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by

M. Costello
Aerospace Laboratory
Georgia Tech Research Institute
Georgia Institute of Technology

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OUTPUT FEEDBACK APPLIED TO HIGH ORDER ROTORCRAFT SYSTEMS*

Mark F. Costello †
Aerospace Laboratory
Georgia Tech Research Institute
Georgia Institute of Technology

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Abstract

This paper develops a method for modifying control law parameters to improve system performance and stability where the control system is general and realized by a set of multiple input and single output filters arranged in an arbitrary fashion. The method is based on reformulating the original control problem as a constant gain optimal output feedback tracking problem where the parameters in the control system that are designated as variable are separated into a constant diagonal matrix. Results using the method to modify control system parameters for a higher harmonic rotor control system are presented.

1. Introduction

The rotorcraft control law design process, like most other design processes, is iterative in nature and is initiated with some form of a baseline design. The baseline design is typically created by seasoned engineers using past experience to furnish the control law structure and initial parameter values. As a result, the baseline control laws tend to evolve from an already existing control system. Indeed, for aircraft modification programs, this is generally the case. A generic flow chart of the control law design process in given in Figure 1. Once the baseline control laws are defined, the combined plant and control system is modeled, simulated, and tested in a wide variety of ways including batch simulation using a simple six degree of freedom linear or non linear fuselage model, batch simulation using a linear or non linear high fidelity and high order dynamic model, real time pilot in the loop simulation, hardware in the loop simulation, full scale component simulation, full scale combined plant and control system functional testing, and/or a combination of the above. If system performance is not satisfactory, control law parameters and possibly control law structure are modified and reevaluated. This loop continues until satisfactory control laws are achieved.

A common format for presenting detailed control laws is through scalar block diagrams. It is important to note that even modern control strategies, such as model following controllers, are ultimately cast into scalar block diagram form. In scalar block diagram form, the main purpose of individual control law elements are obvious and separate from other control law elements. Also, implementation of the block diagram into hardware is straightforward and efficient.

Due to the complex geometric and dynamic nature of rotorcraft systems, mathematical models depicting the behavior of such systems tend to involve a large number of state variables. Moreover, improvements in the fidelity of a particular mathematical model are usually accompanied by an increase in the number of state variables describing the system. For example, a mathematical model of a single main rotor helicopter which includes rigid body dynamics of fuselage motion, main rotor blade flap and lag rotation, main rotor inflow velocity, drive train flexibility, actuator motion, and control systems electrical signals could easily contain in excess of 75 state variables. If fuselage structural dynamics, main rotor blade elasticity, or tail rotor blade flap and lag motion are also reflected in the mathematical model, the order of the system could climb to well over 150.

There are a multitude of control law design techniques available to the control system engineer, however few are particularly well suited to the control system design process mentioned above. For example, optimal observers seek to minimize an integral quadratic performance index based on weighted state and control deviations. Unfortunately, the resulting estimators are of the same order as the plant, an obvious disadvantage when considering the order of a reasonably sophisticated rotorcraft model. More importantly, the method is not conducive to a fixed control law structure nor to decentralized and evolutionary design processes since all control system parameters are modified each design iteration.

Standard practice in industry is to perform exhaus-

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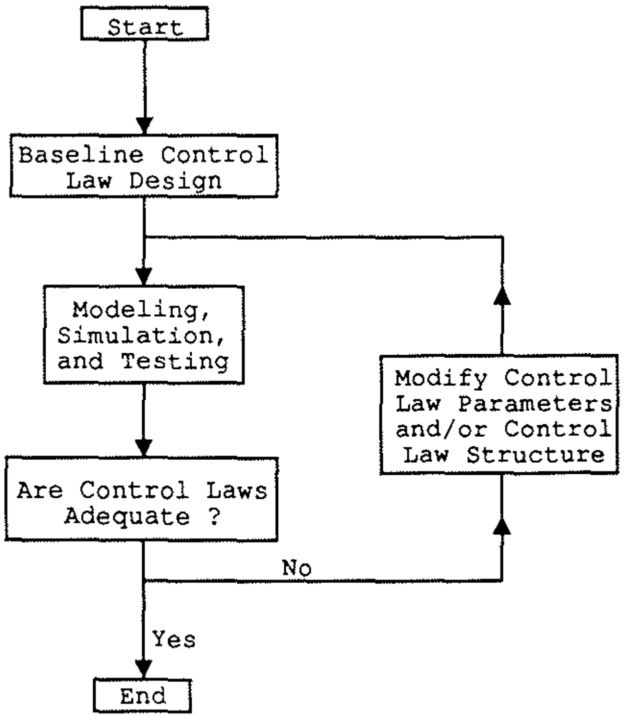

