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# HELICOPTER VIBRATION REDUCTION THROUGH STRUCTURAL MANIPULATION

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## SUMMARY

The problem of modifying a given airframe so as to improve the structural response to given inputs from the head is seen to consist of two parts. In the first, it is necessary to identify those parts of the airframe where useful changes to stiffness may be made. In the second, the precise resizing of members is to be effected. In this paper, the first of these two tasks is examined with the aid of the Vincent Circle Theorem.



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## 1 INTRODUCTION

The helicopter vibration problem with which we shall be concerned is that of the airframe responding to rotor induced loads. We seek a means of adjusting the airframe for a satisfactory response at one or more places in the airframe to vibratory inputs from the head. We shall not be concerned with the origins at the rotor of periodic loads, nor of attempts to isolate dynamically the airframe from the rotor system. It will be assumed that loads of known frequency but not necessarily known magnitude or direction are impressed on to the airframe from above. Moreover, we shall assume a point input of the vibratory load on to the airframe, despite the evident existence of several attachment points of the rotor and gearbox system.

The simplification which we thus introduce allows a specific analysis of the airframe to be developed. Whilst the rotor loading affects the airframe, we assume that what we do to the airframe does not, primarily, influence the generation of load at the rotor itself.

## 2 DYNAMIC MODEL OF THE AIRFRAME

In order to describe the response of an elastic structure to given oscillatory inputs, it is essential to have a valid dynamic model. As a basis for manipulation we seek a matrix of receptances $G_{fr}$ say, where $G_{fr}$ is the response along a given direction at node r to forcing at a point f in a given direction. We shall assume that the structure is conservative so that $G_{fr} = G_{rf}$.

One possible source of receptance is direct measurement, although this is often difficult and for obvious experimental reasons, the number and location of the survey points have to be limited. However, the measured receptance has the merit that its value is not in dispute and may be measured on a structure which has been suitably preloaded.

A second approach is to construct the matrix of receptances from a set of elastic modes which may be either calculated, as for example in a NASTRAN analysis (Cronkhite, 1), or may be measured, using for example, the MAMA method (Taylor *et al*, 2). Neither course is altogether reliable for different reasons.

In spite of strenuous efforts in the application of NASTRAN, it has not as yet been possible to predict precisely the location of elastic modes and frequencies to that accuracy necessary for the application of the ideas described below. Furthermore, the receptances have to be based upon an assumption regarding damping and although we shall, in general, be dealing with non-resonant excitation, the errors introduced by assumed - and therefore spurious - values of damping may be significant.

Experimental modes are determinate only to within a certain error since it is only possible in an experiment to maintain a small relative phase error between the displacements at observation points. Nevertheless, reasonable estimates of the undamped normal modes can be obtained with an accurate determination of the modal damping. There is usually a great difficulty in measuring modal inertia and, as a rule, the errors here increase with mode number. Finally we may note that unless a very large number of modes are measured, which is costly, that the influence of modes above the cut-off frequency has to be allowed for. If these ignored modes are all at frequencies well in excess of excitation frequency, the effect may be ignored, but if responses are known at points of interest in a range of frequencies around the excitation frequency, an allowance may be made for this effect.

Thus, from whatever source we acquire the matrix of receptances, there will be some question of confidence. However, for the present purposes, we shall assume that the matrix of receptances is known and is not in dispute. However, when drawing conclusions on the value of any observations we make, it is essential to recall this limitation in the data.

## 3 STRUCTURAL CHANGES TO BE MADE

The only change to the structure which will be considered is the introduction of a direct spring, or strut, between points on the structure which are included in the analysis set used to form the receptance matrix. Thus we exclude alterations to member size of existing structure unless the relevant structure is itself a strut. Even in those circumstances where a join by a strut is physically impossible, we still include that join as one meriting consideration since, in an actual design, we are as much concerned with the problem of where to make changes as with the precise fixing of the size of members. This latter problem is, of course, ultimately to be solved. Changing, for example, a skin thickness, or stringer pitch, will alter the direct stiffness between several, if not all, of the analysis points. It is important to note that our survey of the structure aims at a preliminary task in optimization rather than the complete specification of member sizes. Indeed, once we have identified those strut locations where stiffening or unstiffening is effective, we may apply a finer tuning procedure to optimize the structure.

