}. Both sets of eigenvectors can be expressed as square matrices [Q] and [S]. Then,

$$[Q] = [q_1, q_2, \ldots q_n] \tag{14}$$

$$[S] = [s_1, s_2, \ldots s_n] \tag{15}$$

$$[S]^T[M]\{Q\} = [I] \tag{16}$$

and

$$[S]^T[K][Q] = [-\Lambda] \tag{17}$$

where $[-\Lambda]$ is the eigenvalue matrix.

Formulation of the Optimization Problem

The continuous problem has now been discretized using finite element formulation. The blade with a total span L is considered to be composed of N number of equal size elements, each of length L_e with possibly differing values of box beam dimensions. The subscript e is used to indicate elemental quantitites. The weight of the resulting discretize blade is given by

$$W = \sum_{i=1}^{n} \rho_{e_i} \; L_{e_i} \; A_{e_i} \; + \; W_o \tag{18}$$

where ρ_e denotes the elemental density L_e the elemental length and A_e the elemental cross sectional area. The quantity W_o refers to the contribution of the leading edge tuning mass distribution.

The optimization problem can now be posed as follows:

minimize

$$W = \sum_{i=1}^{n} \rho_{e_i} L_{e_i} A_{e_i} + W_o \tag{19}$$

subject to the following conditions:

equilibrium equations:

$$([K] - \alpha [M]) \{q\} = \{o\} \tag{20}$$

and

$$([K]^T - \alpha [M]^T) \{s\} = \{o\} \tag{21}$$

Normalization conditions

$$\{q\}^T[M]\{s\} - [I] = \{o\} \tag{22}$$

frequency constraints

$$\text{Im}\ (\{q\}^T[K]\{s\}) = \omega_o \tag{23}$$

where ω_o is a given quantity.

Aeroelastic constraints:

$$\text{Re}\{q\}^T[K]\{s\} < 0 \tag{24}$$

At this stage, the problem has combined frequency and aeroelastic constraints. This has necessitated the consideration of (a) biorthognal eigenvector due to unsymmetry and (b) complex eigenvalues resulting from the considerations of the c matrix. It is necessary to understand the effects of the two constraints, i.e. aeroelastic and frequency constraints, separately and together on the minimum weight optimum design of the blade. As a first step only the problem of unsymmetric matrices and the associated frequency constraints are considered in this paper. Then, the matrix c has been considered to be zero. The reduced problem of optimization is posed as follows

$$\text{minimize } W = \sum_{i=1}^{n} \rho_{e_i} L_{e_i} A_{e_i} + W_o \tag{25}$$

Subject to the following constraints

$$([K] - \omega^2 [M])\{q\} = \{o\} \tag{26}$$

$$([K]^T - \omega^2 [M]^T\{s\} = \{o\} \tag{27}$$

$$\{q\}^T[M]\{s\} - 1 = \{o\} \tag{28}$$

$$\{q\}^T[K]\{s\} - \bar{\omega}^2 = \{o\} \tag{29}$$

and bounds on magnitudes of the design variables

$$\phi_{min} < \phi_i < \phi_{max} \tag{30}$$

In these equations [K] and [M] are nxn matrices. Similarly $\{\hat{q}\}$ and $\{\hat{s}\}$ are nx1 biorthogonal eigenvectors, $\bar{\omega}$ is the prescribed natural frequency and ϕ_i refers to the i^{th} design variable which can be either of the box dimensions t_i, t_2, t_3 (see Fig. 2) or any combinations of them.

Unconstrained Optimization

The constrained optimization problem is now converted into an unconstrained one using Lagrange multipliers. The modified objective function assumes the following form:

$$W^* = \sum_{i=1}^{n} \rho_{e_i} \ell_{e_i} A_{e_i} \; W_o - \mu\left(\{q\}^T[K]\{s\} - \bar{\omega}^2\right)$$

$$-\nu(\{q\}^T[\hat{M}]\{s\}-1) - \{\Omega\}_1^T([K] - \hat{\omega}^2[M])\{q\}$$

$$-\{\Omega\}_2^T\left([K]^T - \hat{\omega}^2[M]^T\right)\{s\} \qquad (31)$$

The problem now is to minimize W* subject to the constraints on the design variables (Equation (30)). In the optimality criteria approach, This is done by obtaining stationary value of the objective function W* using full resources of variational techniques , while staying within the bounds on the design variables.

The necessary condition for stationarity is given by

$$\Delta W^* = 0 \qquad (32)$$

This leads to the condition $\partial W^*/\partial q = \partial W/\partial S = 0$. These conditions together with substitutions from equations (26) and (27) yield

$$\{\Omega_1\} = -2\,\mu\{q\}$$

$$\{\Omega_2\} = -2\mu\,\{s\} \qquad (33)$$

and

$$\nu = -\mu\omega^2$$

The next requirement is that $\partial W^*/\partial \phi_i = 0$. From this follows a set of optimality criteria conditions given below

$$\frac{\partial W^*}{\partial \phi_i} = \rho_{e_i} L_{e_i}\left(\frac{\partial A_e}{\partial \phi_i}\right) - \mu\left(\{q\}^T\left[\frac{\partial K}{\partial \phi_i}\right]\{s\}\right)$$

$$-\nu\left(\{q\}^T\left[\frac{\partial M}{\partial \phi_i}\right]\{s\}\right) - \{\Omega_1\}^T\left(\left[\frac{\partial K}{\partial \phi_i}\right] - \omega^2\left[\frac{\partial M}{\partial \phi_i}\right]\right)\{q\} \qquad (34)$$

$$-\{\Omega_2\}^T\left(\left[\frac{\partial K}{\partial \phi_i}\right]^T - \omega^2\left[\frac{\partial M}{\partial \phi_i}\right]^T\right)\{s\}$$