Figure 1: Control Law Design Flow Chart

tive control system parametric studies to arrive at satisfactory control system performance and stability. This paper provides a methodology rooted in optimal control theory that can be used in concert with current control law design practice to help facilitate control law parametric studies. The control system is general and realized by a set of multiple input and single output filters arranged in an arbitrary fashion. With this realization of the control system, practical rotorcraft control systems with any predefined structure can be modeled with a minimum of input data. The parameter optimization scheme is based on reformulating the original control problem as a constant gain optimal output feedback tracking problem where the parameters in the control system that are permitted to vary are separated into a constant diagonal matrix. The method builds on current control law design practice, starting from a baseline set of control laws and modifying specified parameters to improve performance without altering hardware implementation.

The paper is organized as follows. First, the different systems which are used throughout the development are given followed by the control system parameter modification algorithm. In Section 4, the methodology is highlighted with the application of the control law modification process to a higher harmonic rotor control system.

2. System Realization

The uncontrolled rotorcraft system plant is assumed to be given by a linear time invariant system of the

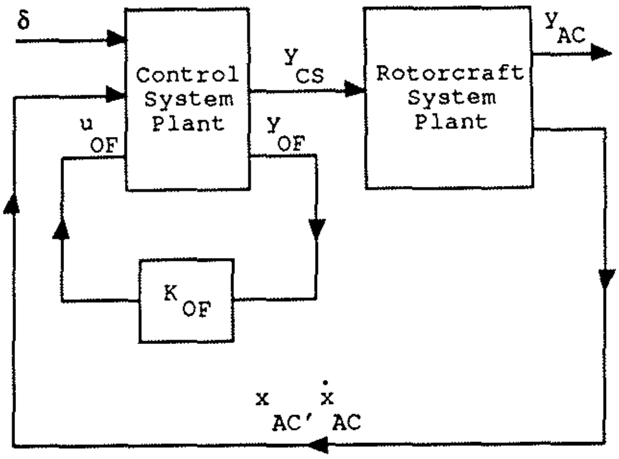

Figure 2: Plant and Control System Connection

form,

$$\dot{x}_{AC} = A_{AC}x_{AC} + B_{AC}u_{AC} \tag{1}$$

$$y_{AC} = C_{AC}x_{AC} + D_{AC}u_{AC} \tag{2}$$

where, x_{AC} is the state vector of the system plant, u_{AC} is the vector of physical control movement, and y_{AC} is the vector of rotorcraft system outputs which are desired to be tracked by control inputs. The order of the uncontrolled rotorcraft system is $nsac$ while the system has $niac$ inputs and $noac$ outputs.

The state space model for the control system can be written as,

$$\dot{x}_{CS} = J_1x_{CS} + J_2\delta + J_3x_{AC} + J_4\dot{x}_{AC} + J_5u_{OF} \tag{3}$$

$$y_E = Q_1x_{CS} + Q_2\delta + Q_3x_{AC} + Q_4\dot{x}_{AC} + Q_5u_{OF} \tag{4}$$

$$y_{OF} = L_1x_{CS} + L_2\delta + L_3x_{AC} + L_4\dot{x}_{AC} + L_7u_{OF} \tag{5}$$

$$u_{OF} = K_{OF}y_{OF} \tag{6}$$

where, x_{CS} is the control system state vector, y_E is the control system output vector, y_{OF} is the output feedback output vector, u_{OF} is the output feedback input vector, and δ is the control system input vector. The details for converting the scalar block diagram data into the above equations are given in Appendix A. Notice that all the parameters in the control system to be modified by the control law algorithm are segregated from the plant matrices and are located on the diagonal of the K_{OF} matrix. The fact that the matrix K_{OF} is diagonal does not represent a simplifying assumption, rather it is simply a byproduct of working with the scalar block diagram form of the control system data.

