ROBUST NONLINEAR CONTROL OF A MINIATURE HELICOPTER FOR AEROBATIC MANEUVERS

L. Marconi and R. Naldi

Center for research on complex Automated SYstems (CASY)
DEIS-University of Bologna, ITALY. [1]

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Abstract: The paper addresses the design of an autopilot for small scale helicopters to let the vertical, lateral, longitudinal and yaw attitude dynamics to track aggressive references. The paper is mostly focused on robust control aspects of helicopters. In particular the distinguish feature of the control law proposed in the paper is the ability to deal with the presence of possible severe uncertainties characterizing the physical parameters of the plant. This is achieved by using non linear control design techniques.

This work enriches a number of recent works focused on autopilot design for small scale helicopters (see, besides others, (Shakernia et al., 1999), (Sira-Ramirez et al., 2000), (Snell et al., 1992.), (Isidori et al., 2003)), motivated, on one hand, from the interest that this kind of air-vehicle has in practical applications, (Sprague et al., 2001), and, on the other hand, from some features of the helicopter model, such as the non linearity of the dynamics and the strong coupling between the forces and torques produced by vehicle actuators, which render the system in question an ideal test-bed for testing and comparing nonlinear design techniques. Inspired by the stabilizing techniques proposed in (Isidori et al., 2003), the control structure is designed as a mix of feed-forward actions, computed according to the reference signals to be tracked and a nominal model inversion (see also (Lane et al., 1998)), (Prasad et al., 1991)), and feedback terms obtained by combining high gain and nested saturation control laws. More specifically the control law proposed is based on a cascade control structure, composed by an inner loop and an outer loop governing respectively the attitude dynamics and the lateral-longitudinal dynamics.

It is shown, also by means of simulation results, how the proposed control structure is able to enforce aggressive manoeuvres, in particular the case of maneuvers performed with aggressive attitude angles is considered, showing how, even in this challenging scenario, the proposed controller is able to achieve asymptotic robustness to a number of physical parameters which are typically affected by strong uncertainties.

1 INTRODUCTION

In this work we focused on autopilot design for small scale helicopters in order to let the dynamics of the system to follow arbitrary trajectories. A crucial feature of the helicopter with respect to others air-vehicles is to be functionally controllable in the lateral/longitudinal and vertical directions with arbitrary yaw-attitude. This feature guarantees high maneuverability to the helicopter and it is somehow the distinguishing feature of this kind of systems with respect

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to fixed-wing air vehicles. In this paper we wish to employ this feature and investigate the design of an autopilot controlling the helicopter in the lateral/longitudinal and vertical direction and in yaw attitude. The emphasis of the results here proposed is on the asymptotic robustness of the proposed control law to a number of physical parameters which are typically affected by strong uncertainties.

The paper is organized as follows. Section 2 describe the helicopter dynamical model used to derive the control law and shows preliminary details about the control problem. Section 3 describe the proposed controller whereas in Section 4 simulation results are presented. Final remarks are reported in Section 5.

Notations: for a bounded function $s : I\!R \rightarrow I\!R^r$ we denote $\|s\|_\infty = \sup_{t\geq 0} \|s(t)\|$ and $\|s\|_a = \lim_{t\rightarrow\infty} \sup \|s(t)\|$ in which $\|\cdot\|$ denotes the Euclidean norm. We use the compact notation C_a, S_a, T_a with $a \in I\!R$ to indicate respectively $\cos a$, $\sin a$ and $\tan a$. For a real-valued function $f : I\!R \rightarrow I\!R$ and $x = (x_1, \ldots, x_n)^{\mathrm{T}}$, $f(x) = \mathrm{col}(f(x_1), \ldots, f(x_n))$. For a vector $\omega = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$, Skew$(\omega)$ denoted the 3×3 skew-symmetric matrix with the first, second and third row respectively given by $[0, -\omega_3, \omega_2]$, $[\omega_3, 0, -\omega_1]$ and $[-\omega_2, \omega_1, 0]$.

2 PROBLEM STATEMENT

2.1 Helicopter model

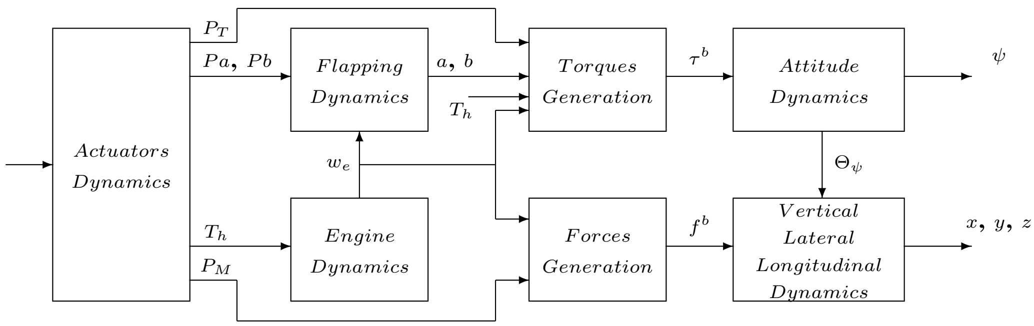


Figure 1: Small Scale Helicopter four level dynamics: rigid body equations, force and torque generation mechanisms, rotor and engine dynamics and actuator dynamics.

A mathematical model of a miniature helicopter could be derived considering the overall system as a rigid body, as shown in figure (1), driven by a wrench vector generated by lower level dynamics which include aerodynamics, engine dynamics, rotor wing dynamics and actuator dynamics.
Rigid body dynamics are modeled according to Newton-Euler equation of motion in the configuration space $SE(3) = I\!R^3 \times SO(3)$. In particular, by fixing an inertial coordinate frame $F_i = \{O_i, \overrightarrow{i}_i, \overrightarrow{j}_i, \overrightarrow{k}_i\}$ and a coordinate frame $F_b = \{O_b, \overrightarrow{i}_b, \overrightarrow{j}_b, \overrightarrow{k}_b\}$ attached to the body, the model of the helicopter with respect to the inertial framework is described as

$$\begin{array}{rcl} M\ddot{p} & = & Rf^b \\ J\dot{w} & = & -Skew(w)Jw + \tau^b \end{array} \tag{1}$$

where f^b and τ^b represents respectively the vector of forces and torques applied to the helicopter expressed in the body frame, M and J the mass and the inertia matrix, the vector

$p = \mathrm{col}(x, y, z)$ the position of the center of mass and R the rotation matrix relating the two reference frame. Rotation matrices has been parameterized by means of roll (ϕ), pitch (θ) and yaw (ψ) euler angles

$$\Theta := \begin{pmatrix} \phi \\ \theta \end{pmatrix} \qquad \Theta_\psi := \begin{pmatrix} \Theta \\ \psi \end{pmatrix}$$

where

$$\dot{\Theta}_\psi = Q(\Theta)\omega \qquad Q(\Theta) = \begin{pmatrix} 1 & S_\phi T_\theta & C_\phi T_\theta \\ 0 & C_\phi & -S_\phi \\ 0 & S_\phi / C_\theta & C_\phi / C_\theta \end{pmatrix} \tag{2}$$