Using equations (33) in equation (34) the optimality exiterion condition is expressed as follows:

$$\frac{\partial A_i}{\partial \phi_i}\rho_{ei}L_{ei} + \omega^2\{q_e\}^T [\frac{\partial M_e}{\partial \Phi_i}]\{s_e\} - \{q_e\}^T [\frac{\partial K_e}{\partial \Phi_i}]\{s_e\} = \frac{1}{\mu} \tag{35}$$

$$i = ,1, 2, , , , N_t$$

where N_t denotes the number of elements over which the design variable ϕ_i does not reach the limiting values posed by equation (19). Note that the global mass and stiffness matrices [M] and [K] respectively have been replaced by the corresponding elemental quantities $[M_e]$ and $[K_e]$ respectively and the global biorthogonal eigenvectors {q} and {s} have been replaced by the corresponding elemental eigenvectors $\{q_e\}$ and $\{s_e\}$ respectively. This has been done since there exists a one to one correspondence between an element and a design variable or in other words, ϕ_i only appears in the element stiffness and mass matrices of the i^{th} element. A simultaneous solutions of equations (25)-(30) and equation (35) will result in possible optimum designs.

Recursion Relations

The optimization procedure begins with a set of feasible initial values for the design variables. For this initial design, solutions of equations (25)-(30) provide the eigenvalue ω and the associated eigenvectors {q} and {s}. Next, it is necessary to solve for the Lagrange multiplier μ. From equation (35) it is seen that there exists N_t equation involving a single unknown μ. An exact solution of μ is therefore not possible. In fact, only on rare occasion a solution of the optimality conditions provides immediate solutions of the Lagrange multipliers. Hence, an approach for finding an estimated or best value of the Lagrange multiplier, which will be denoted by $\bar{\mu}$, is necessary. Once the value $\bar{\mu}$ is obtained, it is necessary to obtain a set of recursion relations of redesign equations which will provide an updated set of values for the design variables. An iterative scheme is developed, based upon these recursion relations, for moving through the design space in such a manner as to eventually locate a stationary design that satisfies the optimality criteria exactly and therefore is an optimum design. This is done as follows:

Denote

$$\{q_e\}^T [\frac{\partial M_e}{\partial \Phi_i}]\{s_e\} = A_i$$

and (36)

$$\{q_e\}^T [\frac{\partial K_e}{\partial \Phi_i}]\{s_e\} = B_i$$

$$\rho_{ei}L_{ei}\frac{\partial A_i}{\partial \Phi_i} = C_i$$

The optimality criterion of equation (24) can now be written as

$$C_i - \mu\,(\omega^2 A_i - B_i) = 0 \qquad i = 1, 2 \,.., N_t \tag{37}$$

Defining

$$Z_i = \mu^2 A_i - B_i \tag{38}$$

equation (26) is written as

$$\frac{C_i}{\mu} - Z_i = 0 \qquad i = 1, 2 \,.., N_t \tag{39}$$

At the optimum design, there exists a single value of μ which will satisfy equation (39) exactly. However, for a non-optimum design since there is no such single value for μ, an approach for finding an estimated value for μ has been derived. A least square type of approach has been used for this purpose. Since equations (39) are exactly satisfied for a unique value of μ only at the optimum design, for nonoptimum designs, a residual R_i is defined where

$$R_i = \frac{C_i}{\bar{\mu}} - Z_i \qquad i = 1, 2, \ldots, N_t \tag{40}$$

At the optimum design $R_i = 0$ for $i = 1, 2 \ldots, N_t$. At nonoptimal designs it is necessary to make $\bar{\mu}$ as close to the exact μ as possible. Hence, the idea is to take the sum of the squares of the residuals and minimize it with respect to $\bar{\mu}$.[32] That is, set

$$\frac{d}{d\bar{\mu}}\left(\sum_{i=1}^{N_t} R_i\right) = 0 \tag{41}$$

which gives

$$\bar{\mu} = \frac{N_t \sum_{i=1}^{N_t} C_i}{\sum_{i=1}^{N_t} (\omega^2 A_i - B_i)} \longrightarrow \textit{Lagrange Mutiplier} \tag{42}$$

At the $(n+1)^{th}$ iteration, the design process requires that

$$B_i^{n+1} = \omega^2 A_i^{n+1} - \frac{1}{\bar{\mu}^n} C_i^{n+1} \qquad \text{if} \qquad \bar{\mu}^n < 0$$

and (43)

$$\omega^2 A_i^{n+1} = B_i^{n+1} + \frac{1}{\bar{\mu}^n} C_i^{n+1} \qquad \text{if} \qquad \bar{\mu}^n > 0$$

The above equations are written in such a manner as to ensure positive quantities on either side of the equality sign and are necessary to develop the recursion relations.

The recursion relations are used to obtain improved value of the design variables and are presented below. The details can be found in Reference 33.

$$\Phi_i^{n+1} = \Phi_i^n [(\frac{\bar{\omega}}{\omega^n})^{\alpha} \{ \omega^2 A_i^n / (B_i^n + \frac{1}{\bar{\mu}^n} C_i) \}^{\beta}] \qquad \bar{\mu}^n > 0$$

and (44)

$$\Phi_i^{n+1} = \Phi_i^n [(\frac{\bar{\omega}}{\omega^n})^{\alpha} \{ B_i^n / (\omega^2 A_i^n - \frac{1}{\bar{\mu}^n} C_i) \}^{\beta}] \qquad \bar{\mu}^n < 0$$

where α is a positive exponent and β_n is a relaxation parameter. The rationale for using a scaling factor of $\bar{\omega}/\omega^n$ in the recursion relations has been explained in Reference 26.