Since the number of possible structural changes is considerable, we rule out of the question the possibility of a formal search of design space for optimum changes. Indeed, we replace this formidable - and probably impracticable - task with a simple, but nonetheless useful, procedure for ranking the effectiveness of strut changes.

To change stiffness we may also need to change structure mass. This is, however, in practice, a minor aspect as we shall show. Thus we may concern ourselves with the effects of ideal stiffness change throughout the structure without mass penalty. This we do with the aid of a theorem due to Vincent[3].

## 4 VINCENT'S CIRCLE THEOREM

If an elastic structure, loaded by a sinusoidal force of fixed frequency acting at node $f$ along a fixed direction is supplemented by a spring of stiffness $k$ joining nodes $p$, $q$ of the structure, then the locus of the response at a node $r$ in a given direction as $k$ varies is a circle. The circle theorem has been applied in a number of papers by Done, Hughes and Webby[4,5,6] who have broadly followed the recommendations of Hughes[7], as to the ideal criteria. Three which are discussed in these papers are: (i) the diameter of the circle; the larger the better; (ii) occurrences of a member in a survey of all spring pairs which, taken together, may lead to exactly zero response; the more frequent the better; (iii) the smallness of minimum response; the smaller the better. These early applications of the method have been followed by an interesting comparison by Hanson and Calapodas[8] of the rival merits of the Vincent circle method and methods based on strain-energy density developed by Sciarra of Boeing-Vertol[9].

The use of circle diameter as a criterion is rejected. Whilst it appears that a small circle diameter is useless (Fig 1a), largeness of diameter is a negative asset for the locus of response may not pass near the origin and so the best choice of $k$ may not improve the situation significantly, as is illustrated in Fig 1b. Fig 1 also shows a number of unsatisfactory situations. For example, in Fig 1c a suitably small response is achieved but for an excessively large value of $k$, which may not be realisable in practice. The use of smallness of response is also inappropriate, for all values as small as, say 1% of the

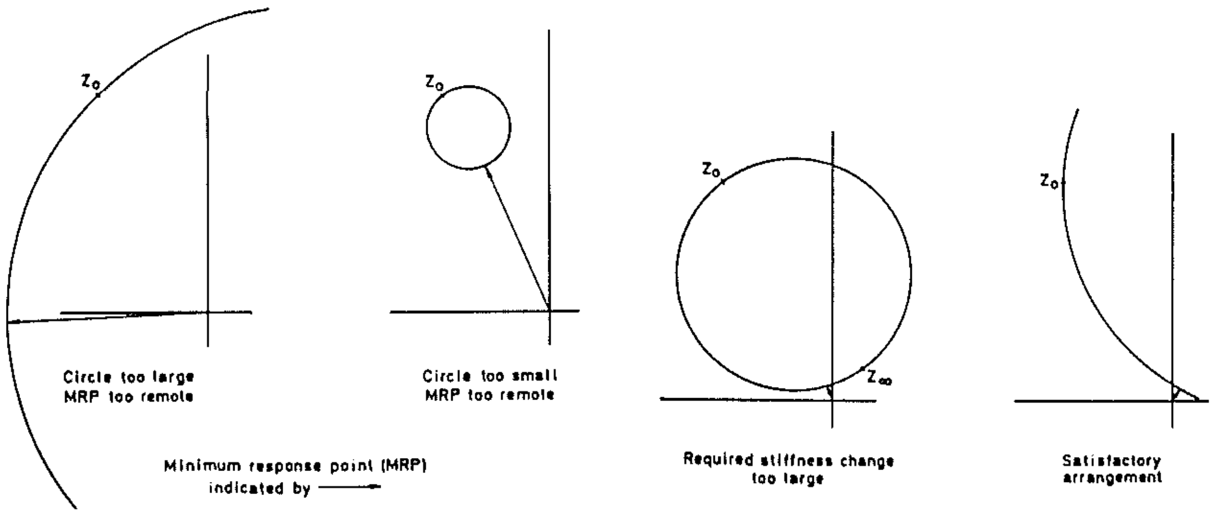


Fig 1 Typical circle plots showing some adverse features

original response are equally meritorious and amongst these equivalently successful changes, the ones realised with smallest stiffness change are best.

## 5 SOME COMMENTS ON THE CIRCLE

Provided that the model of the structure remains valid after a stiffness change $k$, the response of the structure at $r$ is known. Although the basic mathematical model is linear, response is a nonlinear function of $k$, but, because of the simplicity of the circle, the minimum response point is immediately identified without searching formally as a function of $k$. This is a very powerful property that enables us to avoid formal optimization.