Accordingly R is given as

$$R = \begin{pmatrix} C_\psi C_\theta & -S_\psi C_\theta + C_\psi S_\theta S_\phi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\psi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\psi S_\phi + S_\psi S_\theta C_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{pmatrix}$$

The external wrench vector applied to the helicopter is modelled as a nonlinear function of five *control inputs*

$$u = \mathrm{col}\begin{pmatrix} P_M & P_T & a & b & T_h \end{pmatrix} \tag{3}$$

where P_M and P_T denotes respectively the collective pitch of the main and of the tail rotor, a and b are respectively the longitudinal and lateral inclination of the tip path plane of the main rotor imposed controlling flapping dynamics by means of cyclic pitches P_a and P_b and, finally, T_h is the throttle controlling the main engine power. In particular, following (Sastry et al., 1998), it turns out that total force/torque can be modelled as

$$f^b = \begin{pmatrix} X_M \\ Y_M + Y_T \\ Z_M \end{pmatrix} + R^T \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix} \tag{4}$$

$$\tau^b = \begin{pmatrix} R_M \\ M_M \\ N_M \end{pmatrix} + \begin{pmatrix} Y_M h_m + Z_M y_m + Y_T h_t \\ -X_M h_m + Z_M l_m \\ -Y_M l_m - Y_T l_t \end{pmatrix}$$

in which g is the force of gravity, (l_m, y_m, h_m) and (l_t, y_t, h_t) denote respectively the coordinates of the main and tail rotor shafts relative to center of mass expressed in F_b, and

$$\begin{array}{rclcrcl} X_M & = & -T_M S_a & \quad & Y_M & = & -T_M S_b \\ Z_M & = & -T_M C_a C_b & & Y_T & = & -T_T \end{array} \tag{5}$$

and

$$\begin{array}{rcl} R_M & = & c_b^M b - Q_M S_a \\ M_M & = & c_a^M a + Q_M S_b \\ N_M & = & -Q_M C_a C_b \, . \end{array} \tag{6}$$

In the previous expressions c_a^M, are physical parameters modelling torques effect on the rigid body due to the stiffness of the main rotor, Q_M is the total main rotor torque and T_M and T_T are the thrusts generated respectively by the main and the tail rotor given by the following simplified expressions

$$T_M = K_{T_M} P_M w_\mathrm{e}^2 \quad T_T = K_{T_T} P_T w_\mathrm{e}^2 \tag{7}$$

where w_e denotes the angular velocity of the main rotor in the body frame and the coefficients K_{T_M} and K_{T_T} denote aerodynamic constants of the rotor's blades. It is supposed that the main

rotor velocity may differ from the tail one only by a multiplicative constant coefficient $n_{we} > 0$ included in K_{T_T}.
We approximate flapping dynamics of the main rotor as follow

$$\begin{aligned}\tau_e \dot{a} &= -a - w_y \tau_e + A_f P_a \\ \tau_e \dot{b} &= -b - w_x \tau_e + B_f P_b\end{aligned}$$

in which τ_e denotes the time constant of the rotor dynamics which is affected by the presence of the stabilizer bar and A, B are constant parameters.
The angular velocity w_e of the main rotor is governed by the engine dynamic model which is modeled as

$$\dot{w}_\mathrm{e} = \frac{Q_e - Q_M^R}{I_{rot}} \tag{8}$$

in which Q_e is the total engine torques and Q_M^R is a reaction torque due to aerodynamic resistance of rotor's blades given by

$$Q_M^R = c w_\mathrm{e}^2 + d P_M^2 w_\mathrm{e}^2 \tag{9}$$

with c and d physical parameters depending on wing geometry and other rotor's characteristics. As an approximation torque acting on main rotor is assumed to be equal to engine torque, namely

$$Q_M = Q_e$$

The expression of engine (and main) torque is then given

$$Q_e = P_e / w_\mathrm{e} \,. \tag{10}$$

where P_e denotes engine power which is assumed to be proportional to throttle

$$P_e = \bar{P}_e T_h \tag{11}$$

with $0 < T_h < 1$.

This completes the model of the helicopter which is a fifteenth order system with five control inputs. In the expressions above, however, we will make a number of assumptions in order to simplify the model to derive a control law. First of all, as far as the external force f^b is concerned, we neglect the contribution of main rotor thrust along the x^b direction and assume that the contribution of tail rotor thrust along y^b is matched by the corresponding main rotor thrust component obtaining, in this way,

$$f^b = \begin{pmatrix} 0 \\ 0 \\ -T_M \end{pmatrix} + R^T \begin{pmatrix} 0 \\ 0 \\ Mg \end{pmatrix} . \tag{12}$$

Moreover, since the tilt angles a and b are small, we shall assume

$$S_a \approx a \,, \; S_b \approx b \,, \; C_a \approx 1 \,, \; C_b \approx 1 \tag{13}$$

which, along with (7), (11) and (10), yield a simplified total torque (4) given by

$$\tau^b = A\left(P_M, T_h, w_\mathrm{e}\right) v + B\left(P_M, T_h, w_\mathrm{e}\right) \tag{14}$$

with

$$v = \mathrm{col}(a,\ b,\ P_T) \tag{15}$$

in which $A(\cdot)$ and $B(\cdot)$ are a matrix and, respectively, a vector of affine functions of the inputs P_M and T_h and of the angular rotor velocity w_{e}.
Since in small scale helicopters the flapping dynamics is usually fast with respect to the vertical, lateral, longitudinal and attitude dynamics, and could be rendered even faster by removing the stabilizer bar as shown in (Mettler et al., 2002), we consider only the steady state relation between cyclic pitches and tilt angles a and b, obtaining

$$a = K_a P_a \quad b = K_b P_b$$

This allow to consider the tilt angles a and b as the effective inputs for the attitude dynamics.

One of the main goal of the controller to be designed is to deal with possibly large parameters uncertainties, including mass M and the inertia matrix J of the vehicle, the aerodynamic coefficients in (7) and the coefficients in (12), (14) and in engine model (8). In the following we shall denote with the subscript "0" and "Δ" respectively the nominal and the uncertain values of these parameters, namely

$$\begin{array}{rcl} M & = & M_0 + M_\Delta\,, \qquad J = J_0 + J_\Delta \\ K_{T_i} & = & K_{T_i0} + K_{T_i\Delta}\,, \qquad i \in \{M, T\} \\ c & = & c_0 + c_\Delta\,, \qquad d = d_0 + d_\Delta \\ K_a & = & K_{a0} + K_{a\Delta}\,, \qquad K_b = K_{b0} + K_{b\Delta} \\ \overline{P}_e & = & \overline{P}_{e0} + \overline{P}_{e\Delta} \end{array}$$

