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Abstract

The validation of coupled flapping/inflow rotor models has received much recent attention. The present paper concentrates on the analysis of flight conditions close to hover in order to resolve some of the difficulties encountered in the earlier studies. New light is shed on the fundamental problems of identifiability by designing optimal experiments for the parameters of a variety of coupled flapping/inflow models. The models include the Pitt and Peters formulation of the induced flow equations and both first and second order flapping is considered. From the design of optimal experiments it is possible to determine theoretically the minimum possible variance of parameter estimates for a given set of experimental conditions. It is thus possible to determine if the available instrumentation can provide estimates of a specified quality. Using this approach careful attention is given both to the question of whether flapping measurements alone are sufficient for the reliable identification of coupled flapping/inflow models and to the suitability of test inputs currently employed. It is concluded that for the models considered, in the absence of direct measurements of inflow, and despite the relatively short time constants of the models, it is important to retain low frequency information in the system identification process. Finally, it is shown that within the limitations of the flight data available, a simple flapping model with no induced flow dynamics cannot be bettered and gives a good fit to measured data for all frequencies up to that of the rotor.

Nomenclature

n_β Blade inertia number
λ_β Normalised flapping frequency
a_0 Lift slope of blade
s Rotor solidity
M_{11} Time constant associated with λ_0 in Pitt and Peters inflow model
Ω Rotor angular velocity
τ Normalised time (Ωt)
μ_t Normal component of hub velocity
$\bar{v}$ Steady inflow due to rotor thrust
λ_0 Uniform component of inflow through rotor
θ_0 Rotor collective pitch angle
β_0 Rotor coning angle
R Measurement noise covariance matrix
θ Vector of parameters to be estimated
u, u_0, u_n u_{n+1} Test inputs
$\underline{u}$ Set of allowable test inputs
y Response of system to test input u
M, M_0, M_n M_{n+1} Fisher information matrices associated with inputs u, u_0, u_n, u_{n+1} respectively
D, D_n, D_{n+1} Dispersion matrices associated with inputs u, u_n, u_{n+1} respectively
S_{uu}, $S_{u_0 u_0}$, $S_{u_n u_n}$, $S_{u_{n+1} u_{n+1}}$ Auto-spectrum of inputs u, u_0, u_n, u_{n+1} respectively
$G(\omega)$ Transfer function matrix
$F(\omega)$ Derivative of G(u) with respect to parameters, θ
q Number of parameters
α Weighting used at each iteration of input design algorithm
ω_0 Frequency chosen at each iteration of input design algorithm
θ_j Pitch angle of rotor blade j
β_j Flap angle of rotor blade j
a_1, a_2 .. a_7 Model parameters

Introduction

Coupled flapping/inflow models are an important component of any study of the flight dynamics of rotorcraft. In the recent past a fairly simple representation was adequate for piloted and off-line simulation and, typically, models were based on a centre-sprung rigid blade with either momentum theory or a Glauert formula to provide the induced flow [1]. This type of model was adequate for the handling qualities requirements of the day and in any event the limited computing power which was available for real-time simulations lessened the urgency for establishing more ambitious models. Recently, a number of related factors have given cause for developing interest in more advanced models. An important stimulant has been the publication of more stringent handling qualities requirements [2], which together with the need for greater agility has made high bandwidth control systems a practical necessity. At the same time, fully aeroelastic blade models are being prepared for real-time simulation [3,4], which in turn require an aerodynamic model of dynamic inflow of equivalent fidelity. With the processing power now available to make the inclusion of advanced rotor models in piloted simulation a realistic proposition, the simulationist, or flight dynamicist, expects to call up models which have been validated over a wide range of flight test conditions. The validation experiments should have shown a consistent identification of the model structure and have produced parameter estimates which are credibly close to theoretical values.

An exercise in the validation of rotor models using flight data has been the subject of a workshop study at the Royal Aerospace Establishment (RAE), Bedford [5]. Different groups applied a variety of techniques to the flight data resulting from a longitudinal cyclic control input to the RAE Aerospatiale Puma trimmed at 100 knots. Measurements of the flap and pitch for each of the four blades, together with the fuselage kinematics were used to validate simple flap and inflow models. Subsequent work carried out at Glasgow University in defining a strategy for the validation programme and describing enhancements to the basic model has been presented at European Rotorcraft Forum [6,7] and published elsewhere [8]. This work concentrated on relatively fast forward flight and, more recently, endeavours have been made to extend the validation to medium and low speeds and hovering flight, using data from the same flight test programme at RAE Bedford. Data sets for a variety of inputs, mainly doublets and frequency sweeps have been available for the cyclic and collective controls.

The investigation of this wider range of flight conditions proved more difficult than anticipated. The use of parameter estimation software requires some expertise and an appreciation of the physical system to be most effective, but the difficulties with non-convergence and inconsistent values exceeded those normally experienced. As the work progressed, accumulated evidence suggested that there were underlying factors at work which needed a rational explanation. In addition, part of the original strategy had been to operate in the frequency domain in order to target validation on that frequency range most relevant to rotor dynamics. The aim was to exclude the fuselage response from the validation in order to extract the rotor parameters with more confidence but a puzzling feature of even the earliest results [7] was that low frequencies always seemed to be needed for a successful identification.

Against this background of uncertainty, it was decided to concentrate on a simple situation in order to identify the features of the model, control input or identification method which were the source of the difficulty. The hover condition, with collective input, is a dynamical situation which is simple enough to be analysed in some detail. Houston and Tarttelin, for example, consider the validation a coning/inflow/body representation [9]. The course followed at Glasgow is described in the sections below. It depends on examining the criterion for optimal estimates by minimising variation in estimated parameter values. As a consequence of this approach it has been possible to predict the difficulties associated with the hovering situation and to begin to understand the problems experienced in the more general case. The principles behind these explanations are ones which have general applicability and should be given consideration in any validation exercise.

2. Identifiability and Experiment Design for Coupled Flapping/Inflow Models

In any identification, the parameter estimates obtained are random variables with a given mean and standard deviation. The smaller the standard deviation, the more accurate the parameter estimates will be, on average, assuming that they are unbiased (the expected value of the estimates being equal to the 'true' parameter value).