If the structure is loaded (along a specified direction) at $f$, if response is measured (along a specified direction) at $r$ and if a spring of stiffness $k$ joins nodes $p$ and $q$, then the response at $r$, say $z$, is given by

$$z = G_1 - kG_2/(1 + kG_3) \qquad (1)$$

where $G_1 = G_{rf}$,

$$G_2 = (G_{rp} - G_{rq})(G_{fp} - G_{fq})$$

and

$$G_3 = G_{pp} - 2G_{pq} + G_{qq}$$

where the associated freedoms at p and q , say $x_p$ and $x_q$ are along the line from p to q .

The quantities $G_{pq}$ etc are all complex numbers so that z is a point in the complex plane whose real part is in phase with the forcing and whose imaginary part is the (leading) quadrature displacement. As k is a real scalar, the locus of z for variable k is a circle centre at $G_1 - G_2/(G_3 - \bar{G}_3)$ , a bar denoting complex conjugate. The radius of the circle is $|G_2/(G_3 - \bar{G}_3)|$ . Corresponding expressions for the minimum response point and the associated value of k are too cumbersome for presentation but follow at once from the circle geometry.

## 6 EFFECT OF THE MASS OF A STRUT ON RESPONSE

The light strut which has been described in previous sections leading to the circle locus is an ideal concept. In reality, changes in stiffness are always associated with a mass change, at least in an efficient design. Let us, then, examine the associated mass penalty.

In Appendix 1, the case is examined where the mass change is divided between the two ends of the strut and these are considered as point masses with effective inertia only along the direction of the spring. In reality, the masses will be isotropic and the strut would have rotary inertia, but the given simplification is adequate for discussion purposes.

The parameter $\mu$ is defined as $1 - m\omega^2$ where 2mk is the added mass associated with the strut of stiffness k and $\omega$ is the circular frequency of excitation.

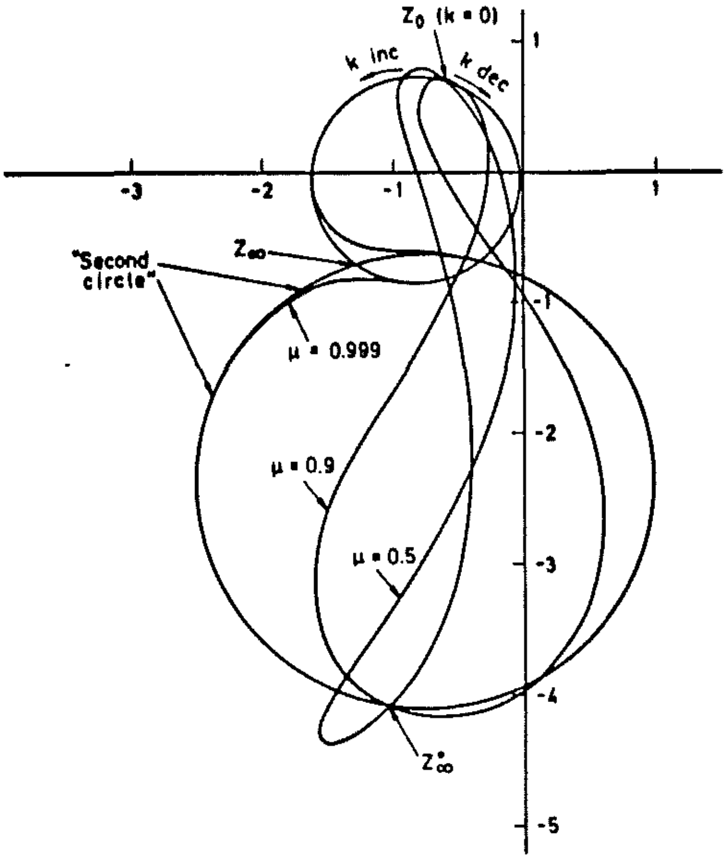


Fig 2 Effect of mass on response loci for stiffness change

In place of the simple circle, a figure-of-eight is obtained (Fig 2). For most practical struts the value of the parameter $\mu$ is 0.999 or higher so that the revised locus is close to parts of two circles. Around $k = 0$, the effects of mass addition are small enough to be ignored: it is only when the effective increase in total inertia is significant that the locus departs significantly from the elementary circle. Since we shall be confining our interest to smallish values of $k$ we shall ignore the effect of mass on the response of a light strut.