These uncertainties reflect into uncertainties of the matrix $A(\cdot)$ and vector $B(\cdot)$ introduced in (14) which will be accordingly written as

$$\begin{array}{c} A\left(P_M, T_h, w_{\mathrm{e}}\right) = A_0\left(P_M, T_h, w_{\mathrm{e}}\right) + A_\Delta\left(P_M, T_h, w_{\mathrm{e}}\right) \\ B\left(P_M, T_h, w_{\mathrm{e}}\right) = B_0\left(P_M, T_h, w_{\mathrm{e}}\right) + B_\Delta\left(P_M, T_h, w_{\mathrm{e}}\right)\ . \end{array}$$

The ranges of the uncertainties of the physical parameters will be not constrained to be "small" but will be allowed in general to be "arbitrarily large" (fulfilling only physical constraints). The only mild requirement needed to support the results presented in this paper, is a restriction on the relative variation of $A(\cdot)$ with respect to its nominal value $A_0(\cdot)$. In particular it is required the existence of a positive number $m^\star$ such that

$$A_\Delta\left(P_M, T_h, w_{\mathrm{e}}\right) A_0\left(P_M, T_h, w_{\mathrm{e}}\right)^{-1} \leq m^\star I \tag{16}$$

for all possible values of P_M, T_h, w_{e} within physical ranges.

2.2 Overview of the control problem

The main control purpose addressed in this paper is to design the five control inputs (3) in order to asymptotically track vertical, lateral, longitudinal and yaw attitude time references $x_{\mathrm{r}}(t)$, $y_{\mathrm{r}}(t)$, $z_{\mathrm{r}}(t)$ and $\psi_{\mathrm{r}}(t)$. The latters are supposed to be known arbitrary time profiles with the only restrictions dictated by the *functional controllability* of the system and by the fulfillment of physical constraints on the control inputs. Bearing in mind the force model simplification of (12) we could rewrite the first of (1) along the desired trajectory $x_{\mathrm{r}}(t)$, $y_{\mathrm{r}}(t)$, $z_{\mathrm{r}}(t)$ and $\psi_{\mathrm{r}}(t)$ as

$$M a_{\mathrm{r}} \begin{pmatrix} n_{x_{\mathrm{r}}} \\ n_{y_{\mathrm{r}}} \\ n_{z_{\mathrm{r}}} \end{pmatrix} = -T_{M_{\mathrm{r}}} R(\Theta_\psi)_{\mathrm{r}} e_z \tag{17}$$

with $e_z = col(0,0,1)$ and

$$\begin{aligned} a_{\mathrm{r}} &= \sqrt{\ddot{x}_{\mathrm{r}}^2 + \ddot{y}_{\mathrm{r}}^2 + (\ddot{z}_{\mathrm{r}} - g)^2} \\ n_{x_{\mathrm{r}}} &= \frac{\ddot{x}_{\mathrm{r}}}{a_{\mathrm{r}}} \\ n_{y_{\mathrm{r}}} &= \frac{\ddot{y}_{\mathrm{r}}}{a_{\mathrm{r}}} \\ n_{z_{\mathrm{r}}} &= \frac{\ddot{z}_{\mathrm{r}} - g}{a_{\mathrm{r}}} \end{aligned}$$

in which $n = col(n_{x_{\mathrm{r}}}, n_{y_{\mathrm{r}}}, n_{z_{\mathrm{r}}})$ define a unit norm vector in the space $I\!R^3$ which represent the orientation of the desired acceleration vector of amplitude a_{r}. From (17) it turns out the desired reference angles ϕ_{r} and θ_{r} and the main thrust $T_{M_{\mathrm{r}}}$ necessary to track the refences are given by

$$\begin{aligned} T_{M_{\mathrm{r}}} &= M a_{\mathrm{r}} \\ \phi_{\mathrm{r}} &= \text{atan2}\left(-C_{\theta_{\mathrm{r}}} S_{\psi_{\mathrm{r}}} n_{x_{\mathrm{r}}} + C_{\theta_{\mathrm{r}}} C_{\psi_{\mathrm{r}}} n_{y_{\mathrm{r}}}, -n_{z_{\mathrm{r}}}\right) \\ \theta_{\mathrm{r}} &= \text{atan2}\left(-S_{\psi_{\mathrm{r}}} n_{y_{\mathrm{r}}} - C_{\psi_{\mathrm{r}}} n_{x_{\mathrm{r}}}, -n_{z_{\mathrm{r}}}\right) \end{aligned} \tag{18}$$

and also

$$\begin{aligned} T_{M_{\mathrm{r}}} &= -M a_{\mathrm{r}} \\ \phi_{\mathrm{r}} &= \text{atan2}\left(C_{\theta_{\mathrm{r}}} S_{\psi_{\mathrm{r}}} n_{x_{\mathrm{r}}} - C_{\theta_{\mathrm{r}}} C_{\psi_{\mathrm{r}}} n_{y_{\mathrm{r}}}, n_{z_{\mathrm{r}}}\right) \\ \theta_{\mathrm{r}} &= \text{atan2}\left(S_{\psi_{\mathrm{r}}} n_{y_{\mathrm{r}}} + C_{\psi_{\mathrm{r}}} n_{x_{\mathrm{r}}}, n_{z_{\mathrm{r}}}\right) \end{aligned} \tag{19}$$

The two different solutions indicate two possible configuration of the helicopter in order to obtain the same resultant acceleration. This property reflects the possibility of small scale helicopters of changing main thrust direction by means of negative collective pitch values.

In this paper we will limit the analysis of the system to the choice of (18), and since fuctional controllability of the system requires that

$$|\theta_r(t)| \leq \frac{\pi}{2} \quad |\phi_r(t)| \leq \frac{\pi}{2} \quad \forall\, t \geq 0 \tag{20}$$

it turns out that the instantaneous desired accelerations are restricted. Observe that (20) is a necessary condition in order to avoid singularities in attitude representations due to the choice of the euler angle parametrization of the group $SO(3)$. Moreover, by bearing in mind (1), (8) and (14), it is readily seen that the five desired control inputs compatible with the tracking references are given by

$$P_{M\mathrm{r}} = M \frac{g - \ddot{z}_{\mathrm{r}}}{K_{T_M} w_{\mathrm{er}}^2 C_{\phi_{\mathrm{r}}} C_{\theta_{\mathrm{r}}}} \qquad T_{h\mathrm{r}} = \frac{w_{\mathrm{er}}^3}{\bar{P}_{\mathrm{e}}} \left(c + d P_{M\mathrm{r}}^2\right)$$

$$v_{\mathrm{r}} = \begin{pmatrix} a_{\mathrm{r}} \\ b_{\mathrm{r}} \\ P_{T\mathrm{r}} \end{pmatrix} = A^{-1}(\cdot)\left[J\dot{\omega}_{\mathrm{r}} + \text{Skew}\omega_{\mathrm{r}} J \omega_{\mathrm{r}} - B(\cdot)\right]$$

having denoted by w_{er} the desired (constant) value of the angular velocity of the main rotor and by ω_{r} the angular velocity of the helicopter compatible with the tracking references given by (see (2))

$$\omega_{\mathrm{r}} = Q^{-1}(\Theta_{\mathrm{r}}) \begin{pmatrix} \dot{\Theta}_{\mathrm{r}} \\ \dot{\psi}_{\mathrm{r}} \end{pmatrix}. \tag{21}$$

Thus we assume as a physical constraint that the reference inputs are bounded, limiting the class of admissible reference signals. From this, we assume that the reference inputs satisfy

$$\begin{array}{cc} \max_{\mu\in\mathcal{I}} \|v_{\mathrm{r}}(t)\|_\infty \le v^U & \max_{\mu\in\mathcal{I}} \|P_{M\mathrm{r}}(t)\|_\infty \le P_M^U \\ \max_{\mu\in\mathcal{I}} \|T_{h\mathrm{r}}(t)\|_\infty \le T_h^U & \end{array}$$

where v^U, P_M^U, T_h^U denotes upper bounds on the amplitude of the inputs v, P_M and T_h imposed by physical constraints. This, in turn, imposes further constraints which limit the class of admissible reference signals. Apart these natural limitations and other minor restrictions specified throughout the paper, the reference signals are assumed to be completely arbitrary.