Convergence Criteria

Once a set of new values of the design variables is obtained, the same convergence criteria as used in Reference 26 are used. They are:

$$\Delta\omega = \left| (\bar{\omega} - \omega)/\bar{\omega} \right| < \epsilon_1 \qquad (45)$$

$$\Delta W = \left| (\widehat{W} - W^{n+1})/W^{n+1} \right| < \epsilon_2 \qquad (46)$$

where ε_1 and ε_2 are small prescribed tolerances w is a weight obtained from a previous design that satisfies condition (45) and W^{n+1} denotes the weight at the $(n+1)^{th}$ iteration.

The iteration scheme used is the same as in Reference 26.

Results and Discussion

In this section, numerical results have been presented for selected cases under conditions of hover. The initial blade configuration, before the start of the optimization process has been assumed to be a blade of uniform cross sectional mass and stiffness distributions. The off-set between the aerodynamic center and the mass center has been assumed to be equal to zero to start with. The following initial dimensions and material properties have been assumed for the box beam shown in Fig. 2.

$h/b = 0.75$
$t_2/b = 0.85$
$t_3/b = 0.025$

In the optimization process t_1/b is varied from an initial uniform distribution of 0.05.

The nondimensionalized blade frequencies corresponding to the initial design have been calculated to be as follows:

$\omega_{flap} = 1.2$
$\omega_{lag} = 0.6$
$\omega_{torsion} = 2.41$

The frequency constraint has been imposed on the first natural frequency of the coupled system and has been assumed to be equal to a numerical value of 1.2. For response calculations, the exciting frequency has been assumed to be equal to 1.0 (refer Appendix).

The autorotational constraint can be reduced to a minimum gage restriction. This minimum gauge constraint imposed on the design variable t_1/b has been assumed to be equal to 0.01. W_0 has been set equal to zero in the present analysis.

The uniform blade has been discretized by using 10 equal sized elements. The coupled flap-lag-torsion element and the forces and moments acting on the hub have been shown in Fig. 3. These forces and moments have been calculated following the method described in Appendix for both the initial and optimum configurations.

The results of the optimum design with reference to the uniform blade are summarized in Table 1. In this table, all the forces, moments and frequencies are nondimensionalized with respect to $m_o\Omega^2L^2$ where m_o is the mass per unit length, Ω is the R.P.M. and L is the span of the initial uniform blade.

The optimum thickness distribution has been presented in Fig. 4 and in Table 1. A 24.9% reduction in blade weight, as compared to the weight of the uniform blade, has been achieved with the design. The shear force distributions in y and z directions (refer Fig. 3) along the rotor blade, corresponding to the optimum configuration, are presented in Figs. 5 and 6. As is noted in Table 1, between 42.69-52.4% reduction in root shears have been obtained with the optimum configuration. The reductions are with reference to the root shears of the blade with initial uniform configuration.

Several different variations of the assumed frequency constraint have been considered. A particular question of interest follows. Does the inclusion of only the minimum weight objective without the requirement of minimum shear forces always result in lowering the shear forces? Is it necessary that some inequality constraints be imposed on shear forces? Alternatively is the consideration of the case of multiple objective functions necessary?

From the results it is noted that the application of design optimization procedure has a significant effect on blade design. With a single design variable the reduction in weight can be considered to be significant.

Conclusions

In this paper, the problem of minimizing the weight of a given rotor blade subject to frequency constraints has been studied by using an optimality criterion approach. In the cases studied, a minimization of the weight resulted in lowering the shear forces. However, it has not been noted that the shear forces will not always decrease. It is not possible to conclude that minimization of weight results in minimum rotor shear forces.

The work in the paper has been done with equality constraints on frequencies. This has been difficult to implement with some mathematical programming procedures. Before selecting an appropriate design, it is necessary to examine cases of multiple objective functions and the resulting optimum designs. In this paper, an optimality criterion approach, for structural dynamics problem, has been extended to consider unsymmetric matrices.

APPENDIX

Calculation of Vibratory Hub Loads

Reduction of helicopter vibration is much related with the reduction of vibratory hub loads. For low vibration throughout the airframe, a rotor which produces inherently low hub loads is needed.

It is helpful for the discussion that follows to review the origin of the vibratory loads. Rotor blade dynamic equations discretized by Galerkin finite element method for undamped forced vibrations are written as

$$[M]\{a\}+[K]\{a\} = \{Q_0\} \sin\omega t \qquad \text{(A1)}$$

Where $\{Q_0\}$ represents the nodal aerodynamic load vector which is assumed to be acting sinusoidaly with frequency ω. Again [M] and [K] are the global mass and stiffness matrices respectively and {a} denotes the nodal displacement vector. Inherently for the rotor blade dynamics, [M] and [K] are real unsymmetric banded matrices. Response of general dynamical system including damping has been discussed in Reference (34). By introducing state vector approach and by making appropriate matrix decompositions and transformations, the eigenvalue problem is reduced to the following form.

$$[A]\{X\} = \omega^2\{x\} \qquad \text{(A2)}$$

A similar approach can also be applied to equation (A1) to convert the eigenvalue problem into a form as given by equation (A2).

By using LU decomposition the mass matrix can be written as

$$[M]=[L][U] \tag{A3}$$

where [L] is the lower triangle and [U] is upper triangle matrix, introducing the linear transformation

$$[U]\{a\}=\{z\},\ \{a\}=[U]^{-1}\{z\} \tag{A4}$$

Equation (A1) can be reduced to

$$\{z\}+[A]\{z\}=\{P_o\}\ \sin\omega t \tag{A5}$$

where

$$[A]=[L]^{-1}[K][U]^{-1} \tag{A6}$$

is real nonsymmetric matrix and

$$\{p_0\} = [L]^{-1}\{q_0\} \tag{A7}$$

The eigenvalue problem associated with the system (A5) has the form

$$[A]\{u\}=\omega^2\{u\} \tag{A8}$$

The solution consists of n eigenvalues w_i^2 and eigenvectors $\{u_i\}$ (i=1,2, .., n) by assuming that the eigenvalues are distinct so that the Jordan matrix is diagonal