## 7 LIMITATIONS ON THE USE OF THE CIRCLE

The supposition that the stiffness $k$ is a free-ranging variable is false. In practice the promise of improved response may violate the assumptions upon which the description of the structure by a given matrix of receptances $G_{pq}$ is based.

For example, the effect of a stiffness change may be interpreted in terms of the original modal description (assuming one is available) and the new modes and frequencies of the associated undamped system calculated. In Appendix 2 we show how the response for a given $k$ may be interpreted in terms of modes and modify the stiffness matrix accordingly. (For completeness, particularly in the case where several springs may be added, we include the effect of mass change.) We find thereby the way in which the new response is assembled in terms of the original modes and similarly we may examine the revised modes of the undamped structure.

For $k$ positive, the effect of adding stiffness is to raise the modal frequencies and for small values of $k$ the effect is small. However, for large $k$ the modal frequencies may change considerably and the original order be lost. In this case we have clearly lost relevance with the original structure. The added strut is supposedly behaving in a way difficult to realize in a light shell structure and, very probably, attempts to include such a strut would lead to local deformations of the airframe around the ends of the strut which are not describable in terms of the original modes. Thus the credibility of the dynamic base is destroyed. For some value of $k$, then, the model ceases to be valid: exactly what the value is may not be easy to assess but one can recognize when it has been exceeded.

In a similar way, removal of direct stiffness ($k$ negative) will cause frequencies to lower and, for some sufficiently large negative $k$ the lowest frequency will fall to zero, when the structure is dynamically unstable. Long before that point is reached, the model has ceased to be valid.

We must, then, for helicopter applications, restrict our interest to modest, if imprecisely limited, values of $k$. Thus, in a situation like that represented by Fig 1d, we must ensure that the minimum response is achieved with acceptably small values of $k$.

## 8 PROPOSED CRITERION

If we wish to survey the potential for change in a single structural member, a criterion is needed in order to identify which, of all possible candidates for change, will be the most effective. Thereby we can identify a group of struts which admit of effective change. The criterion establishes which, of all possible changes, are the most effective ones. It is further argued that where change is to be effected not by changing one member alone but by smaller changes in a number of struts, that those most effective as a single change will be effective as a group.

Such a criterion has to recognize that relative improvement in response has to be weighed against the spring size needed to achieve it. It must thus produce significantly different values in the four cases illustrated in Fig 1. The following proposal seems appropriate:

$$C = (1 - |z_{opt}/z_0|)/\log |k| . \qquad (2)$$

In equation (2), $z_{opt}$ and $z_0$ are, respectively, the minimum response and initial one ($k = 0$) whilst the value of $k$ used in the criterion is the one producing minimum response. Thus when $z$ does not reduce much the first factor is small, when $k$ is too large the denominator is excessive. Hence the higher the value of $C$, the more effective the proposed change.

Even in the case where the optimum $k$ is outside the acceptable range the criterion is useful as it is a form of sensitivity indicator. A study has been made of the change in response of a system when definite limits on each allowed stiffness are introduced. For moderate limits the group of struts with the higher values of $C$ correspond - broadly - with the group with the higher sensitivities at the limit. As long as there is significant movement around the circle at the limits (say $10^8$ N/m) the groups correspond and the criterion is useful in this case too.

A different situation prevails, however, where the viable portion of the circle is very close to the origin. We may envisage a situation in which we consider changes to dozens of members, but allow each to have only a very small change (say $10^6$ N/m or less). In this case the evidence of the circle may be quite difficult to interpret.