For the case of an efficient estimator the Cramer-Rao bound [10] relates the variance of estimates to elements of the dispersion matrix throught the equation

$$\mathrm{cov}(\theta) = D = M^{-1} \qquad (1)$$

where θ is the vector of parameter estimates

D is the dispersion matrix
M is the information matrix

The dispersion matrix therefore provides a basis for experiment design and by designing optimal identification experiments, it is possible to determine the minimum possible standard deviations of the parameter estimates for a given set of experimental conditions.

This provides useful information about the identifiability of a model since, if the best possible information matrix has been found, any indication of ill-conditioning or singularity suggests that unique parameter estimates are unobtainable using the available measurements.

In the case of coupled flapping/inflow models one of the most important difficulties encountered has been in the estimation of inflow dynamics using flapping measurements alone. In order to get a better understanding of this problem ten different models have been investigated in terms of identifiability. All these models are based on standard flapping equations [1] coupled with appropriate inflow descriptions [11]. Details of all the models considered may be found in Appendix A.

2.1 Optimal Experiment Design

The time constants associated with flapping and inflow dynamics are small (typically less than 1 second) in comparison with the period of typical measured response data sets available (typically 120 seconds). Since these time histories are relatively long, frequency domain methods of input design, with their acknowledged simplicity, may be used with confidence.

The experiment design problem is well documented (e.g. [10], [12], [13], [14]). For the case of output-error identification methods and the design of inputs which are energy constrained the problem can be stated in the following form:-

$$\underset{u}{\text{minimise}} \; |D|$$

where $D = M^{-1}$

and $$M = \int_{-\infty}^{\infty} \frac{dy(t)}{d\theta}^{T} R^{-1} \frac{dy(t)}{d\theta}\, dt \qquad (2)$$

By Parseval's theorem

$$M = \int_{-\infty}^{\infty} \frac{dY(\omega)}{d\theta}^{T} R^{-1} \frac{dY(\omega)}{d\theta}\, d\omega \qquad (3)$$

$$= \int_{-\infty}^{\infty} F^{*}(\omega)\, R^{-1}\, F(\omega)\, S_{uu}(\omega) d\omega \qquad (4)$$

where R is the noise covariance matrix, $S_{uu}(\omega)$ is the autospectrum of the input and the quantity $F(\omega)$ is a matrix of sensitivity coefficients

$$F(\omega) = \frac{dG(\omega)}{d\theta} \qquad (5)$$

where $G(\omega)$ is the system transfer function matrix.

This shows that the only information required to calculate $|D|$, for the case of an infinitely long record, is the autospectrum of the input ($S_{uu}(\omega)$) and the form of $F(\omega)$. It should be noted that for practical systems $|F(\omega)|$ becomes negligible above some frequency ω_c and the limits of integration therefore become finite.

In order to obtain an algorithm for finding the autospectrum of the input which minimises $|D|$ consider an input u_{n+1} formed from a combination of inputs u_0 and u_n according to the expression

$$S_{u_{n+1}u_{n+1}}(\omega) = \alpha\, S_{u_0u_0}(\omega) + (1-\alpha) S_{u_nu_n}(\omega) \qquad (6)$$

From equation (2) it can be shown that the information matrix M_{n+1} of the input u_{n+1} can be related to the information matrix M_0 of the input u_0 and the information matrix M_n of input u_n by the equation

$$M_{n+1} = \alpha M_0 + (1-\alpha) M_n \qquad (7)$$

The corresponding dispersion matrix is given by

$$D_{n+1} = M_{n+1}^{-1} \qquad (8)$$

It is known [13] that for any square matrix Δ the relation

$$\frac{d \log|\Delta|}{dx} = \mathrm{Tr}\left[\Delta^{-1} \frac{d\Delta}{dx}\right] \qquad (9)$$

is true. Hence, from equation (7), we have in this case

$$\frac{d \log|D_{n+1}|}{d\alpha} = -\mathrm{Tr}(M_n^{-1}M_0 - I) = -\mathrm{Tr}(M_n^{-1}M_0) - q \qquad (10)$$

where q is the number of parameters considered. Also, for a sufficiently small value of α

$$\log|D_{n+1}| = \log|D_n| - \alpha\,(T_r(M_n^{-1}M_0) - q) \qquad (11)$$

If the term $\alpha\,(T_r(M_n^{-1}M_0) - q)$ is positive it follows that

$$|D_{n+1}| < |D_n|$$

and u_{n+1} is clearly a better test input than u_n. This result provides a basis for an optimisation algorithm that successively improves upon a test input until an optimum is reached.

For an input consisting of a pure sine wave of frequency ω_0, a similar approach may be employed, but one must use

$$M = R_e\{F^{*}(\omega_0)\, R^{-1}\, F(\omega_0)\} \text{ or } I_m\{F^{*}(\omega_0)\, R^{-1}\, F(\omega_0)\}$$

in place of equation (4). The use of a discrete set of such inputs produces a significant simplification of the algorithm [15].

2.2 Optimal Experiments for a Second Order Flapping/First Order Inflow Model

In order to investigate the importance of inflow data for identifying the parameters of each model, optimal experiments were designed in each case both with and without inflow measurements. The approach is best illustrated using the second order flapping/first order inflow model.

$$\begin{bmatrix} \dfrac{d^2\beta_0}{d\tau^2} \\ \dfrac{d\beta_0}{d\tau} \\ \dfrac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ a_4 & 0 & a_5 \end{bmatrix} \begin{bmatrix} \dfrac{d\beta_0}{d\tau} \\ \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} a_6 \\ 0 \\ a_7 \end{bmatrix} \theta_0 \qquad (12)$$

where details of the model structure and parameters may be found in Appendix A. Using the theoretical parameter values from Appendix A (with the corrected value for M_{11}) we have

$a_1 = -1.171$ $\quad a_5 = -0.648$
$a_2 = -1.06$ $\quad a_6 = 1.171$
$a_3 = -1.561$ $\quad a_7 = 0.1666$
$a_4 = -0.1666$

Case 1 - inflow and coning-rate measurements available, and all model parameters estimated.