We will assume that all state is accessible for control purpose, in particular w_{e} for engine dynamic, vectors Θ_ψ and w for the attitude dynamic, vectors p and $\dot{p}$ for the vertical, lateral and longitudinal dynamics. Furthermore the initial state is supposed to belong to any (arbitrarily large) compact set with the only restriction that $-\pi/2 < \phi(0) < \pi/2$ and $-\pi/2 < \theta(0) < \pi/2$ (which implies that the helicopter is not overturned in the initial condition).

3 DESIGN OF THE CONTROL LAW

In this section we show how it is possible to design a control law which is able to satisfy the desired requirements. The design is carried out considering explicitly the uncertainties which characterize the miniature helicopter dynamics.

3.1 Vertical controller

Looking at (1), (4) and (7), vertical dynamics are described by

$$M\ddot{z} = -(C_\phi C_\theta)P_M K_{T_M} w_{\mathrm{e}}^2 + Mg \tag{22}$$

We choose the following preliminary feedback for the collective pitch P_M

$$P_M = \frac{-P'_M + M_0 g - M_0 \ddot{z}_{\mathrm{r}}}{K_{T_M 0}\, w_{\mathrm{es}}^2\, C_{\phi_{\mathrm{s}}}\, C_{\theta_{\mathrm{s}}}} \tag{23}$$

in which $w_{\mathrm{es}} := \max\{w_{\mathrm{e}}, \underline{w}_{\mathrm{e}}\}$, $C_{\phi_{\mathrm{s}}} := \max\{C_\phi, C_{\overline{\phi}}\}$, $C_{\theta_{\mathrm{s}}} := \max\{C_\theta, C_{\overline{\theta}}\}$, with

$$\underline{w}_{\mathrm{e}} \in (0, w_{\mathrm{er}}) \quad \overline{\phi} \in (\|\phi_{\mathrm{r}}(t)\|_\infty, \pi/2) \quad \overline{\theta} \in (\|\theta_{\mathrm{r}}(t)\|_\infty, \pi/2) \tag{24}$$

and P'_M is an auxiliary control input, whose goal is to decouple the vertical dynamics from the attitude and engine dynamics. The variables $(w_{\mathrm{es}}, \phi_{\mathrm{s}}, \theta_{\mathrm{s}})$ are clearly introduced to avoid singularities in the expression of (23) and, according to (20), to guarantee $(\phi_{\mathrm{s}}(t), \theta_{\mathrm{s}}(t)) \equiv (\phi(t), \theta(t))$ so long as $(\phi(t), \theta(t)) = (\phi_{\mathrm{r}}(t), \theta_{\mathrm{r}}(t))$. Defining $e_z = z - z_{\mathrm{r}}$ the vertical error dynamics is thus described by

$$M\ddot{e}_z = \Psi_1(\Theta)\Psi_2(w_{\mathrm{e}})\mu_1(P'_M + M_0\ddot{z}_{\mathrm{r}} - M_0 g) - M\ddot{z}_{\mathrm{r}} + Mg \tag{25}$$

in which $\Psi_1(\Theta) = (C_\phi C_\theta)/(C_{\phi_{\mathrm{s}}} C_{\theta_{\mathrm{s}}})$, $\Psi_2(w_{\mathrm{e}}) = w_{\mathrm{e}}^2/w_{\mathrm{es}}^2$ and $\mu_1 = K_{T_M}/K_{T_M 0}$ is a positive uncertain parameter. Clearly, if $\phi \le \overline{\phi}$, $\theta \le \overline{\theta}$ and $w_{\mathrm{e}} \ge \underline{w}_{\mathrm{e}}$, then necessarily $\Psi_1(\Theta) = \Psi_2(w_{\mathrm{e}}) = 1$ and system (25) simplifies as

$$M\ddot{e}_z = \mu_1 P'_M + (M - \mu_1 M_0)(g - \ddot{z}_{\mathrm{r}}) \tag{26}$$

Since we will be able to show, through a suitable design of the others control inputs, that $\Psi_2(w_\mathrm{e}(t)) = 1$ and $\Psi_1(\Theta(t)) = 1$ in finite time, we design the residual input P'_M focusing on the simplified system (26). In particular we design P'_M as a *PID* controller of the form

$$\begin{aligned} P'_M &= \xi - k_2\dot{e}_z - k_2k_1e_z \\ \dot{\xi} &= -k_2\dot{e}_z - k_2k_1e_z + M_0\dot{e}_z \end{aligned} \tag{27}$$

where k_1 and k_2 are design parameters. Then the following proposition holds:

Proposition 1 *Let $z_\mathrm{r}(t)$ be fixed so that $z_\mathrm{r}^{(3)}$ exists and is bounded. Let $T^\star > 0$ be a finite time such that $\Psi_1(\Theta(t)) = 1$ and $\Psi_2(w_\mathrm{e}(t)) = 1$ for all $t > T^\star$ and let $k_1 > 0$ fixed. There exists a $k_2^\star > 0$ such that for all $k_2 \geq k_2^\star$ the following holds:*

- *there exist fixed positive numbers $\bar{P}_M$ and $\bar{X}$ such that $P_M(t) \leq \bar{P}_M$ and $|(e_z(t), \dot{e}_z(t), \xi(t))| \leq \bar{X}$ for all $t \in [0, T^\star]$;*
- *there exist $M > 1$, $C > 0$, $\lambda > 0$ and $r > 0$ such that for all $t > T^\star$*

$$\|(e_z(t), \dot{e}_z(t), \chi(t))\| \leq Me^{-\lambda t}\|(e_z(T^\star), \dot{e}_z(T^\star), \chi(T^\star))\| + r\,\|\dot{\varrho}(\cdot)\|_\infty$$

having defined $\chi := \xi - (M_0 - M/\mu_1)(g - \ddot{z}_\mathrm{r})$ and $\varrho(t) = (M_0 - M/\mu_1)(g - \ddot{z}_\mathrm{r}(t))$.