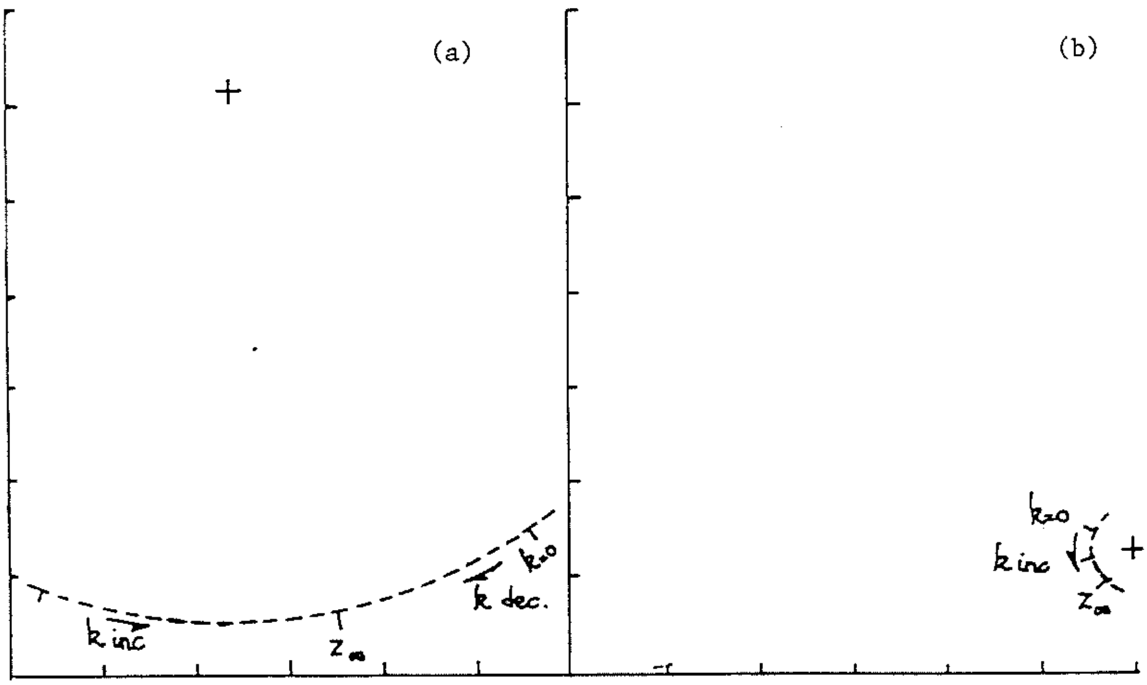


Fig 3 Two illustrative circle plots

Consider, for example, the two situations illustrated in Fig 3. In Fig 3a the circle is large and the criterion is about 0.1. The minimum response is obtained for a stiffness of about $4 \times 10^8$ N/m but around the $k = 0$ point it is necessary to reduce stiffness to improve response and the reduction is about 0.2% per MN/m. In Fig 3b, the circle is small and at the minimum response point an improvement of less than 4% is realized. However the initial sensitivity is better than in Fig 3a being 0.5% per MN/m.

Such small changes are not, of themselves, of great account. Taken collectively, however, a series of such changes could be very advantageous.

## 9 SEARCH FOR A GROUP OF STRUTS SENSITIVE TO CHANGE

From whatever source our receptance matrix is compiled, the number of candidate struts may be large. Including notional struts with no real prospect of realization, there may be hundreds.

For each strut there are appropriate limits on $k$ which are either imposed by limitations of design or by appropriateness of the theory. We investigate each strut in turn for its sensitivity to change using either the criterion $C$ or a direct application of equation (1) for $k = 1$ MN/m say. Either calculation will require the evaluation of eight receptances. We may sample response at one point to each of six head loads (fore and aft, lateral and vertical shear, pitch, roll and yaw couple). We look for common occurrences in the strut-rankings we establish for each load case. Occurrence in several load rankings indicates - if the indicated sign of stiffening is consistent - a strut worth changing.

We argue that partial progress towards a minimum response for each strut is to be compounded as a group change leading to effective response reduction.

## 10 RESPONSE AT SEVERAL POINTS

In a typical helicopter there may be several areas of the structure at which it is desired to have reasonably small response. Accordingly we may investigate, using the heirarchical method outlined above, how the candidate struts are ranked for sensitivity for each of the $N$, say, response points of interest.

For each case we survey response to six loads and thus have $6N$ sets of rankings. Since the only questions to put thus far are: (i) do we stiffen or unstiffen? and (ii) which struts are the most sensitive to change?, we are able to look for struts which appear consistently with a $\pm k$, and with acceptably high sensitivity.

Let us assume that among the $6N$ sets of sensitivities there are identified some struts consistently indicating change in one sense. Then we can be certain that such a change will be beneficial and is probably associated with the detuning of a mode that is sensitive, either by frequency change or by reducing the effective head loads. On the other hand, promising indications of a profitable change in some of the sets may be countered by contradictory changes in others. Let us, then, relate response improvement to stiffness change.