The coning, coning-rate, and inflow measurements were assumed to have noise with unity covariance, for convenience, giving a matrix R^{-1} having unit elements in the leading diagonal and zero elements elsewhere. Application of the optimal input design algorithm gave an optimal value of $|D|$ of 3.24×10^6. Standard deviations for parameter estimates for an experiment involving this optimal input are shown in Table 1a. The results showed that the estimate of parameter a_3 will, on average, be much less accurate than those of other parameters. The optimal test input for this case has its energy distributed as follows:

frequency	0.15	0.50	0.60	0.62	0.97
Percentage energy	20.2	31.2	3.5	10.2	34.9

Case 2 - inflow measurements available, but not coning-rate measurements: all parameters estimated.

In order to remove the coning-rate measurement, its noise covariance was set to 10^{12} i.e. effectively to infinity. This gives R^{-1} as

$$R^{-1} = \begin{bmatrix} 10^{-12} & 0 & 0 \\ 0 & 1 & 0 \\ 0 &) & 1 \end{bmatrix} \qquad (13)$$

The optimal $|D|$ for this case is 1.879×10^6, giving the following parameter standard deviations shown in Table 1b. These standard deviations are larger than in Case 1, as expected, since less information is available as there is no coning-rate measurement. The optimal test input for this case has an energy distribution which is the same as that for Case 1 since the information provided by the coning-rate measurements is also present in the coning measurements.

Case 3 - no inflow measurements, but coning-rate measurements available: all parameters estimated.

In order to remove the inflow measurement for the experiment design, its noise covariance was set to 10^{12} — gives R^{-1} as

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-12} \end{bmatrix} \qquad (14)$$

The optimal $|D|$ in this case was 5.464×10^{18} i.e. the optimal dispersion matrix is effectively infinite, and so unique parameter estimates cannot be obtained.

Case 4 - no inflow or coning rate measurements: all parameters estimated.

In order to remove the coning-rate and inflow measurements, their noise covariances were set to 10^{12}, giving

$$R^{-1} = \begin{bmatrix} 10^{-12} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-12} \end{bmatrix} \qquad (15)$$

The optimal $|D|$ in this case was found to be 4.09×10^{29} i.e. the dispersion matrix was effectively infinite, and so unique parameter estimates cannot be obtained.

These results for the four cases presented above can be assessed further using the transfer function matrix G(s) for this model. It can be shown that in terms of the state space description of equation (12) the transfer function matrix is

$$G(s) = \frac{1}{s^3 + s^2(-a_5 - a_1) + s(a_1 a_5 - a_2 + a_3 a_4) + a_2 a_5} \qquad (16)$$

$$\times \begin{bmatrix} a_6 s^2 - a_6 \left[a_5 - \dfrac{a_3 a_7}{a_6} \right] s \\ a_6 s - a_6 \left[a_5 - \dfrac{a_3 a_7}{a_6} \right] \\ a_7 s^2 + (a_4 a_6 - a_1 a_7) s - a_2 a_7 \end{bmatrix}$$

The system responses therefore give information about the following quantities:

A. $(-a_5 - a_1)$

B. $(a_1 a_5 - a_2 + a_3 a_4)$

C. $a_2 a_5$

D. a_6

E. $a_6 \left[a_5 - \dfrac{a_3 a_7}{a_6} \right]$

F. a_7

G. $a_4 a_6 - a_1 a_7$

H. $a_2 a_7$

Values for a_6 and a_7 can be found from (D) and (F): a_2 can then be found from (H) and a_5 from (C). From (A) a_1 can then be found, and finally a_3 and a_4 from (E) and (G), or (B) and (E), or (B) and (G). The model is therefore identifiable when both coning and inflow measurements are available, as was found in Case 1 above.

If coning-rate measurements are not available, then this makes no difference to the identifiability of the model, since the pole/zero information provided by the coning-rate data are also provided by the coning data. As noted in Case 2, however, the standard deviations of the parameter estimates will be larger when there are no coning-rate measurements.

If inflow measurements are not available, then values for quantities (F), (G) and (H) above will not be available. A value for a_6 can be found from (D), but we then have four remaining equations (A), (B), (C) and (E), and six unknowns, so cannot solve these uniquely i.e. the model is unidentifiable when inflow measurements are not available and all of the parameters have to be estimated. This can be overcome if some a priori information about the parameters is known. In particular, if any one of the three parameters a_1, a_2 or a_3 is known and so does not need to be estimated, then it is possible to obtain estimates of the remaining two from (A) and (C). Further, if one of a_5, a_7 or a_4 is then known, the others can be found from (B) and (E). The model is then identifiable. Three special cases also exist: if one of the pairs a_5, a_4, or a_5, a_7, or a_4, a_7 is known, then the model is also identifiable. With a_5 and a_4 known, (A), (B), and (C) have three unknowns (i.e. a_1, a_2 and a_3) and so can be solved. Expression (E) can then be used to obtain an estimate of a_7, since a_6 can be found from (D). With a_5, a_7 known, (E) can be used to obtain a_3 since a_6 can be found from (D), and we can then use (A) and (C) to obtain a_1 and a_2. A value for a_4 can then be found from (B). Finally, with a_4, a_7 known, there are four equations (A), (B), (C) and (E) with four unknowns which can be solved to obtain a_1, a_2, a_3 and a_5, with a_6 found from (D).

In order to verify these identifiability predictions, optimal experiments were designed for all the possible combinations of one and two model parameters known beforehand and so not estimated. The results were found to be in complete agreement with the predictions. On the basis of this work, it therefore appears that it is not possible to identify all of the unknown parameters in this model simultaneously, without inflow measurements being available. This is an important result, and goes a long way towards explaining the difficulties encountered by researchers using system identification techniques to investigate flapping/inflow models.