The first claim of the previous proposition guarantees the existence of a bound on the closed-loop trajectories which is *independent of the attitude and the engine dynamics* and only dependent on $T^\star$. This, as shown in the following, will play a crucial role in the stability analysis of the overall system. Furthermore note that, as a consequence of the integral action in (27), the second claim asserts that the asymptotic properties of the vertical error dynamics are governed by the *time derivative* of ϱ which, in turn, is dependent on the jerk of the vertical reference trajectory ($z_\mathrm{r}^{(3)}$) and on the difference $(M - M_0\mu_1)$. To this respect either in the case of reference trajectories characterized by $z_\mathrm{r}^{(3)}(t) \equiv 0$ or of an accurate knowledge of the system's parameters (namely $(M - M_0\mu_1) = 0$), the proposed vertical control law yields a perfect asymptotic tracking of the reference signal (provided that $\Psi_1(\Theta(t)) = 1$ and $\Psi_2(w_\mathrm{e}(t)) = 1$ in finite time).

3.2 Engine controller

We consider engine dynamics (8) rewritten as following

$$I_{rot}\dot{w}_\mathrm{e} = \frac{\bar{P}_e T_h}{w_\mathrm{e}} - \left(c + dP_M^2\right) w_\mathrm{e}^2$$

and we look for a control input T_h able to keep the value of w_e close to a desired constant value w_er and to guarantee that $w_\mathrm{e}(t) \geq \underline{w}$ in finite time as desired by the previous analysis. We choose a preliminary feedback, aiming to compensate for nominal value of Q_M^R, as

$$T_h = \frac{w_\mathrm{e}^3}{\bar{P}_{e0}}(T'_h + c_0 + d_0P_M^2) \tag{28}$$

in which T'_h is an additional control input designed as the nonlinear *PI* control law

$$\dot{\xi} = k_3 w_\mathrm{e}^2\, \tilde{w}_\mathrm{e} \qquad \tilde{w}_\mathrm{e} = w_\mathrm{e} - w_\mathrm{er} \qquad T'_h = -k_3\tilde{w}_\mathrm{e} - k_4\xi \tag{29}$$

where k_3 and k_4 are design parameters. The result underlying the tuning of the engine controller (28), (29) can be given as follows.

Proposition 2 *Let ϵ, $T^\star$, $\bar{P}_M$, ℓ_w, ℓ_ξ be arbitrary positive numbers with $\ell_w < w_{\rm er}$. There exists a $k_4^\star > 0$ and for all $k_4 \geq k_4^\star$ there exists a $k_3^\star > 0$ such that for all $k_3 \geq k_3^\star$ and for all initial conditions $|\tilde{w}_{\rm e}(0)| \leq \ell_w$ and $|\xi(0)| \leq \ell_\xi$ and for all $\|P_M\|_\infty \leq \bar{P}_M$ the following holds:*

- *the closed loop trajectories are such that $w_{\rm e}(t) > 0$ for all $t \geq 0$ and $|\tilde{w}_{\rm e}(t)| \leq \epsilon$ for all $t \geq T^\star$;*
- *the closed loop system is input to state stable with respect to the input $P_M^2(t)$*

Observe that by joining the results of propositions 1 and 2 it is possible to conclude that *if there exists a time $T^\star$ such that $\Psi_1(\Theta(t)) = 1$ for all $t \geq T^\star$, then there exist a k_3 and k_4 (obtained according to Proposition 2) such that also $\Psi_2(w_{\rm e}(t)) = 1$ for all $t \geq T^\star$*. As a matter of fact suppose that, besides $\Psi_1(\Theta(t)) = 1$, also $\Psi_2(w_{\rm e}(t)) = 1$ for $t \geq T^\star$. According to Proposition 1, this guarantees the existence of a number $\bar{P}_M'$ such that $\|P_M\|_\infty \leq \bar{P}_M'$ and, according to Proposition 2, this implies the possibility of tuning the parameters k_3 and k_4 in such a way that $\Psi_2(w_{\rm e}(t))$ is indeed equal to one for $t \geq T^\star$. According to this in the following section we will show, besides others, that $\Psi_1(\Theta(t)) = 1$ in finite time so that to recover the ideal vertical dynamics (26) and, as a consequence, to enjoy the asymptotic bound indicated in the second item of Proposition 1.

3.3 Lateral and longitudinal controller

Considering that attitude dynamics could be seen as a "virtual" control inputs for the lateral and longitudinal dynamics, we concentrate on a *cascade control structure* constituted by an *inner-loop* controlling the attitude dynamics and an *outer-loop* governing the lateral and longitudinal dynamics. Inner-loop regulator is in charge to control the attitude dynamics in such a way that the helicopter does not overturn and the lateral-longitudinal dynamics (having the attitude variables as virtual inputs) asymptotically approach the desired references. To achieve this objective we consider both feedforward control terms, based on the references to be tracked, and high-gain feedback control actions processing the actual attitude measures and the output of the outer lateral-longitudinal controller. The design of the inner and outer controllers leads to an overall loop characterized by two-time scale dynamics, with the inner attitude and outer lateral/longitudinal loops playing the role respectively of *fast* and *slow dynamics*. A crucial role in imposing the two-time scale behavior, is given by the use, in the design of outer loop, of nested saturation functions providing a decoupling between the attitude and lateral-longitudinal dynamics.

We consider first the preliminary choice

$$v = A_0^{-1}\left(P_M, T_h, w_{\rm e}\right)\left[\tilde{v} - B_0\left(P_M, T_h, w_{\rm e}\right)\right] \tag{30}$$

with $\tilde{v}$ is a residual control input, designed according to the inner-outer loop paradigm discussed above, which is meant to compensate for the nominal part of $A_0^{-1}(\cdot)$ and $B_0^{-1}(\cdot)$

$$\begin{array}{rcl}\tilde{v} & = & -K_P K_D(\omega - \omega_{\rm r}) - K_P \begin{pmatrix} T_\Theta - T_{\Theta_{\rm r}} \\ \psi + K_\psi \eta_\psi - \psi_{\rm r} \end{pmatrix} \\ & + & K_P \begin{pmatrix} A(\Theta_\psi)\Theta_{\rm out} \\ 0 \end{pmatrix} + J_0\dot{\omega}_{\rm r} + \text{Skew}(\omega_{\rm r}) J_0 \omega_{\rm r} \end{array} \tag{31}$$

in which K_P, K_D and K_ψ are design parameters,

$$A(\Theta_\psi) := \begin{pmatrix} -C_\psi & S_\psi C_\theta / C_\phi \\ S_\psi / C_\theta & C_\psi / C_\phi \end{pmatrix}$$