## 11 THE NATURE OF RESPONSE CHANGE

At a given point, $r$, say, the response to unit load at forcing node $f$ in a given direction may be written

$$z = \sum_n \frac{d_{rn} d_{fn} \left(\omega_n^2 - \omega^2 - 2\omega\omega_n c i\right)}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\omega\omega_n c\right)^2} \tag{3}$$

where $d_{rn}$, $d_{fn}$ are the generalized displacements at r and f (for the appropriate directions) in the nth orthonormalized mode, and summation is over the six rigid-body and all elastic modes.

In a typical analysis, the components of z from each mode consist of one or more real vectors from rigid-body modes and elastic contributions in any of the four quadrants. These tend to lie reasonably close to the real axis. The vector sum of all these individual vectors comprises z .

Thus a change in a single strut stiffness k may alter all the elastic responses through changes to (a) $d_{rn}$ , (b) $d_{fn}$ , (c) $\omega_n$ , for each elastic mode.

Where a beneficial change is brought about by changes to $d_{rn}$ , the benefit may be local and at the expense of increased response elsewhere. This is the case where the effect of change is to reduce the response in one mode by bringing a node of that mode closer to r . Where the change is due primarily to reduction in $d_{fn}$ , there will be a system benefit, as the generalized force in the mode is reduced. In the cases where we have a near resonant condition (which would be bad design anyway) or where we have dominant contributions to response from an adjacent mode changes in $\omega_n$ are likely to be effective. These would have widespread benefit throughout the structure.

## 12 BOEING-VERTOL STRATEGY

The methods which have been described by Sciarra[9] and applied to the Boeing-Vertol type 347 with considerable success are related to the above. Using a finite-element model, the forced response is calculated for assumed damping and given load. Thus the modal coefficients can be calculated and the contributions from each element to the strain-energy in each mode identified. Sciarra searches for elements with the prospect of improvement in the total response. If, for example, there are two modes astride the excitation frequency $\omega$ (the nearest two) with frequencies $\omega_\ell$ and $\omega_u$ where $\omega_\ell < \omega < \omega_u$ , then we seek parts of the structure in which the strain energy contributions from a particular element of real structure is high in one mode, low in the other and stiffen or unstiffen as appropriate. Provided that other modes have not been brought into the excitation area, this strategy is successful. It attacks primarily $\omega_n$ and to a lesser extent $d_{fn}$ . The former is objectively monitored but generalized load (or response) is less readily dealt with.

The procedure is keyed closely to a finite-element (*eg* NASTRAN) model which may require considerable bringing into line with the observed behaviour of the aircraft. It has the great merit that it modifies real structure but its value is contingent on that model being appropriate.

A diagnostic method based upon response attacks the heart of the problem - response. What has been outlined above is conceptual - a diagnosis. The detailed redesign is a later exercise.

## 13 APPLICATION

The diagnostic techniques outlined above have been applied to a computer study of a helicopter for which experimental modes and frequencies have been measured.

A set of preferred struts has been identified with the promise of improved response and a test of the likelihood that such a set would be generated randomly in the numerical experiments which have been used is negative. It is concluded that the indications are positive, that the set of struts so identified is a group worth changing.

## REFERENCES


| | | |
|---|---|---|
| 1 | J.D. Cronkhite | Development, Documentation and Correlation of a NASTRAN Vibration of the AH-1G Helicopter Airframe. NASA TM X-3428, pp 273-294 (1976) |
| 2 | G.E. Taylor<br>D.R. Gaukroger<br>C.W. Skingle | MAMA - A Semi-Automatic Technique for Exciting the Principal Modes of Vibration of Complex Structures. RAE Technical Report 67211 (1967) |
| 3 | A.H. Vincent | A Note on the Properties of the Variation of Structural Response with Respect to a Single Structural Parameter when Plotted in the Complex Plane. Westland Helicopters Ltd. Report GEN/DYN/RES/010R (1973) |
| 4 | G.T.S. Done<br>A.D. Hughes | The Response of a Vibrating Structure as a Function of Structural Parameters. J. Sound and Vib. 38, 2, pp 255-266 (1976) |
| 5 | G.T.S. Done<br>A.D. Hughes | Reducing Vibration Through Structural Modification. Vertica, 1, pp 31-38 (1976) |
| 6 | G.T.S. Done<br>A.D. Hughes<br>J. Webby | The Response of a Vibrating Structure as a Function of Structural Parameters - Application and Method. J. Sound and Vib. 49, 2, pp 149-159 (1976) |
| 7 | A.D. Hughes | Some Dynamic Aspects of Structural Manipulation. PhD Thesis, University of Edinburgh (1975) |
| 8 | H.W. Hanson<br>N.J. Calapodas | Evaluation of the Practical Aspects of Vibration Reduction using Structural Optimisation Techniques. Am. Hel. Soc. preprint, 79-21 (1979) |
| 9 | J.J. Sciarra | Use of Finite-Element Damped Forced Responses Train Energy Distribution for Vibration Reduction. Boeing-Vertol Report D210-10819-1 (1974) |