Lacking inflow measurements, identification results can still be obtained if suitable parameters can be fixed at known values. Moreover, if the justification given above for these identifiability problems is valid, knowledge of relationships between the model parameters may be used as an alternative to fixing parameters, or in combination with it. For example, the following relationships are present between the model parameters:

$$a_4 = -a_1 \;;\quad a_5 = \frac{4}{3}a_1 \;;\quad a_7 = -a_4$$

If the relationship $a_4 = -a_1$ is included in the identification, then a_1 can be found from quantity (D) given above. Values for a_2 and a_3 can then be found from (A) and (C). If $a_5 = 4/3\ a_1$ is then used, a_4 can be found from (B) and a_7 from (E). Hence, by assuming the two relationships given, the model should become identifiable. Any suitable combination of relationships and/or fixing of parameter values could alternatively be used. Therefore, despite a lack of inflow data, results can therefore still be obtained if certain assumptions concerning the model can be made. While limited by the assumptions used, such results may well still provide valuable information.

2.3 Identifiability Results for First-Order Flapping/First Order Inflow Model

In this case, neglecting μ_z and labelling the parameters as a_1, a_2 .. a_5 leads to:

$$\begin{bmatrix} \dfrac{d\beta_0}{d\tau} \\ \dfrac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} a_5 \\ 0 \end{bmatrix} \theta_0 \qquad (17)$$

Using the theoretical parameter values given in Appendix A (with the corrected value for M_{11}) we have,

$a_1 = -0.9052$; $a_2 = -1.333$; $a_3 = 0.15086$

$a_4 = -0.42477$, $a_5 = 1.000$

The values for the optimal {D} which result from an application of the optimal experimental design approach outlined above, indicated that difficulties will be encountered if an attempt is made to estimate all of the parameters of the model. On the other hand, if only parameters a_1, a_3 and a_4 are estimated successful identification may be possible, even in the absence of inflow measurements.

The transfer function matrix for this model is

$$G(s) = \frac{1}{s^2 - (a_1+a_4)s + (a_1 a_4 - a_2 a_3)} \begin{bmatrix} a_5(s-a_4) \\ a_5\, a_3 \end{bmatrix} \qquad (18)$$

It is noted that the poles and zeros of a system can be directly related to the characteristics of its response, and that this can be extended to the coefficients of the numerator and denominator transfer function polynomials, since these are in turn directly related to the poles and zeros. Hence, for the above transfer function, information is available from the response as to the values of the following quantities:

A. $(a_1 + a_4)$

B. $(a_1\, a_4 - a_2\, a_3)$

C. a_5

D. $a_5\, a_4$

E. $a_5\, a_3$

Given the value of a_5 from (C), a_3 and a_4 can be found from (D) and (E). From (A), a_1 can then be found, and finally from (B), a_2 can be obtained. Hence, with both coning and inflow information, the model is identifiable.

If inflow information is not available, then a value for quantity (E) above is not available. In this case, a_5 can be found from (C), a_4 from (D), a_1 from (A); leaving a_2 and a_3 to be found from (B). There is only one expression (B), but two unknowns, and so the model is unidentifiable in this situation.

If values for parameters a_2 and a_3 are known beforehand then the arguments above suggest that the model will always be identifiable.

2.4 The Frequency Content of Test Inputs

At present, the test inputs used in flight trials are largely general-purpose rather than specially designed for the current work on rotor models. By comparing these general test inputs with the optimal inputs for identifying rotor models, some indication of their suitability can be obtained.

In Section 2.3 consideration is given to the identification of Model II of Appendix A when parameters a_1, a_3 and a_4 are being estimated. When inflow data are available, the optimal input has 90% of its energy at frequency 0.38 units, and 10% at frequency 0.41 units (corresponding to 1.67 Hz and 1.80 Hz respectively for the Puma). These are relatively low frequencies, when compared with the rotor frequency of 1.0 units (4.39 Hz for the Puma). Table 2 gives the optimal inputs for Model V of Appendix A when various sets of the model parameters are to be estimated and inflow measurements are available. It can be seen that these largely concentrate on exciting three sets of frequencies: around 0.2 units, 0.5 units and 0.9 units (corresponding to 0.87 Hz, 2.19 Hz, and 3.95 Hz respectively for the Puma). The optimal inputs for Model V excite much higher frequencies than those for Model II because Model V is a more accurate representation of the rotor, in theory, and includes high frequency dynamics, whereas Model II is a simpler representation that includes only lower frequency dynamics.

A typical manually applied frequency sweep input has little energy above 1 Hz, and so is perhaps of doubtful use for rotor identification work. The bandwidth of such manually applied inputs is severely limited by simple physical constraints, in particular, how fast the pilot can move the controls. In order to overcome this, some form of automatic control input device is necessary, and it is suggested that until such a device is available only limited rotor identification work will be possible. Unfortunately, even with a control input device, the dynamics of the rotor actuators may restrict the frequency content of any inputs applied. For example, in the Puma helicopter, the actuators can be modelled as first-order lags with a nominal time constant of around 50 ms. This corresponds to a cut-off frequency of around 3.2 Hz, and so lies within the frequency range for rotor identification work. Nevertheless, even being able to excite the rotor at this sort of frequency would be a considerable improvement on the present situation.

Turning now to the case when inflow measurements are not available, in Section 2.3 the optimal input for Model II has 33% of its energy at d.c. and 67% at frequency 0.62 units (corresponding to 2.72 Hz) when estimating parameters a_1, a_3 and a_4. The optimal inputs for Model V without inflow, corresponding to those given in Table 2, were all approximately the same, and had 19.2% of their energy at zero frequency, 31.2% at frequency 0.50 units, 16.2% at frequency 0.66 units, and 33.4% at frequency 1.06 units (0 Hz, 2.19 Hz, 2.89 Hz 4.65 Hz respectively for the Puma).

Clearly, when inflow data is not available the optimal inputs have rather different characteristics to when it is available. For both Models II and V, the inputs contain a zero frequency component that was not present previously, and excite higher frequencies. In order to investigate these differences in more detail, for Model V the optimal inputs were re-designed subject to restrictions on their frequency content. The result showed that when low frequencies are excluded, sufficient information can still be obtained about the model for identification to be successful, albeit with much less accurate parameter estimates than if the full frequency range was available. It was also found that low frequency information alone is insufficient to identify the model parameters.