$$[\Lambda]=diag[\ w_1\ ^2] \tag{A9}$$

The eigenvectors u_i are known as right eigenvectors of [A] and can be arranged in the square matrix form as follows

$$[u]=[\{u_1\},\ \{u_2\},\ \{u_3\}\ ..\ \{u_n\}] \tag{A10}$$

The adjoint eigenvalue problem is

$$\{v\}^T[A]=\omega^2\{v\}^T,\ [A]^T\{v\}=\omega^2\{v\} \tag{A11}$$

and its solution consists of the same eigenvalues w_i^2 as well as the left eigenvectors $\{v_i\}$ (i=1,2, .., n). They also can be arranged in a square matrix, as follows:

$$[V]=[\{v_1\},\ \{v_2\},\ ...,\ \{v_n\}] \tag{A12}$$

Eigenvectors $\{u_i\}$ are related to the set of eigenvectors $\{v_i\}$ through a property known as biothogonality. This means that the eigenvectors can be normalized in the following manner.

$$[v]^T [u]=[u]^T[v]=[I] \tag{A13}$$

and in this case the Jordan matrix is simply

$$[\Lambda]=[v]^T[A][u] \tag{A14}$$

Solution of equation (A) can be obtained by the method described above. But as is mentioned in Reference (34) and, practically experienced, this solution procedure is computationally very sensitive and even unstable. Therefore, an alternative approach is introduced. Eigenvalue equations (A8) and (A11) are written as

$$[K]\{q\}=\omega^2[M]\{q\} \tag{A15}$$

and

$$[K]^T\{s\}=\omega^2[M]^T\{q\} \tag{A16}$$

similarly

$$[Q]=[\{q_1\}, \{q_2\}, \quad ,\{q_n\}] \tag{A17}$$

and

$$[S]=[\{s_1\}, \{s_2\}, \quad , \{s_n\}] \tag{A18}$$

Eigenvectors $\{q_i\}$ and $\{s_i\}$ are normalized such that

$$[s]^T[M][Q]=[I], [Q]^T[M]^T[S]=[I] \tag{A19}$$

and in this case the Jordan matrix is simply

$$[\Lambda]=[s]^T[K][Q]=[\Lambda], [Q]^T[K]^T[S] \tag{A20}$$

Introducing linear transformation

$$\left\{ a(t) \right\} = \left[Q \right] \left\{ \mu(t) \right\} \tag{A21}$$

and substituting equation (A21) into equation (A1) and post multiplying by $[s]^T$ equation (A1) can be reduced to

$$\left\{ \ddot{\mu}(t) \right\} + \left[\Lambda \right] \left\{ \mu(t) \right\} = \left\{ F_o \right\} \sin\omega t \tag{A22}$$

where

$$\left\{ F_o \right\} = \left[S \right]^T \left\{ Q_o \right\} \tag{A23}$$

Equation (A22) represents a set of n independent equations which are time dependent and the solution is given by

$$\left\{ \mu_i(t) \right\} = F_i \frac{\sin\omega t}{\omega_i^2 - \omega^2} \tag{A24}$$

By back substitution, the nodal displacement vector is recovered as follows

$$\left\{ a(t) \right\} = \left[Q \right] \left\{ \mu(t) \right\} \tag{A21}$$

and the nodal forces and moments can be obtained using the following equations

$$\left\{ R(t) \right\} = \left[K \right] \left\{ a(t) \right\} \tag{A25}$$

References

1. Hirsh, H., R.E. Dutton, and A. Rasumoft, "Effects of Spanwise and Chordwise Mass Distributions on Rotorblade Cyclic Stresses," Journal of American Helicopter Society, Vol. 1, No. 2, 1956.

2. Peters, D.A., Timothy Ko, Rossow Korn, Mark P. Rossow, "Design of Helicopter Rotor Blades for Desired Placement of Natural Frequencies," Proc. of the 39th Annual Forum of the AHS, May 9-11, 1983, St. Louis

3. Bielewa, R.L., "Techniques for Stability Analysis and Design Optimization with Dynamic Constraints of Nonconservative Linear Systems," AIAA/ASME 12th Structures, Structural Dynamics and Materials Conference, Anaheim, California, April 19-21, 1971, AIAA Paper No. 71-388.

4. Bennett, R.L., "Optimum Structural Design," Proc. of the 38th Annual forum of the American Helicopter Society, May 4-7, 1982, Anaheim, CA, pp. 90-101.

5. Bennett, R.L., "Application of Optimal Design Techniques to Helicopter Design Problems, Report prepared by Bell Helicopter Textron, Inc., under contract NAS2-11666.

6. Friedmann, P.P., "Response Studies of Rotors and Rotor Blades with Application to Aeroelastic Tailoring," Semi-Annual Progress Report on Grant-NSG-1578, Dec. 1981 (also unpublished U.S.C. Doctoral Thesis, Dec. 1982).

7. Friedmann, P.P., and P. Shauthakumaran, "Optimum Design of Rotor Blades for Vibration Reduction in Forward Flight," Proc. of the 39th Annual Forum of the AHS, May 9-11, 1983, St. Louis, MO.

8. Friedmann, P., "Application of Modern Structural Optimization to Vibration Reduction in Rotorcraft," Vertica, Vol. 9, No. 4, pp . 363-376, 1985.

9. Taylor, R.B., "Helicopter Vibration Reduction by Rotor Blade Modal Shaping," Proceedings of the 38th Annual Forum of the American Helicopter Society, May 4-7, 1982, Anaheim, CA, May 1982.

10. Zeleny, M., Linear Multiobjective Programming, Lecture Notes in Economics and Mathematical Systems, Managing Editors: M. Beckmann, Providence, and

11. Niordson, F.I., "On the Optimal Design of Vibrating Beam," Quarterly of Applied Mathematics, Vol. 23, No. 1, 1965, pp. 47-63.