## Appendix 1

## EFFECT OF MASS PENALTY ON STIFFNESS CHANGE

If the introduction of a stiffness $k$ between nodes $p$ and $q$ of the structure invokes a mass penalty of, say, $2mk$, we may simplify the associated analysis by taking mass $mk$ at each of $p$ and $q$ with inertial effect only along $pq$.

With the notation:

$x_i$ = effective displacement at node $i$ in allowable direction;

$G_{pq}$ = complex receptance as in text;

then we may write for force $F_f$ at node $f$,

$$x_p = G_{pf}F_f - G_{pp}k(x_p - x_q) + G_{pq}k(x_p - x_q) + G_{pp}mk\omega^2x_p + G_{pq}mk\omega^2x_q ,$$

$$x_q = G_{qf}F_f - G_{qp}k(x_p - x_q) + G_{qq}k(x_p - x_q) + G_{qp}mk\omega^2x_p + G_{qq}mk\omega^2x_q ,$$

$$x_r = G_{rf}F_f - G_{rp}k(x_p - x_1) + G_{rq}k(x_p - x_q) + G_{rp}mk\omega^2x_p + G_{rq}mk\omega^2x_q .$$

Thus

$$\begin{vmatrix} -\mu G_{pp} + G_{pq} - 1/k , & G_{pp} - \mu G_{pq} , & G_{pf} \\ Gqq - \mu G_{qp} , & G_{qp} - \mu G_{qq} - 1/k , & G_{qf} \\ G_{rq} - \mu G_{rp} , & G_{rp} - \mu G_{rq} , & G_{rf} - x_r/F_f \end{vmatrix} = 0$$

where $\mu = 1 - m\omega^2$, $\omega$ being the exciting frequency.

On expansion we find

$$z = \frac{x_r}{F_f} = G_{rf} + \frac{(1 - \mu^2)k^2G_1 + (G_2 + G_3)k}{(1 - \mu^2)k^2G_4 + (G_5 + G_6)k + 1}$$

where $G_1 = G_{pf}(G_{pr}G_{qq} - G_{qr}G_{qq}) + G_{qf}(G_{pp}G_{qr} - G_{pr}G_{pq})$

$G_2 = G_{pf}G_{qr} + G_{pr}G_{qf}$

$G_3 = G_{pf}G_{pr} - G_{qf}G_{qr}$

$G_4 = G_{pq}^2 - G_{pp}G_{qq}$

$G_5 = - 2G_{pq}$

and

$$G_6 = G_{pp} + G_{qq} .$$

For the case of no mass penalty we have the familiar result in equation (1) of the text, which passes through the point

$$z_\infty = G_{rf} + \frac{G_2 + G_3}{G_5 + G_6} .$$

For $\mu \neq 1$ , the locus passes through the points

$$z_0 = G_{rf} \quad \text{for } k = 0$$

and

$$z_\infty^* = G_{rf} + \frac{G_1}{G_4} \quad \text{for } k \to \infty$$

Typical behaviour for $\mu \neq 1$ is given in Fig 2 which has been computed for the following receptance values:

$$G_{pp} = 9.350 \times 10^{-9} - 1.251 \times 10^{-8}\, i ,$$
$$G_{pq} = 4.199 \times 10^{-9} - 8.062 \times 10^{-9}\, i ,$$
$$G_{pe} = 1.575 \times 10^{-8} - 7.312 \times 10^{-9}\, i ,$$
$$G_{pf} = 3.250 \times 10^{-10} - 1.334 \times 10^{-10} i ,$$
$$G_{qq} = 4.187 \times 10^{-10} - 1.060 \times 10^{-8}\, i ,$$
$$G_{qr} = 2.148 \times 10^{-8} - 8.821 \times 10^{-9}\, i ,$$
$$G_{qf} = 2.197 \times 10^{-9} - 8.205 \times 10^{-11} i ,$$