η_ψ is an integrator variable governed by

$$\dot{\eta}_\psi = \psi - \psi_{\rm r} \tag{32}$$

and $\Theta_{\rm out}$ represents the output of the outer lateral-longitudinal loop. The latter is designed using a nested saturation control law of the form

$$\Theta_{\rm out} = \lambda_3 \sigma(\frac{K_3}{\lambda_3}\xi_3) \tag{33}$$

with

$$\begin{array}{rcl}\xi_3 & := & \begin{pmatrix} \dot{y} - \dot{y}_{\rm r} \\ \dot{x} - \dot{x}_{\rm r} \end{pmatrix} + \lambda_2 \sigma(\frac{K_2}{\lambda_2}\xi_2) \\ \xi_2 & := & \begin{pmatrix} y - y_{\rm r} \\ x - x_{\rm r} \end{pmatrix} + \lambda_1 \sigma(\frac{K_1}{\lambda_1}\xi_1) \\ \xi_1 & := & \begin{pmatrix} \eta_y \\ \eta_x \end{pmatrix} \end{array} \tag{34}$$

where η_y and η_x represent integrator variables governed by

$$\dot{\eta}_y \;=\; y - y_{\rm r} \qquad \dot{\eta}_x \;=\; x - x_{\rm r}\,. \tag{35}$$

In the definition of the outer controller, (λ_i, K_i), $i = 1, 2, 3$, represent design parameters while $\sigma(\cdot)$ is a *saturation function* defined as any differentiable function $\sigma : I\!R \to I\!R$ satisfying:

$|\sigma'(s)| := |d\sigma(s)/ds| \leq 2$ for all s,
$s\sigma(s) > 0$ for all $s \neq 0$, $\sigma(0) = 0$.
$\sigma(s) = \mathrm{sgn}(s)$ for $|s| \geq 1$.
$|s| < |\sigma(s)| < 1$ for $|s| < 1$.

3.4 Inner loop analysis

After the preliminary compensation (30) and the addition of the integrator dynamic (32), the inner loop could be rewritten as

$$\begin{array}{rcl}\dot{\eta}_\psi & = & \psi - \psi_{\rm r} \\ \dot{Q}_{\rm e} & = & Q(\Theta)\omega \\ J\dot{\omega} & = & -\mathrm{Skew}(\omega)J\omega + L(P_M, T_h, \omega_{\rm e})\tilde{v} + \\ & + & \Delta(P_M, T_h, \omega_{\rm e}) \end{array} \tag{36}$$

with $\tilde{v}$ as in (31), in which $L(\cdot)$ and $\Delta(\cdot)$ are defined as

$$\begin{array}{rcl} L(\cdot) & = & I + A_\Delta(\cdot)A_0^{-1}(\cdot) \\ \Delta(\cdot) & = & B_\Delta(\cdot) - A_\Delta(\cdot)A_0^{-1}(\cdot)B_0(\cdot) \end{array}$$

Proposition 3 *Consider the inner attitude loop with $|\phi(0)| < \pi/2$ and $|\theta(0)| < \pi/2$ and suppose assumption (16) holds true. For any $K_\psi > 0$, $T^\star > 0$ and $\varepsilon > 0$ there exists $K^\star_{D1} > 0$ and, for any positive $K_D < K^\star_{D1}$, there exist $K^\star_{P1}(K_D)$, $\lambda^\star(K_D)$, both depending on K_D, such that for all $K_P \geq K^\star_{P1}(K_D)$ and $0 < \lambda < \lambda^\star(K_D)$ the following holds:*

(i) $|\phi(t)| < \pi/2$ *and* $|\theta(t)| < \pi/2$ *for all* $t \geq 0$ *;*

(ii) $\| \, (\; \eta_\psi(t)\,, \quad \Theta(t) - \Theta_{\rm r}(t)\,, \quad \psi(t) - \psi_{\rm r}(t) \;)^{\rm T} \, \| \leq \varepsilon$ *for all* $t \geq T^\star$.

An immediate consequence of Proposition 3, joined to Propositions 1, 2, is that it is possible to tune the design parameters K_D, K_P and λ_3 so that $|\Psi(\Theta(t))| = 1$, namely $|\phi(t)| \leq \bar{\phi}$ and $|\theta(t)| \leq \bar{\theta}$, in finite time.

3.5 Outer loop analysis

Looking at system 1, due to results previously discussed about the tuning of the inner loop, the lateral/longitudinal dynamics for all time $t \geq T^\star$ is described by

$$M \begin{pmatrix} \ddot{y} \\ \ddot{x} \end{pmatrix} = MD(\Theta_\psi, \ddot{z}_\mathrm{r}) \begin{pmatrix} T_\phi \\ T_\theta \end{pmatrix} + n\left(\Theta_\psi, \chi, e_z, \dot{e}_z\right) \tag{37}$$

in which

$$\begin{aligned} D(\Theta_\psi, \ddot{z}_\mathrm{r}) &= R_\psi^T \begin{pmatrix} -1/C_\theta & 0 \\ 0 & 1 \end{pmatrix} (g - \ddot{z}_\mathrm{r}) \\ n(\Theta_\psi, \chi, e_z, \dot{e}_z) &= R_\psi \begin{pmatrix} T_\phi / C_\theta \\ T_\theta \end{pmatrix} y_z(e_z, \dot{e}_z, \chi) \end{aligned} \tag{38}$$

and R_ψ is a two dimensional rotation matrix

$$R_\psi = \begin{pmatrix} C_\psi & -S_\psi \\ S_\psi & C_\psi \end{pmatrix}$$

For the overall closed-loop system the following proposition holds:

Proposition 4 *Consider system (37), (36), (30)-(35). Let $K_i^\star$ and $\lambda_i^\star$, $i = 1, 2, 3$, be such that the following inequalities are satisfied*

$$\begin{aligned} \frac{\lambda_2^\star}{K_2^\star} &< \frac{\lambda_1^\star}{4}, & \frac{\lambda_3^\star}{K_3^\star} &< \frac{\lambda_2^\star}{4}, \\ 4K_1^\star\lambda_1^\star &< \frac{\lambda_2^\star}{4}, & 4K_2^\star\lambda_2^\star &< \mu_3^L \frac{\lambda_3^\star}{8} \end{aligned} \tag{39}$$

and with $\mu_3^L := M^L(g - \|\ddot{z}_\mathrm{r}(\cdot)\|_\infty) > 0$ and $\mu_3^U := M^U\|g - \ddot{z}_\mathrm{r}(\cdot)\|_\infty > 0$. Moreover, let (K_i, λ_i) be chosen as $\lambda_i = \epsilon^{i-1}\,\lambda_i^\star$ and $K_i = \epsilon\, K_i^\star$, $i = 1, 2, 3$ in which ϵ is a positive design parameters. Let R_Δ an arbitrary positive number. There exist $r_1 > 0$, $r_2 > 0$, R_n, $\epsilon^\star > 0$ and $K_{D2}^\star > 0$ such that, for any positive $K_D \leq K_{D2}^\star$ and $\epsilon \leq \epsilon^\star$, there exists $K_{P2}^\star(\epsilon, K_D)$ such that for any $K_P \geq K_{P2}^\star(\epsilon, K_D)$ the system in question is ISS with restrictions $(R_n\epsilon^2, R_\Delta)$ on the inputs $(n(\cdot), \Delta(\cdot))$ and linear asymptotic gains; in particular if $\|n\|_\infty < R_n\epsilon^2$ and $\|\Delta\| \leq R_\Delta$ then the overall state satisfies the following asymptotic bound

$$\|(\xi_1, \xi_2, \xi_3, \eta_\psi, \Theta - \Theta_r, w - w_r)\|_a \leq \max\{r_1\,\|n\|_a\; \frac{r_2}{K_P}\|\Delta_e\|_a\}\,. \tag{40}$$

In summary, in case of perfect knowledge of the helicopter dynamics, it turns out that $n(\cdot) \equiv 0$ and $\Delta_e(\cdot) \equiv 0$ by which it is possible to conclude that perfect asymptotic tracking of the references $(y_\mathrm{r}, x_\mathrm{r}, z_\mathrm{r}, \psi_\mathrm{r})$ is achieved. On the other hand, in presence of uncertainties, a residual tracking error is observed which, though, can be rendered arbitrarily small by increasing the parameter K_P and by enforcing vertical reference signals characterized by small jerk.