12. Brach, R.M., "On the External Fundamental Frequencies of Vibrating Beams," International Journal of Solids and Structures, Vol. 4, 1968, pp. 667-674.

13. Vepa, K., "On the Existence of Solutions to Optimization Problems with Eigenvalue Constraints," Quarterly of Applied Mathematics, 31, 1973-1974, pp. 329-341.

14. Prager, W., and J.E. Taylor, "Problems of Optimal Structural Design," Journal of Applied Mathematics, Vol. 35, No. 1, 1968, pp. 102-106.

15. Taylor, J.E., "Optimum Design of a Vibrating Bar with Specified Minimum Cross-Section," AIAA Journal, Vol. 6, No. 7, July 1968, pp. 1379-1381.

16. Kamat, M.P., and G.J. Simitses, "Optimal Beam Frequencies by the Finite Element Displacement Method," International Journal of Solids and Structures, Vol. 19, 1973, pp. 415-429.

17. Karihaloo, B.L., and F.I. Niordson, "Optimum Design of Vibrating Cantilevers," Journal of Optimization Theory and Applications, Vol. II, No. 6, 1973, pp. 638-654.

18. Olhoff, N., "Optimization of Vibrating Beams with Respect to Higher Order Natural Frequencies," Journal of Structural Mechanics, Vol. 4, No. 1, 1976, pp. 87-122.

19. Olhoff, N., "Maximizing Higher Order Eigen Frequencies of Beams with Constraints on the Design Geometry," Journal of Structural Mechanics, Vol. 5, No. 2, 1977, pp. 107-134.

20. Olhoff, N., "Optimization of Transversely Vibrating Beams and Rotating Shafts," Optimization of Distributed Parameter Structures (Ed., Haug, E. J., and Cea, J.) Sijthoff and Noordhoff, Alphen aan den Rijn, Netherlands, 1980.

21. Turner, M.J., "Design of Minimum Mass Structures with Specified Natural Frequencies," AIAA Journal, Vol. 5, No. 3, March 1967, pp. 406-412.

22. Brach, R.M., "On Optimal Design of Vibrating Structures," Journal of Optimization Theory and Applications, Vol. 11, No. 6, 1973, pp. 662-667.

23. McCant, B.R., E.J. Haug, and T.D. Streeter, "Optimal Design of Structures with Constraints on Natural Frequency," AIAA Journal, Vol. 8, No. 6, 1970, pp. 1012-1019.

24. Sippel, D.L., and W.H. Warner, "Minimum Mass Design of Multielement Structures Under a Frequency Constraint," AIAA Journal, Vol. 11, No. 4, April 1973, pp. 483-489.

25. Pierson, B.L., "An Optimal Control Approach to Minimum-Weight Vibrating Beam Design," Journal of Structural Mechanics, Vol. 5, No. 2, 1977, pp. 147-178.

26. Khan, M.R., and K.D. Willmert, "An Efficient Optimality Criterion Method for Natural Frequency Constrained Structures," Computers and Structures, Vol. 14, No. 5-6, 1981, pp. 501-507.

27. Venkayya, V.B., and V.A. Tischler, "Optimization of Structures with Dynamic Constraints," Computer Methods for Nonlinear Solids and Structural Mechanics, Vol. 54, presented at the Applied Mechanics, Bioengineering and Fluids Engineering Conference, June 20-22, 1983, Houston, TX.

28. Warner, W.H., and D.J. Vavrick, "Optimal Design in Axial Motion for Several Frequency Constraints," Journal of Optimization Theory and Applications, Vol. 15, No. 1, 1975.

29. Pederson, P., "Design with Several Eigenvalue Constraints by Finite Elements and Linear Programming," Journal of Structural Mechanics, Vol. 10, No. 3, 1982-83, pp. 243-271.

30. Khot, N.S., "Optimization of Structures with Multiple Frequency Constraints," AFWAL-TI-83-29-fibr-201, Wright-Patterson Air Force Base, Ohio, 1983.

31. Hanagud, S., Aditi Chattopadhyay, and C.V. Smith, "Optimal Design of Vibrating Beam with Coupled Bending and Torsion," Proceedings of the 27th Annual Structures, Structural Dynamics and Material Conference, April 1985.

32. "Minimum Weigh Design of a Structure with Dynamic Constraints and a Coupling of Bending and Torsion," Proceedings of the 28th Annual Structures, Structural Dynamics and Materials Conference, May 1986, San Antonio, TX.

33. Straub, F.K., and P.P. Friedmann, "Application of the Finite Element Method to Rotary Wing Aeroelasticity, NACA CR 165854, 1982.

34. Meirovitch, L., "Computational Methods in Structural Dynamics," Sijthoff & Noor-hoff, Alphen aan den Rijn, Netherlands, p. 217, 220, 1980.

Table 1. Results of Optimum Design ($\bar{\omega} = 1.2$, $\omega = 1.0$*)

				REACTIONS FOR SINGLE BLADE				
	W_{Flap}	W_{Lag}	$W_{Torsion}$	$\bar{F}$†	$\bar{M}y$†	$\bar{F}_z$†	$\bar{M}_z$†	V/Vo
Initial	1.16	0.6	2.41	-3.9339	9.547	26.2978	-1.8856	1.0
Final	1.51	1.074	5.51	-1.9055	5.4385	15.069	-0.8966	0.75
Reduction				51.6%	43.1%	42.69%	52.4%	25%