and

$$G_{rf} = -6.328 \times 10^{-10} + 7.281 \times 10^{-10} i .$$

In the limiting case as $\mu$ approaches 1, the locus consists of parts of two circles, centres

$$- 8.243 \times 10^{-10} - 4.516 \times 10^{-11} i$$

and

$$- 7.721 \times 10^{-9} - 2.365 \times 10^{-9}\, i ,$$

with respective diameters

$$1.558 \times 10^{-9} \text{ and } 3.498 \times 10^{-9} .$$

The $z_\infty$ point, corresponding to $k = \pm\infty$ , has

$$x_p = x_q .$$

The $z_\infty^*$ point has

$$x_p = x_q = 0 .$$

## Appendix 2

## MODIFICATIONS TO MASS AND STIFFNESS MATRICES OF THE SYSTEM

In the case of orthonormalized modes, the mass matrix is unit diagonal, whilst the corresponding stiffness matrix is diagonal with its first six diagonal elements zero, and each other diagonal element equal to $\omega_i^2$ .

The corresponding damping matrix D of the system is taken to be diagonal, its first six diagonal elements being zero and the others $2\omega_i c_i$ , where $c_i$ is the damping coefficient in the mode with frequency $\omega_i$ .

The corrections to the mass matrix, M say, with general element $M_{ij}$ and to the stiffness matrix, K say, with general element $K_{ij}$ that arise from the introduction of a massive strut of stiffness k between nodes p and q can be written down.

If $\underline{u}$, $\underline{v}$ are displacements at the ends of the strut joining p to q , and $\underline{s}$ is a unit vector along the join from p to q , then the axial extension of the rod is $(\underline{u} - \underline{v}) \cdot \underline{s}$ and the strain energy of the rod due to stretching is

$$U = \tfrac{1}{2}k\left((\underline{u} - \underline{v}) \cdot \underline{s}\right)^2 .$$

Writing $\underline{u}_r$ and $\underline{v}_r$ as the vectors representing the displacements at p and q in the rth mode, we may write

$$\underline{u} = \sum_r q_r \underline{u}_r \qquad \text{and} \qquad v = \sum_r q_r \underline{v}_r .$$

Thus

$$U = \tfrac{1}{2}k \left\{ \sum_r q_r (\underline{u}_r \cdot \underline{s} - \underline{v}_r \cdot \underline{s}) \right\}^2$$

from which the increment to the element $K_{ij}$ follows as

$$k(\underline{u}_i \cdot \underline{s} - \underline{v}_i \cdot \underline{s})(\underline{u}_j \cdot \underline{s} - \underline{v}_j \cdot \underline{s}) .$$

The corresponding changes to the mass matrix are derived from the corresponding increase in kinetic energy, T , say.

If the join of p to q has mass m , then the kinetic energy of the rod is

$$\left(\frac{m}{6}\right) (\dot{\underline{u}}^2 + \dot{\underline{u}} \cdot \dot{\underline{v}} + \dot{\underline{v}}^2)$$

where the dots over u, v denote time derivatives.

The ijth term in the mass matrix M , say $M_{ij}$ is thus increased by

$$\left(\frac{m}{6}\right) \left\{\underline{u}_i \cdot (2\underline{u}_j + \underline{v}_j) + \underline{v}_i \cdot (2\underline{v}_j + \underline{u}_j)\right\}$$

which is, of course, equal to $M_{ji}$ .

The first six rows and columns of M and K are unchanged by the introduction of the massive spring. The new modal equations are, for periodic excitation at frequency $\omega$ ,

$$(K - M\omega^2 + i\omega D)\underline{q} = F ,$$

where F is the in-phase forcing load, and $\underline{q}$ a column vector of generalized complex coordinates. The response then follows from the solution of this equation and confirms the circle plot.

The eigenvalues, that is the new frequencies, of the system augmented by the spring, follow as the roots of $|K - M\omega^2| = 0$ , where K and M here have their augmented values.

Should there be a series of springs fitted to the system, the appropriate new mass and stiffness matrices follow by summation over k , each with its appropriate p and q .