4 SIMULATION RESULTS

We test the proposed control law on two possible desired maneuvers whose executions require aggressive attitude configurations. In the first scenario (see figure 2(a)) we design a reference trajectory simulating an aggressive maneuvers realized with constant heading. Lateral and longitudinal dynamics have to follow a circular fast reference signal whose tracking requires very aggressive roll and pitch angles (near to 60^o). The overall reference trajectory of the first maneuver is precisely set to

$$x_{\rm r}(t) = 3S_{2t}\,, \qquad y_{\rm r}(t) = 3C_{2t}\,, \qquad z_{\rm r}(t) = 0\,, \qquad \psi_{\rm r}(t) = 0 \tag{41}$$

The second reference signals have been chosen to test the control algorithm in a different scenario (see figure 2(b)). More precisely we first ask the helicopter to track a fast ascendent trajectory with constant yaw and a aggressive pitch angle carrying the helicopter rapidly to a certain altitude with zero final speed. Then, in the second part of the trajectory, we ask the helicopter to move backward at a constant altitude without changing yaw attitude. Analytically we designed the following trajectory:

$$x_{\rm r}(t) = \begin{cases} 16t - 2t^2 & t < 4 \\ 32 & 4 \leq t < 8 \\ 32 - \frac{8}{3}(t-8) & t \geq 8 \end{cases} \tag{42}$$

$$z_{\rm r}(t) = \begin{cases} -t^2 & t < 4 \\ -16 & t \geq 4 \end{cases}$$

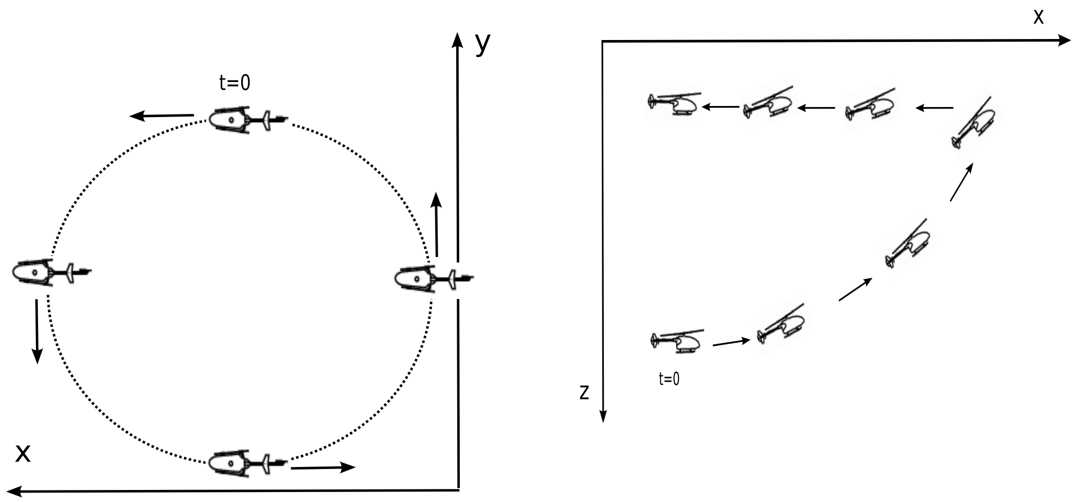
The control algorithm has been tuned according to the procedures described in the previous sections and by using, as helicopter's nominal parameters, the values reported in table (which refer to a specific model of miniature helicopter described in (Mettler et al., 2002)). More precisely the parameters in the vertical control loop have been fixed according to Proposition 1. The design parameters k_3 and k_4 of the engine control loops have been chosen according to Proposition 2 by assuming the desired engine speed to the constant value $w_{\rm er} = 167rad/s$ whereas lateral and longitudinal controller parameters have been chosen according to Proposition 3 and 4. For further details concerning the tuning of the control law reader should refer to (Marconi et al., 2006). The results shows how even in presence of strong uncertainties on helicopter's parameters the control law succeeds in enforcing the desired trajectories robustly with small errors.

$J_x = 0.18\ kg\ m^2$	$J_y = 0.34\ kg\ m^2$	$J_z = 0.28\ kg\ m^2$
$l_m = 0\ m$	$y_m = 0\ m$	$h_m = 0.24\ m$
$l_t = 0.9\ m$	$h_t = 0.1\ m$	$M = 8\ kg$
$c_M^{Q,T} = 52\ N \cdot m/rad$	$K_{T_M} = 5.8 \cdot 10^{-2}\ N \cdot s^2/rad^3$	$K_{T_T} = 1 \cdot 10^{-3}\ N \cdot s^2/rad^3$
$\bar{P}_e = 2000\ W$	$c = 1.6 \cdot 10^{-4}\ N \cdot m$	$d = 1.2 \cdot 10^{-3}\ N \cdot m/rad^2$

Table 1: Helicopter's nominal parameters

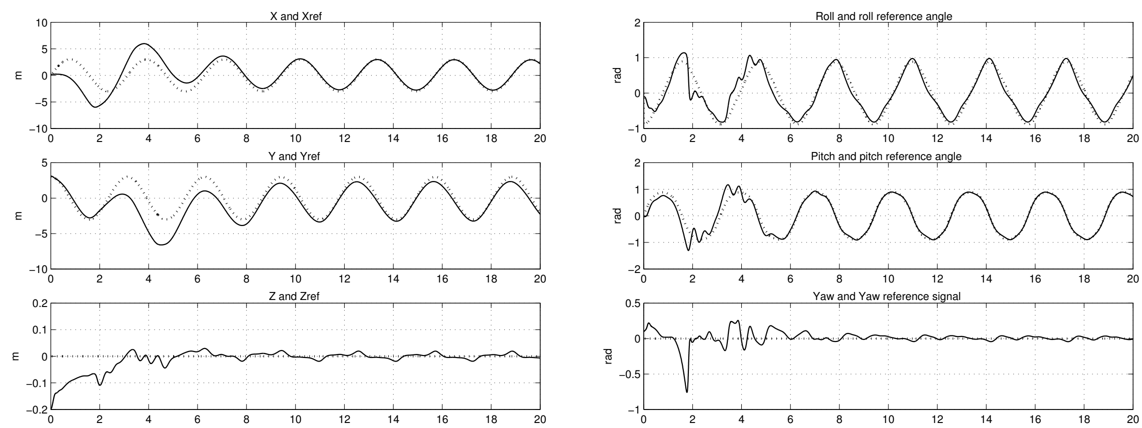
Vertical	$k_1 = 0.8$	$k_2 = 100$	
Engine	$k_3 = 4.5/(w_{\text{er}})^2$	$k_4 = 1/(w_{\text{er}})^2$	$w_{\text{er}} = 167$
Attitude	$K_P = 22$	$K_D = 0.6$	$K_\psi = 0.8$
Nested Saturations	$K_1 = 0.002$ $\lambda_1 = 160$	$K_2 = 0.4$ $\lambda_2 = 8$	$K_3 = 0.5$ $\lambda_3 = 0.4$