* ω: Excitation frequency

† See Fig. 3

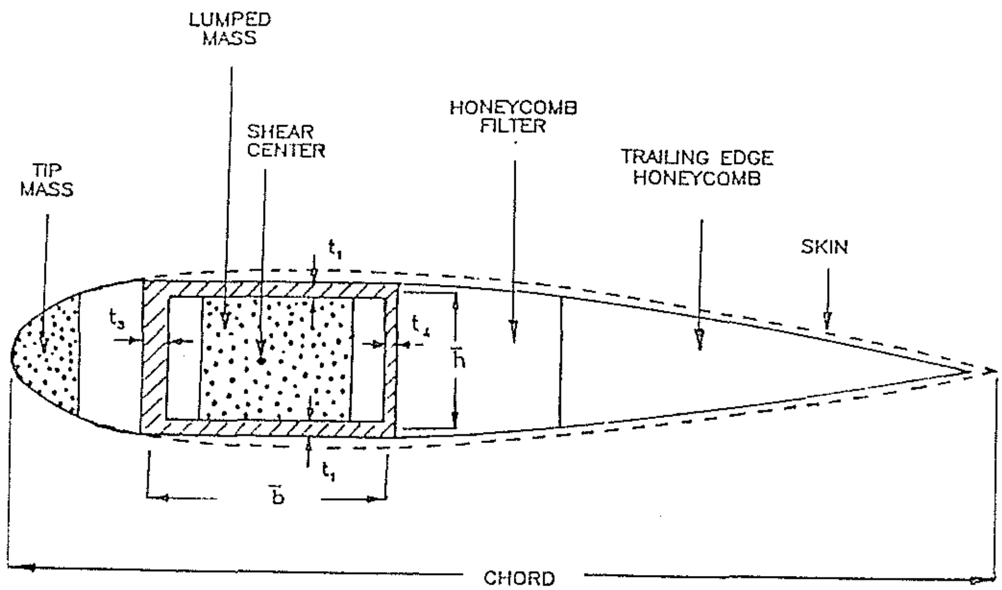

Fig. 1 A Typical Blade Section

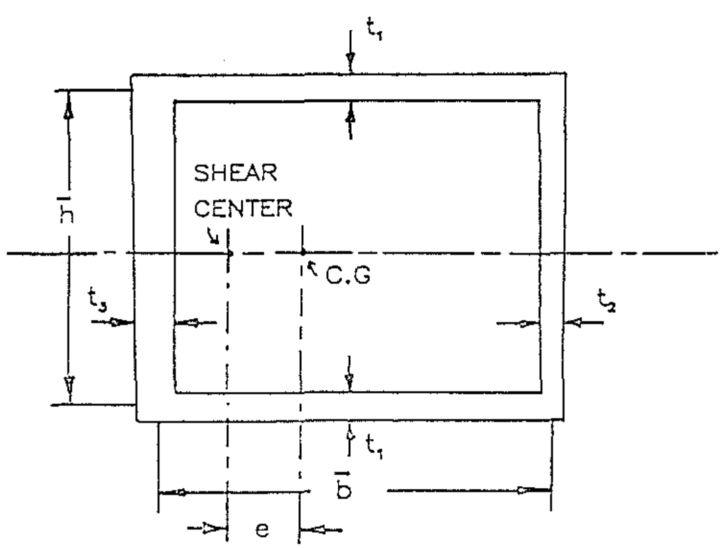

Fig. 2. Cross Section of a Box Beam

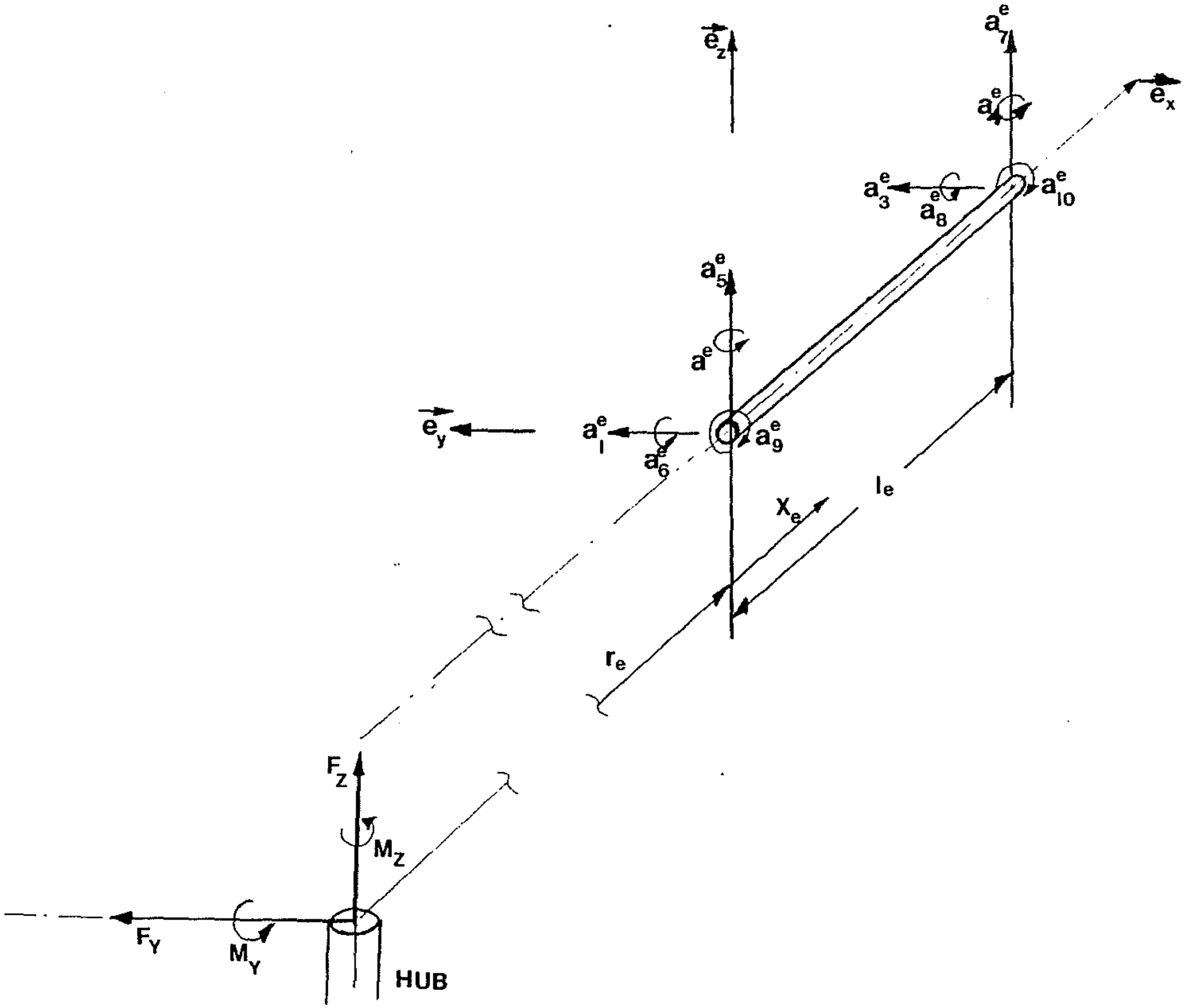

Fig. 3. Coupled Flap-Lag-Torsion Element and Forces and Moments Acting on the Hub in the Undeformed Coordinate System