These results suggest that high frequency information is more important than low frequency information, as expected since we are dealing with the rotor. However, the presence of a zero frequency component in the optimal inputs when inflow data is not available is surprising, particularly since it is not present when inflow measurements are available. No conclusive explanation as to the reason for this zero frequency component can yet be offered, but clearly it is related to the availability of inflow data in some manner. It is possible that, in an identification, very low frequency information aids in separating inflow from flapping effects when only coning measurements are available. This unexpected importance of low frequency information is discussed further in Section 3.

It was mentioned above that the rotor actuators may typically impose an upper limit of around 3.2 Hz (0.72 units for the Puma, in normalised frequency terms) on the frequencies that an input can excite. This is significantly less than the 4.65 Hz upper frequency present in the optimal inputs for Model V without inflow data, although it is only slightly less than the 3.95 Hz upper frequency when inflow data is available, and so could significantly restrict the effectiveness of any identification when inflow measurements are unavailable.

This line of argument is in agreement with the results obtained, and is intuitively reasonable. Moreover, in Section 3.4 below it is found to hold also for the result obtained using Model V. It therefore appears to be quite a useful tool for gaining greater insight into the source of the identifiability problems encountered in this work.

3. Identification Results

3.1 Introduction

Flight data used in the current work were obtained from tests on a Puma helicopter and were provided by the Royal Aerospace Establishment (Bedford). Blade pitch and flap data were available but no form of inflow measurement was provided. The test signal used was a frequency sweep applied by the pilot to the collective input. Test conditions involved hovering flight, out of ground effect.

Parameter identification methods used have involved a frequency-domain output-error approach forming part of an identification package developed for rotorcraft applications [16]. The frequency range used for identification was selected initially from examination of the coherence between the control input and the coning response. It was found that the coherence between β_0 and θ_0 was high (above 0.8) from 0 Hz to about 3.2 Hz and dropped sharply at higher frequencies. The maximum frequency used was therefore 3.2 Hz. The lowest frequency included was 0.011 Hz, the zero frequency component being excluded deliberately so that any bias in measurments of β_0, θ_0 and μ_z could be ignored initially.

The coherence between β_0 and μ_z was found to be very small except at very low frequencies. This suggested that velocity μ_z is relatively unimportant over the frequency range being considered.

Finally, attention is drawn to the use of a delay, τ, to represent the bias in the azimuth measurement. The multiblade values β_0 and θ_0 are calculated as follows for the Puma:

$$\beta_0(i) = \frac{1}{4}\sum_{j=1}^{4} \beta_j(i) \quad ; \quad \theta_0(i) = \frac{1}{4}\sum_{j=1}^{4} \theta_j(i) \tag{19}$$

where

β_j is the flapping measurement for blade j.
θ_j is the pitch measurement for blade j.
i refers to the ith data point.

Clearly, the azimuth measurement is not required in these equations. However, azimuth is essentially a measure of the time at which the measurements were taken and is therefore needed in order to synchronise β_0 and θ_0 with the rigid-body measurements. Any bias in the azimuth will produce a time shift between the rotor measurements and the rigid-body measurements, which can be compensated for by estimating a delay on β_0 and θ_0 as part of the identification. It is important to note that use of a simple delay is only possible for β_0 and θ_0. The multiblade transformations for cyclic measurements involve the azimuth measurement, and so any bias on the azimuth will have a more complex effect than with β_0 and θ_0.

3.2 Identification of First-Order Flapping Models with constant or infinitely-fast inflow

Models I, VII, and VIII in Appendix A have the following general structure when the zero frequency component is excluded from the identification:

$$\frac{d\beta_0}{d\tau} = a_1\beta_0 + a_2\theta_0 + a_3\mu_z \tag{20}$$

The parameters a_1, a_2 and a_3 are to be estimated. The theoretical values and the estimates from identification are shown in Table 3a.

A rank 3 solution was used since this was indicated by examination of the eigenvalues of the information matrix and was found to give the best fit. This was one less than full-rank, since a delay τ was also estimated.

The identification results appear to favour the use of Model VIII, especially for the value of parameter a_1, i.e., infinitely fast inflow dynamics, with the coning inflow effect included.

For comparison, the identification was repeated neglecting μ_Z (i.e. fixing parameter a_3 at zero). A rank 2 solution was used (i.e. full-rank, as indicated by the eigenvalues of the information matrix) and it was not necessary to estimate the delay, τ, since both β_0 and θ_0 are subject to the same azimuth bias. It was found that removing μ_Z from the identification had a negligible effect and the estimates for a_1 and a_2 were virtually the same as those found with μ_Z included.

3.3 Identification of Second-Order Flapping Models with constant or infinitely-fast inflow

Models IV, IX and X have the following general structure when the zero frequency component is excluded from the identification:

$$\frac{d^2\beta_0}{d\tau^2} + a_1\frac{d\beta_0}{d\tau} = a_2\beta_0 + a_3\theta_0 + a_4\mu_Z \qquad (21)$$

The parameters a_1, a_2, a_3 and a_4 are to be estimated, and have values shown in Table 3b.

A rank 3 solution was used. From the estimates of parameters a_3 and a_4, it appears that Models IX and X are preferred to Model IV, and from the estimate of a_1 it appears that Model X is a better match. However, given the large standard deviation associated with the estimate of a_4, little confidence can be attached to this preference of Model X over Model IX. Also, none of the theoretical values for a_2 matches the identified value. On the whole, however, as with the first-order flapping model, the models incorporating infinitely-fast inflow dynamics appear to be preferred to that with constant inflow. This is a particularly interesting result since many of the existing Level 1 flight mechanics models assume constant inflow. Further identification results for the case where μ_Z is neglected showed that μ_Z is again relatively unimportant.