The plant and control system are connected as shown in Figure 2. However, to efficiently apply the optimal output feedback control law strategy below, the plant and control system are combined while still leaving the variable control system parameters separate. This is accomplished by substituting $u_{AC} = y_{CS}$ into equations 1 and 2. Corresponding to Figure 3, the combined rotorcraft and control system plant equations are,

$$\dot{x}_{AC} = H_1x_{AC} + H_2x_{CS} + H_3\delta + H_4u_{OF} \tag{7}$$

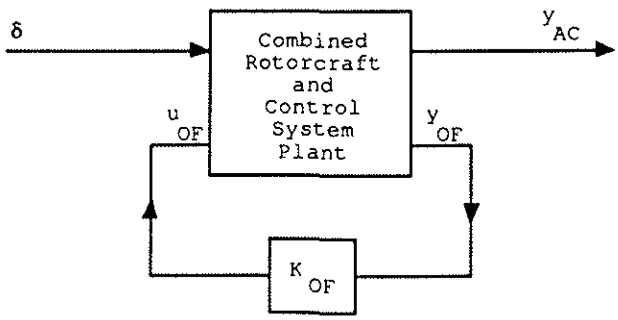

Figure 3: Equivalent Output Feedback System

$$\dot{x}_{CS} = F_1 x_{AC} + F_2 x_{CS} + F_3 \delta + F_4 u_{OF} \tag{8}$$
$$y_{AC} = T_1 x_{AC} + T_2 x_{CS} + T_3 \delta + T_4 u_{OF} \tag{9}$$
$$y_{CS} = R_1 x_{AC} + R_2 x_{CS} + R_3 \delta + R_4 u_{OF} \tag{10}$$
$$y_{OF} = S_1 x_{AC} + S_2 x_{CS} + S_3 \delta + S_4 u_{OF} \tag{11}$$

The expressions for H_1, H_2, H_3, H_4, F_1, F_2, F_3, F_4, T_1, T_2, T_3, T_4, R_1, R_2, R_3, R_4, S_1, S_2, S_3, and S_4 are given in Appendix B.

The closed loop system is derived by substituting $u_{OF} = K_{OF} y_{OF}$ into the combined rotorcraft and control system plant equations. The resulting closed loop plant equations are given by equations 12, 13, and 14.

$$\dot{x}_{OF} = A_{CL} x_{OF} + B_{CL} \delta \tag{12}$$
$$y_{AC} = C_{CL1} x_{OF} + D_{CL1} \delta \tag{13}$$
$$y_{CS} = C_{CL2} x_{OF} + D_{CL2} \delta \tag{14}$$

The vector x_{OF} is equal to $[x_{AC} x_{CS}]^T$. The expressions for A_{CL}, B_{CL}, C_{CL1}, C_{CL2}, D_{CL1}, and D_{CL2} are provided in Appendix C. The closed loop system formulation here differs from traditional optimal control system formulations. Since control system parameters in the feedback and feedforward path may be present in K_{OF}, both the closed loop poles and zeros can be affected by a modification in K_{OF}.

3. Control System Parameter Optimization

This section presents an algorithm for modifying control system parameters and is based on solving an optimal output feedback tracking problem. The development is in line with Reference 1 except for some obvious differences in the inputs considered, plant definition, cost function, and method for computing the cost function. Optimal tracking controllers are known to be functions of the type of input that they are desired to track. For this work, the reference inputs used are step and sinusoidal inputs. It is important to recognize that although the control system parameters will be optimized for step and sinusoidal inputs, the tracker will perform successfully for all inputs.

3.1 Step Response Cost

For the step response cost, the input, δ, is given by a vector of step functions with the individual step function amplitudes contains in r_{OF}. Consider the following transformations.

$$\tilde{x}_{OF}(t) = x_{OF}(t) - \bar{x}_{OF} \tag{15}$$
$$\tilde{y}_{AC}(t) = y_{AC}(t) - \bar{y}_{AC} \tag{16}$$
$$\tilde{y}_{CS}(t) = y_{CS}(t) - \bar{y}_{CS} \tag{17}$$
$$\tilde{\delta}(t) = \delta(t) - \bar{\delta} = -r_{OF} \tag{18}$$

In equations 15, 16, 17, and 18, $\bar{}$ denotes a steady state value while $\tilde{}$ represents a perturbation value. Using equations 15, 16, 17, and 18 the dynamic equations of the perturbation state variables are,

$$\dot{\tilde{x}}_{OF} = A_{CL} \tilde{x}_{OF} \tag{19}$$
$$\tilde{y}_{AC} = C_{CL1} \tilde{x}_{OF} \tag{20}$$
$$\tilde{y}_{CS} = C_{CL2} \tilde{x}_{OF} \tag{21}$$