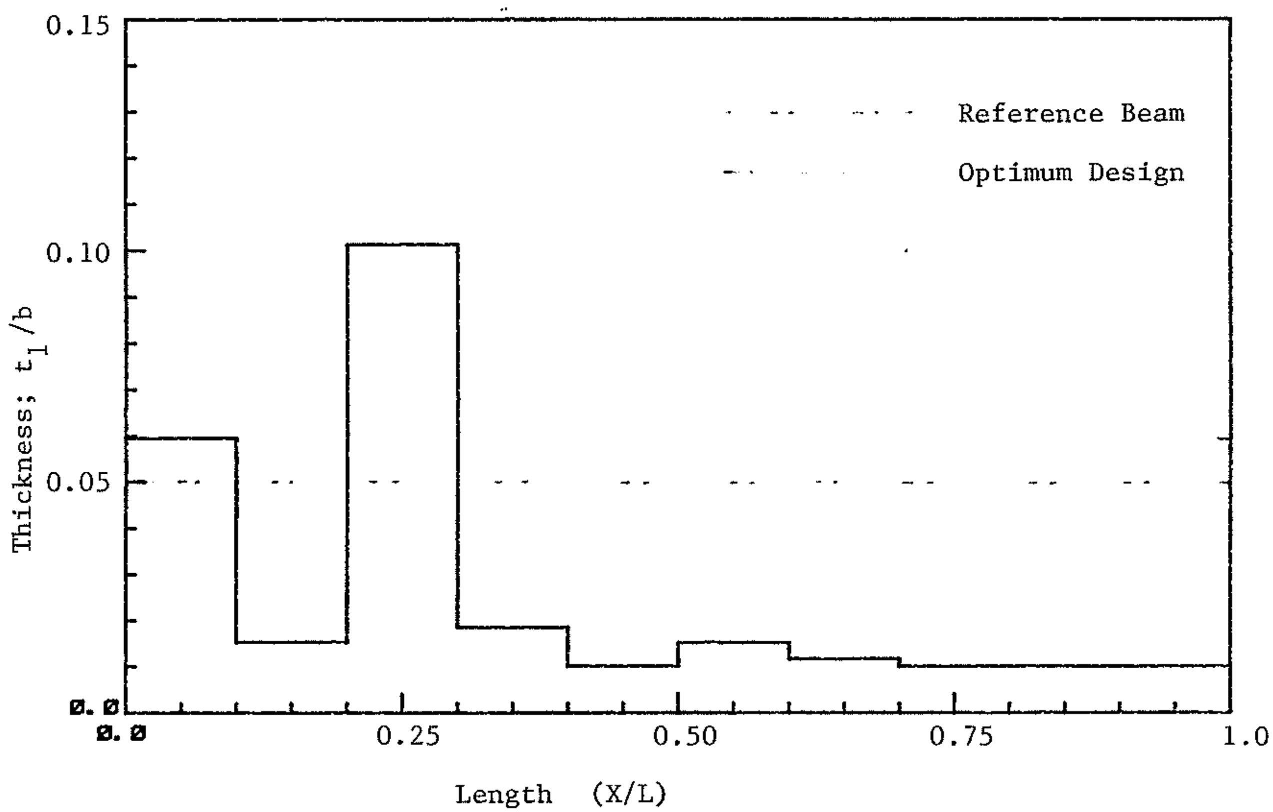

Fig. 4 Optimum Thickness (t_1) Distribution

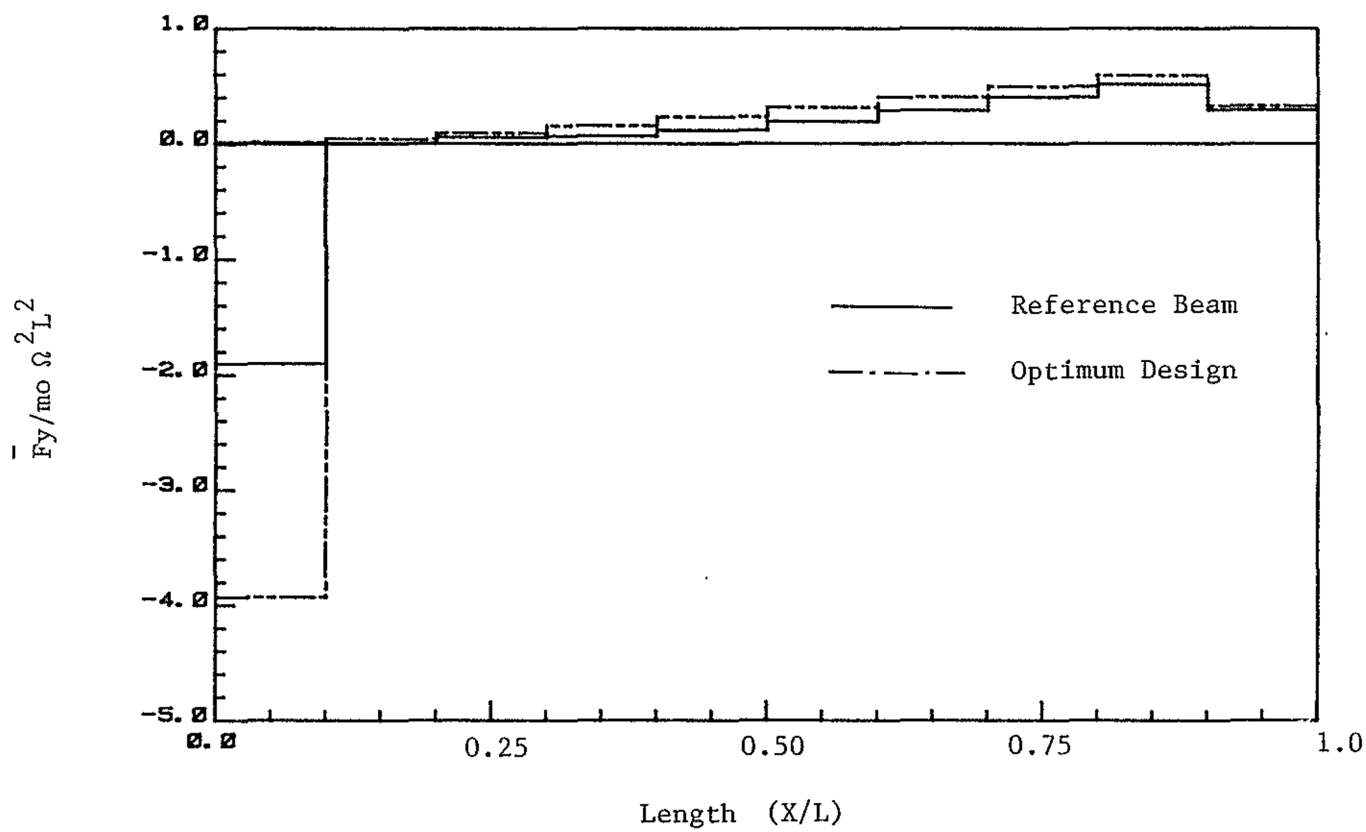