3.4 Identification of First-Order Flapping Models with First-Order Inflow

Models II and III in Appendix A have the following general structure:

$$\begin{bmatrix} 1 & 0 \\ a_5 & a_6 \end{bmatrix} \begin{bmatrix} \frac{d\beta_0}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 0 & a_7 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} a_3 & a_4 \\ a_8 & a_9 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \mu_Z \end{bmatrix} \qquad (22)$$

where a_1, a_2 ... a_9 are the parameters to be estimated. The theoretical and estimated values of these parameters are shown in Table 3c.

A rank 3 solution was used, since the use of higher ranks was found to lead to convergence difficulties in the identification algorithm, and so to much poorer fits. It can be seen that these results are in good agreement with the theoretical values given above, and based on the value obtained for a_7, Model II appears to be favoured. From the values of a_6, the corrected value of M_{11} also appears to be preferred, and in fact a_6 is in excellent agreement with the theoretical corrected M_{11} value. However, an extremely low rank of solution was necessary, and this can be attributed partly to identifiability problems, and partly to the poor frequency content of the input, which has little power above 1 Hz. If the effect of these factors is as stated, then the full-rank identification problem will not produce unique parameter estimates, and so is unidentifiable. Unfortunately, from the data it is not possible to verify the results found in Section 2.2, since it is not possible to disentangle the fundamental identifiability problems caused by having too many parameters and too few measurements, from the identifiability problems arising from the poor input used. Only if an improved input was applied which excited the higher frequencies much more thoroughly could any useful conclusions be drawn.

3.5 Identification of Second-Order Flapping Models with First-Order Inflow

Models V and VI have the following general structure:

$$\begin{bmatrix} \frac{d^2\beta_0}{d\tau^2} \\ \frac{d\beta_0}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ a_4 & 0 & a_5 \end{bmatrix} \begin{bmatrix} \frac{d\beta_0}{d\tau} \\ \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} a_6 & a_7 \\ 0 & 0 \\ a_8 & a_9 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \mu_Z \end{bmatrix} \qquad (23)$$

The theoretical and estimated values of the parameters are shown in Table 3d.

Once again, a rank 3 solution was found best, and azimuth bias was estimated using a delay, τ.

It can be seen that a_4 is underestimated, and the values of a_7, a_8 and a_9 suggest the use of the corrected M_{11} value. However, it was found that a_4 and a_5 did not change from their initial values, suggesting that these parameters were relatively unimportant in terms of the fit obtained in the identification. This is unexpected, if the results given in Section 2.2 are correct, since these suggest that parameter a_4 is an important parameter which can be estimated independently of the other parameters. This again suggests that the test input is inadequate since it does not produce responses which are sensitive to the model parameters. Hence the low rank solution used is likely to have been needed because of the identifiability problems associated with the model combined with the identifiability problems caused by the poor input.

3.6 The Effect of the Frequency-Range used on Identification Results

Based on the coherence between β_0 and θ_0 the frequency range used in the identifications described above was 0.011-3.2 Hz. In addition identifications were carried out keeping the lower frequency at 0.011 Hz and increasing the upper frequency. The model structure given in Section 3.2 for Models I, VII and VIII was used, since this was easier to identify, gave good fits at frequencies up to 3.2 Hz, and would highlight the presence of any interesting dynamics at high frequencies since it is a simple model and does not contain high frequency effects. It was found that the fits did not deteriorate suddenly at higher frequencies, as would be expected if there were unmodelled dynamics, and the parameter estimates remained relatively constant, until a frequency of 4.39 Hz is reached. At this frequency, rotor noise swamps the response, and so distorts the identification results.

These results appear to suggest that a rotor model assuming constant or instantaneous inflow dynamics is valid out to the rotor frequency. This is an unexpected result, since theoretical models such as Models V and VI predict that significant flapping and inflow dynamics are present at these high frequencies.

The most plausible explanation of these results is that the test input used does not excite high frequencies sufficiently, as has been suggested by the findings throughout this report. The high frequencies would then consist largely of noise, which could be fitted equally well by any of the models studied.

Turning now to lower frequencies, the frequency range used in the identifications described so far has started at frequency 0.011 Hz, corresponding to the first data point when 0 Hz is excluded. Using the same simple model as above, the upper frequency was held at 3.2 Hz and the lower frequency increased. It was found that the parameter estimates remained effectively the same, but the correlation coefficient falls quite rapidly, indicating a reduction in the quality of the fit being obtained. This is also shown by the average relative error between the measured data and the model response. The error rises as lower frequencies are excluded from the identification, and this is in agreement with the results obtained when the upper frequency was varied. That is, at high frequencies, there is little excitation by the test input and so the response consists largely of noise. Hence, when low frequencies are removed from the identification, the fit will deteriorate.

A second factor may also be affecting these low frequency results. In Section 2.3 it is noted that without inflow measurements, the optimal inputs excite zero frequency, whereas they do not when inflow data is available. Low frequency information may therefore be important for separating the effects of inflow from flapping in an identification. However, since the model being identified in this work does not contain inflow dynamics, it is unlikely that this second factor is of importance. It should be borne in mind, however, if more complex models, incorporating inflow dynamics, are used.

As a final check on these results obtained for the simple model with no inflow dynamics the model structure given in Section 3.5 for Models V and VI was also identified keeping the lower frequency fixed and varying the upper frequency of the frequency range used. It was found difficult to obtain proper convergence of the identification algorithm. However, as higher frequencies were included it was noted that the identification appeared to become more sensitive to the model parameters. This is obviously to be expected, given that the model dynamics are mainly concentrated at higher frequencies, and in the light of the results obtained in the optimal experiment investigations described in Section 2. Nevertheless, it lends further support to the contention that adequate identification results will not be obtained unless the test input used excites much higher frequencies than at present.

4. Conclusions

There are several types of conclusions which can be drawn from the work described above. As regards the selection of the most suitable rotor model, the identification results indicate that the more complex models give no better predictions than the simple first order flapping with no inflow dynamics. Although the results may be interpreted to suggest how effective a simple model can be under suitable conditions, a more likely explanation is that they are due to the convergence difficulties which were encountered with the more complex models as a result of identifiability problems. The latter explanation again highlights the inadequacy of the data for dynamic inflow identification.