Table 2: Controller parameters



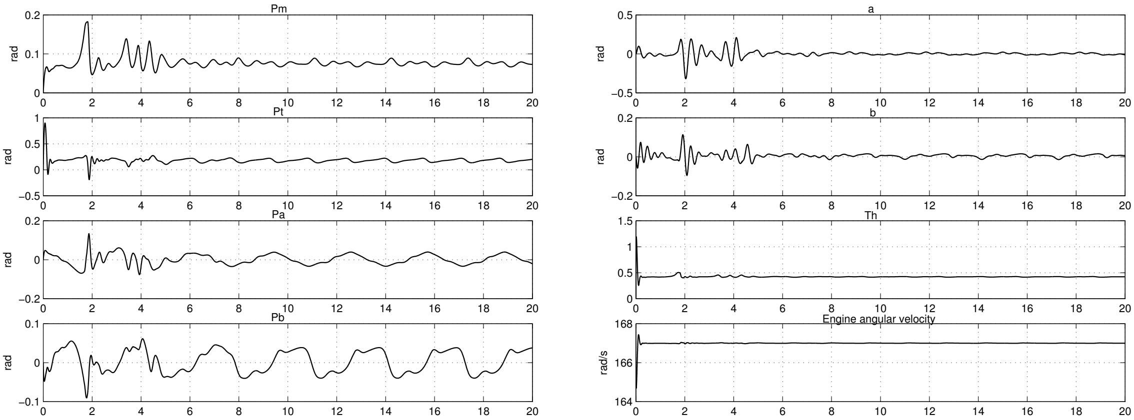
(a) (b)

Figure 2: First and second maneuvers



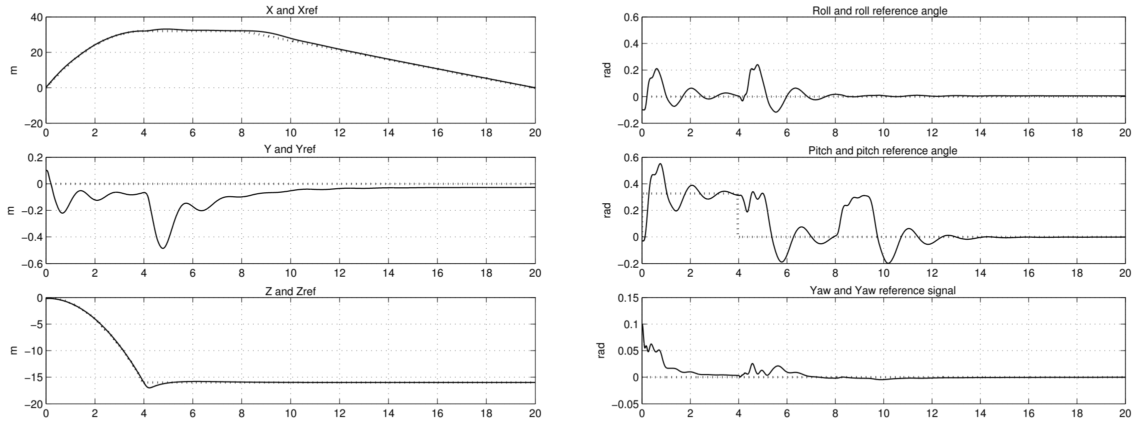
(a) (b)

Figure 3: First maneuver: (a) X-Y-Z trajectories followed compared with reference signals (dotted); (b) Roll, Pitch and Yaw trajectories followed compared with reference signals (dotted).



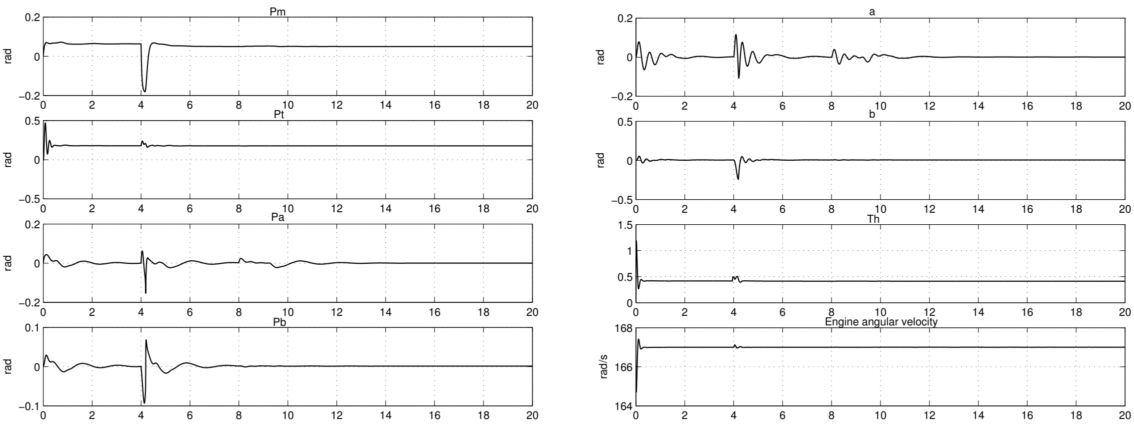
(a) (b)

Figure 4: First maneuver: (a) P_M, P_T, P_a and P_b control inputs; (b) flapping dynamics state variables a and b, engine control input T_h and engine angular velocity w_e.



(a) (b)

Figure 5: Second maneuver: (a) X-Y-Z trajectories followed compared with reference signals (dotted); (b) Roll, Pitch and Yaw trajectories followed compared with reference signals (dotted).



(a) (b)

Figure 6: Second maneuver: (a) P_M, P_T, P_a and P_b control inputs; (b) flapping dynamics state variables a and b, engine control input T_h and engine angular velocity w_e.

5 CONCLUSION

In this paper we have shown an autopilot design for small scale helicopters in order to follow aggressive lateral, longitudinal, vertical and heading attitude references in presence of uncertainties in the controlled system. The references are arbitrary signals with some limitations in the higher order time derivatives. The proposed control structure is composed by a mix of high-gain and nested saturation feedback control laws and feedforward control actions. Future works on this subject are focalized on experimental validation of the proposed design techniques on a small scale helicopter.

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