As can be seen from equations 19, 20, and 21, the transformations changes the tracking problem into a regulator problem with respect to the perturbation state variables. The steady state values of the system are,

$$\bar{x}_{OF} = -A_{CL} B_{CL} r_{OF} \tag{22}$$
$$\bar{y}_{AC} = C_{CL1} \bar{x} + D_{CL1} r_{OF} \tag{23}$$
$$\bar{y}_{CS} = C_{CL2} \bar{x} + D_{CL2} r_{OF} \tag{24}$$

It is desired for the control system input vector, r_{OF}, to track the rotorcraft system output vector, y_{AC}. The tracking error, $e(t)$, is defined as,

$$e(t) = y_{AC}(t) - r_{OF} \tag{25}$$

and the tracking error perturbations and steady state values are,

$$\tilde{e} = C_{CL1} \tilde{x}_{OF} \tag{26}$$
$$\bar{e} = (D_{CL1} - C_{CL1} A_{CL}^{-1} B_{CL} - I) r_{OF} \tag{27}$$

Consider a performance index, J_{SR}, which is comprised of a combination of tracking error perturbations, steady state tracking error, and control system input all excited by the step function defined above.

$$J_{SR} = \int_0^\infty (\tilde{e}^T Q \tilde{e} + \tilde{y}_{CS}^T R \tilde{y}_{CS}) dt + \bar{e}^T V \bar{e} \tag{28}$$

Q, R, and V are positive definite matrices chosen by the designer. Equation 28 is equivalent to,

$$J_{SR} = \int_0^\infty \tilde{x}_{OF}^T \mathcal{Q} \tilde{x}_{OF} dt + \bar{e}^T V \bar{e} \tag{29}$$

where,

$$\mathcal{Q} = C_{CL1}^T Q C_{CL1} + C_{CL2}^T R C_{CL2} \tag{30}$$

The vast majority of optimal output feedback algorithms compute the cost functional by using the solution of the Lyupunov equation, $A_{CL}^T P + P A_{CL} = \mathcal{Q}$ to directly compute J_{SR}. Here, equation 29 is computed by numerical quadrature. While solving equation 29 by numerical quadrature is considerably less efficient than the Lyupunov equation method mentioned above, it does avoid problems with ill conditioned systems. Since $\tilde{x}_{OF}$ is governed by a differential equation, initial conditions for $\tilde{x}_{OF}$ are required to solve for the time integral quadratic portion of the cost function. Using equation 15, the initial conditions for $\tilde{x}_{OF}$ are,

$$\tilde{x}_{OF}(0) = -\bar{x}_{OF} = A_{CL}B_{CL}r_{OF} \qquad (31)$$

In contrast to conventional optimal output feedback regulators, the initial conditions for $\tilde{x}_{OF}$ are specified by the problem definition.

3.2 Harmonic Response Cost

For the harmonic response cost, consider inputs of the form $\delta = r_{HR} \sin \Omega t$. The harmonic cost function is given by equation 32.

$$J_{SR} = \int_0^\infty e^T P e dt + \bar{e}^T W \bar{e} \qquad (32)$$

The matrices P and W are positive definite and chosen by the designer. The vector e is given by equation 25 with appropriate harmonic inputs creating y_{AC} and δ while $\bar{e}$ is the maximum error over on cycle at steady state.

3.3 Cost Minimization

The total cost function is given by,

$$J = J_{SR} + J_{HR} \qquad (33)$$

The control system parameter optimization problem is to minimize the cost function, J, where the independent variables are the variable control system parameters. Thus, the control problem is essentially a multidimensional optimization problem where the dimension of the optimization is equal to the number of variable control system parameters. Improved control system parameters, at least with respect to the cost function, can be computed by lowering the cost through suitable modification of the variable control system parameters. In this work, the simplex method for minimization of multivariable cost functions is used. Details of the method are available in Reference 2.

4. Higher Harmonic Rotor Controller

To illustrate the method above, a 0.4 scale BO105 rotor system used for higher harmonic rotor control experiments in the DNW wind tunnel is considered. Details on the higher harmonic rotor system and related experiments with the system can be found in References 3 and 4. The following presents a brief overview of the higher harmonic rotor system linear model.