Turning next to the general validation problem for hover, it has been shown by a consideration of the optimal control input that inflow measurements are extremely important for determining the parameters of such models. However, in the absence of such data, identification results can still be obtained if a suitable knowledge of the model structure is assumed. Surprisingly, in the latter case, the optimal input has a significant low frequency component which suggests that both low and high frequency information is important for the identification when inflow data is unavailable. These findings are supported by the results obtained using flight data, within the limitations of that data. If these conclusions can be extended to forward flight it is clear now why it has been necessary to include low frequencies in the identification. In the absence of direct measurements of inflow, the low frequency information is essential.

It has also been shown for hover that certain model structures can reduce the determinant of the information matrix to zero and one can predict that identification is impossible. The results obtained from applying system identification procedures to flight data for such cases supports these predictions. It is to be expected that the general principles of the findings for hover extend to other cases, and the observed failure of the identification procedure in certain cases for forward flight could occur for similar reasons.

Finally, there is every reason to expect that the findings described above should be given consideration in any system identification exercise. The finding that the measurement base of a validation exercise may be as significant as the assumed model structure when determining the type of test input to use, needs particularly to be emphasised.

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Table 1a: Standard deviations for parameter estimates for experiment using optimal input for a second order flapping/first order inflow model with inflow and coning-rate measurements available.

Parameter	Standard Deviation
a_1	5.302
a_2	2.107
a_3	18.447
a_4	2.671
a_5	7.966
a_6	4.203
a_7	1.815

Table 1b: Standard deviations for parameter estimates for experiment using optimal input for a second order flapping/first order inflow model with inflow measurements available but no coning-rate measurements.

Parameter	Standard Deviation
a_1	8.001
a_2	2.689
a_3	23.860
a_4	3.003
a_5	9.216
a_6	6.289
a_7	1.986

Table 2: Components of optimal inputs for Model V for some typical combinations of known parameters.

Parameters known	a_1 a_3		a_1 a_4		a_1 a_7		a_2 a_3	
Optimal Input	frequency	% energy	frequency	% energy	frequency	% energy	frequency	% energy
	0.28	23.4	0.09	16.0	0.13	21.3	0.26	32.4
	0.50	31.4	0.50	41.5	0.50	46.7	0.50	32.0
	0.83	10.3	0.80	30.5	0.88	32.0	0.85	35.6
	0.88	34.9	0.88	12.0				

Parameters known	a_1 a_3		a_1 a_4		a_1 a_7		a_2 a_3	
Optimal Input	frequency	% energy	frequency	% energy	frequency	% energy	frequency	% energy
	0.15	32.4	0.50	47.3	0.50	40.2	0.50	47.3
	0.50	32.0	0.91	52.7	0.96	59.8	0.9	52.7
	0.89	35.6						

Table 3a: Theoretical values and estimates of parameters obtained from identification for case of first-order flapping models with constant or infinitely-fast inflow.

Parameter	Model I	Model VII	Model VIII	Estimate
a_1	-0.9052	-1.3776	-2.3567	-2.677 (0.0110)
a_2	1.0	1.0	1.7107	1.224 (0.0234)
a_3	1.333	0.6231	1.0660	0.920 (0.0789)
τ	-	-	-	-2.522 (0.154)

Table 3b: Theoretical values and estimates of parameters for the case of second-order flapping models with infinitely-fast inflow.

Parameter	Model IV	Model IX	Model X	Estimate
a_1	1.171	0.769	0.449	0.433 (0.228)
a_2	-1.06	-1.06	-1.06	-1.423 (0.0598)
a_3	1.171	0.769	0.769	0.817 (0.0361)
a_4	1.561	0.479	0.479	0.249 (0.0479)
τ	-	-	-	-0.003 (0.366 x 10^{-4})

Table 3c: Theoretical values and estimates of parameters for the case of first-order flapping models with first-order inflow.

Parameter	Model II	Model III	Estimate
a_1	-0.9052	-0.9052	-1.024 (0.0268)
a_2	-1.333	-1.333	-1.367 (0.0021)
a_3	1.0	1.0	0.997 (0.0302)
a_4	1.3	1.333	1.299 (0.0074)
a_5	0.16666	0.2993	0.197 (0.0305)
a_6 (M_{11} = 1.0)	1.0	1.0	1.0005 (0.0123)
a_6 (M_{11} = 1.56)	1.5	1.56	
a_7	-0.648	-0.648	-0.582 (0.0219)
a_8	0.1666	0.1666	0.230 (0.0105)
a_9	0.449	0.449	0.518 (0.0169)
τ	-	-	-2.916 (0.00916)

Table 3d: Theoretical values and estimates of parameters for the case of second-order flapping models with first-order inflow.

Parameter	Model V	Model VI	Estimate
a_1	-1.171	-1.171	-1.258 (0.0202)
a_2	-1.06	-1.06	-1.192 (0.0249)
a_3	-1.561	-1.561	-1.588 (0.00536)
a_4(M_{11} = 1.0)	-0.1666	-0.299	0.0306 (0.0456)
a_4(M_{11} = 1.56)	-0.1068	-0.191	
a_5(M_{11} = 1.0)	-0.648	-0.648	-0.623 (0.0234)
a_5(M_{11} = 1.56)	-0.415	-0.415	
a_6	1.171	1.171	1.171 (0.0396)
a_7	1.561	1.561	1.561 (0.00712)
a_8(M_{11} = 1.0)	0.1666	0.1666	0.253 (0.0125)
a_8(M_{11} = 1.56)	0.1068	0.1068	
a_9(M_{11} = 1.0)	0.449	0.449	0.574 (0.0175)
a_9(M_{11} = 1.56)	0.287	0.287	
τ	-	-	0.0187 (0.0135)