Fig. 5 Fy Distribution

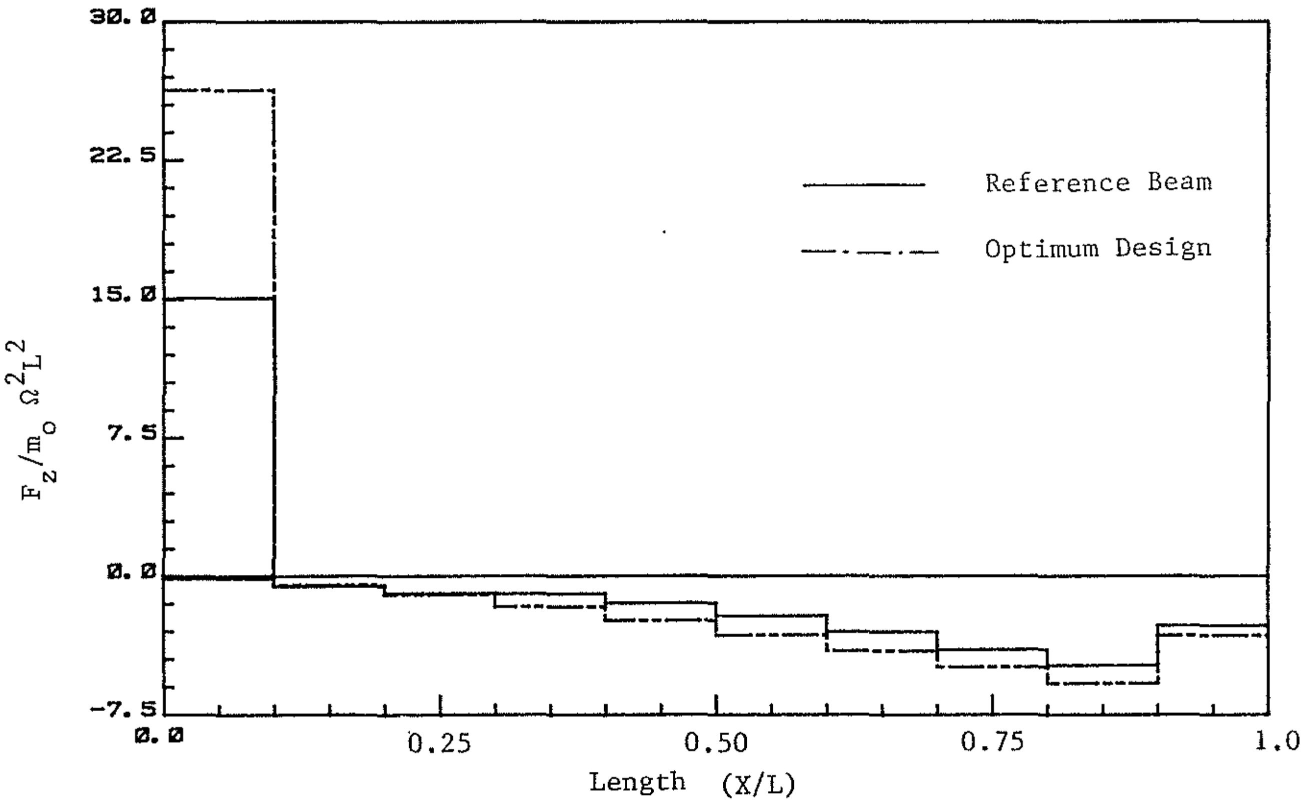

Fig. 6 Fz Distribution