The state space model utilized for design purposes is valid in a hovering flight condition and only represents the vertical axis. The mathematical model of the rotor includes collective flap and torsion blade flexibility modes and a two state servo actuator. The states of the model are the pitch angle at the blade root, vertical deflection of the blade tip, angular rotation of the blade tip, pitch angle of the blade tip, first flapping deflection mode, second flapping deflection mode, third flapping deflection mode, first torsion deflection mode, the derivatives of the above eight states, pressure difference of the servo actuator, and the displacement of the servovalve piston. Thus, the rotor and actuator plant model contains 18 state variables, 1 input which is the commanded actuator force, and 2 outputs to be tracked which are the rotor force and the swashplate displacement. The control system block diagram is given in Figure 4. The actuator force command, $FZSP$, is formed as a combination of the vertical rotor force error, $FRZ - FRZREF$, the swashplate displacement error, $ZSP - ZSPREF$, the swashplate displacement, ZSP, and the derivative of the swashplate displacement, $ZSPDOT$. The first control task is to guarantee an identity of the 4/rev rotor force, FRZ, with the commanded 4/rev rotor force, $FRZREF$, as time approaches infinity. This is achieved by feeding back the error signal, $FRZ - FRZREF$, through a prefilter and 4/rev inverted notch filter as shown in Figure 4. The second control task is to guarantee an identity of the mean swashplate displacement, ZSP, and the static reference signal, $ZSPREF$. This is achieved by feeding back the swashplate displacement error through an integrator. Additionally the swashplate displacement, and the derivative of the swashplate displacement is fed back in order to improve stability of the closed loop system.

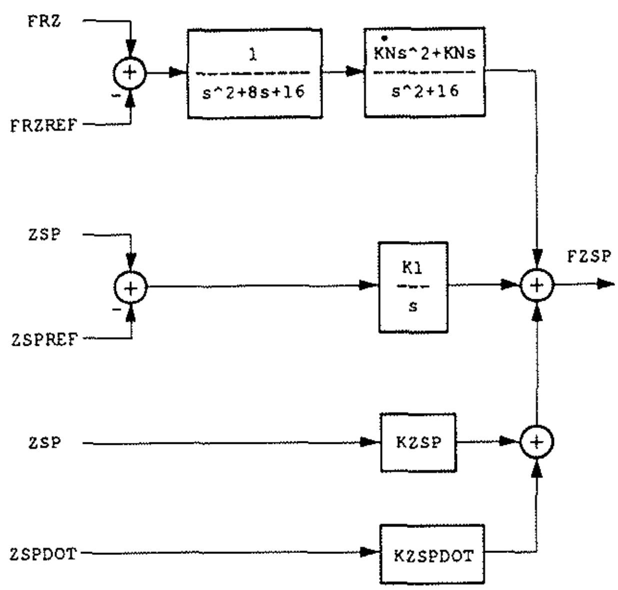

Figure 4: Higher Harmonic Rotor Controller Block Diagram

The control system thus contains a total of 5 state variables. The control system parameters that can be varied are $\dot{K}N$, KN, $K1$, $KZSP$, and $KZSPDOT$

and their baseline values are -12.584, -331.3, -120000, -325000, and 1050000, respectively. For simplicity, the matrices Q, R, V, P, and W have been assumed diagonal. The parameter values for the above matrices and r_{OF} and r_{HR} are given in Table . The resulting baseline total cost function equals 13.82. With the cost function

Matrix/Vector	Element 1	Element 2
Q	0	1
R	1	-
V	0	1
P	1	0
W	0	0
r_{OF}	0	1
r_{HR}	1	0

Table 1: Cost Function Weightings

weightings in Table , a combination of the mean value of the swashplate displacement tracking error perturbation, swashplate displacement steady state tracking error, steady state control effort, and the 4/rev rotor force is minimimized. Figure 29 shows the normalized accumulated cost function versus time. As can be seen in Figure 29, the cost function converges to its final value after approximately 20 non-dimensionalized seconds. The time step used for the time integration was 0.00044 and was based on having 10 integration points for the highest frequency oscillation.