Appendix A

Theoretical Models Considered

Model I: First-order flapping model with constant inflow

$$n_\beta \frac{d\beta_0}{d\tau} = -\frac{4}{3} n_\beta \lambda_0 - \lambda_\beta^2 \beta_0 + \frac{4}{3} n_\beta \mu_z + n_\beta \theta_0 \quad \text{where } \tau = \Omega t$$

Model II: First-order flapping model, with first order inflow. Effect of motion of the tip path plane on inflow neglected

$$\begin{bmatrix} n_\beta & 0 \\ \frac{1}{6} & M_{11} \end{bmatrix} \begin{bmatrix} \frac{d\beta_0}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} -\lambda_\beta^2 & -\frac{4}{3} n_\beta \\ 0 & -L \end{bmatrix} \begin{bmatrix} \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} n_\beta & \frac{4}{3} n_\beta \\ \frac{1}{6} & \frac{L}{2} + \frac{1}{8} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \mu_z \end{bmatrix}$$

where, $L = \frac{1}{4} + \frac{4\bar{V}}{a_0 s}$

Model III: As for Model II, but including effect of coning motion on inflow

$$\begin{bmatrix} n_\beta & 0 \\ \frac{L}{3} + \frac{1}{12} & M_{11} \end{bmatrix} \begin{bmatrix} \frac{d\beta_0}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} -\lambda_\beta^2 & -\frac{4}{3} n_\beta \\ 0 & -L \end{bmatrix} \begin{bmatrix} \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} n_\beta & \frac{4}{3} n_\beta \\ \frac{1}{6} & \frac{L}{2} + \frac{1}{8} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \mu_z \end{bmatrix}$$

Model IV: Second-order flapping model, with constant inflow

$$\frac{d^2\beta_0}{d\tau^2} + n_\beta \frac{d\beta_0}{d\tau} = -\frac{4}{3} n_\beta \lambda_0 - \lambda_\beta^2 \beta_0 + \frac{4}{3} n_\beta \mu_z + n_\beta \theta_0$$

Model V: Second-order flapping, with first-order inflow. Effect of coning motion on inflow neglected

$$\begin{bmatrix} \frac{d^2\beta_0}{d\tau^2} \\ \frac{d\beta_0}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} -n_\beta & -\lambda_\beta^2 & -\frac{4}{3} n_\beta \\ 1 & 0 & 0 \\ -\frac{1}{6M_{11}} & 0 & -\frac{L}{M_{11}} \end{bmatrix} \begin{bmatrix} \frac{d\beta_0}{d\tau} \\ \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} n_\beta & \frac{4}{3} n_\beta \\ 0 & 0 \\ \frac{1}{6M_{11}} & \frac{1}{M_{11}}\left[\frac{L}{2} + \frac{1}{8}\right] \end{bmatrix} \begin{bmatrix} \theta_0 \\ \mu_z \end{bmatrix}$$

Model VI: As for Model V, but including effect of coning motion on inflow

$$\begin{bmatrix} \frac{d^2\beta_0}{d\tau^2} \\ \frac{d\beta_0}{d\tau} \\ \frac{d\lambda_0}{d\tau} \end{bmatrix} = \begin{bmatrix} -n_\beta & -\lambda_\beta^2 & -\frac{4}{3} n_\beta \\ 1 & 0 & 0 \\ -\frac{1}{M_{11}}\left[\frac{L}{3} + \frac{1}{12}\right] & 0 & -\frac{L}{M_{11}} \end{bmatrix} \begin{bmatrix} \frac{d\beta_0}{d\tau} \\ \beta_0 \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} n_\beta & -\frac{4}{3} n_\beta \\ 0 & 0 \\ \frac{1}{6M_{11}} & \frac{1}{M_{11}}\left[\frac{L}{2} + \frac{1}{8}\right] \end{bmatrix} \begin{bmatrix} \theta_0 \\ \mu_z \end{bmatrix}$$

Model VII: First-order flapping model, with infinitely fast inflow dynamics. Effects of coning motion on inflow neglected

$$n_\beta \left[1 - \frac{2}{9L}\right] \frac{d\beta_0}{d\tau} = -\lambda_\beta^2 \beta_0 + n_\beta \left[\frac{2}{3} - \frac{1}{6L}\right] \mu_z + \left[1 - \frac{2}{9L}\right] n_\beta \theta_0$$

Model VIII: As for Model VII, but effect of coning motion on inflow included

$$n_\beta\left[1 - \frac{1}{9}\left[4 + \frac{1}{L}\right]\right]\frac{d\,\beta_o}{d\tau} = -\lambda_\beta^2\,\beta_o + n_\beta\left[\frac{2}{3} - \frac{1}{6L}\right]\mu_z + \left[1 - \frac{2}{9L}\right]n_\beta\,\theta_o$$

Model IX: As for Model VII, but with second-order flapping

$$\frac{d^2\,\beta_o}{d\tau^2} + n_\beta\left[1 - \frac{2}{9L}\right]\frac{d\,\beta_o}{d\tau} = -\lambda_\beta^2\,\beta_o + n_\beta\left[\frac{2}{3} - \frac{1}{6L}\right]\mu_z + \left[1 - \frac{2}{9L}\right]n_\beta\,\theta_o$$

Model X: As for Model VIII, but with second-order flapping

$$\frac{d^2\,\beta_o}{d\tau^2} + n_\beta\left[1 - \frac{1}{9}\left[4 + \frac{1}{L}\right]\right]\frac{d\,\beta_o}{d\tau} = -\lambda^2\,\beta_o + n_\beta\left[\frac{2}{3} - \frac{1}{6L}\right]\mu_z + \left[1 - \frac{2}{9L}\right]n_\beta\theta_o$$

The following theoretical parameter values were used (corresponding to the Puma helicopter used in flight trials [1]).

$\lambda_\beta^2 = 1.06$ $\quad$ $M_{11} = 128/(75\pi a_o s) = 1.00$ (corrected Pitt value [11])

$n_\beta = 1.171$ $\quad$ or $M_{11} = 8/(3\pi a_o s) = 1.56$ (uncorrected Pitt value [11])

$\Omega = 27.6$ rads/sec

$L = 0.648$