The five variable control law parameters were optimized using the optimal output feedback tracking formulation developed above with respect to the weightings defined in Table . The cost function was reduced from 13.82 to 8.02 in 10 iterations. The optimized values for KN, $\dot{K}N$, $K1$, $KZSP$, and $KZSPDOT$ are -17.02, -172.35, -291858.9, -21929.2, 1495190.7, respectively. Figure 6 shows the 4/rev force versus time for the baseline and optimized system for $FRZREF = 100 \sin 4t$ and $ZSP = -0.01$. From Figure 6 it can be seen that both the baseline and optimized systems settle at approximately the same time, however, the FRZ 4/rev response of the baseline system exhibits overshoot which is not present in the optimized design. Figure 7 shows the mean swashplate displacement versus time for the baseline and optimized systems for $FRZREF = 100 \sin 4t$ and $ZSP = -0.01$. The optimized ZSP response tracks commands more quickly. However, the optimized ZSP response does have slight overshoot. It should be noted that due to the control system rigging geometry, negative perturbations in ZSP produce positive perturbations in rotor blade pitch angles.

The cost function weightings in Table are noticably simple. The simple cost function weightings were used since the point of the control system parameter optimization application was to exercise the algorithm. In reality, many modifications to the cost function weightings would be executed with subsequent control system parameter optimizations performed until a truly overall improved design emerged.

5. Concluding Remarks

A method for modification of control law parameters has been presented. The method is rooted in optimal control theory yet can be used in concert with current rotorcraft system control law design practice. The key to the methods utility is realization of the control system plant as a set of multiple input and single output filter arranged in an arbitrary fashion. With this control system arrangement, current rotorcraft system control laws can be modeled and parameters in the control system which are permitted to vary can be isolated into an output feedback matrix. Subsequently, optimal output feedback can be applied to the system and the overall system performance can be improved by minimizing a quadratic cost functional. It is important to stress that control law structure is not determined by the method. The technique has been applied to a 23 state higher harmonic rotor controller successfully.

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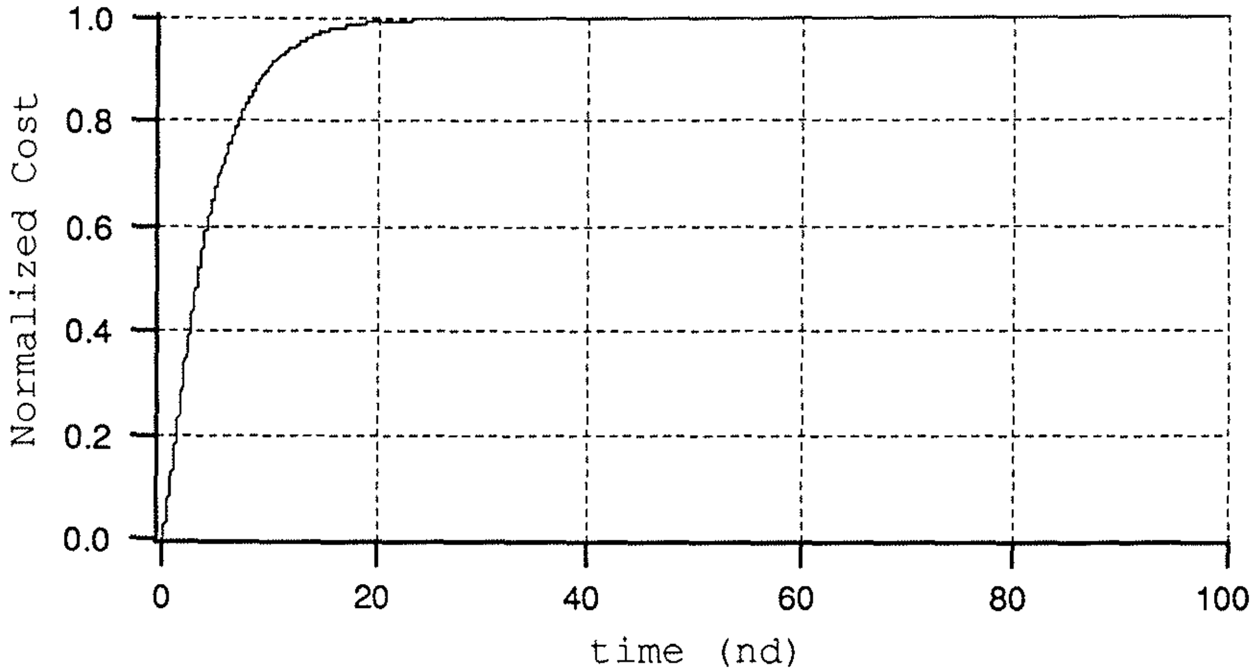

Figure 5: Accumulated Cost Function versus time

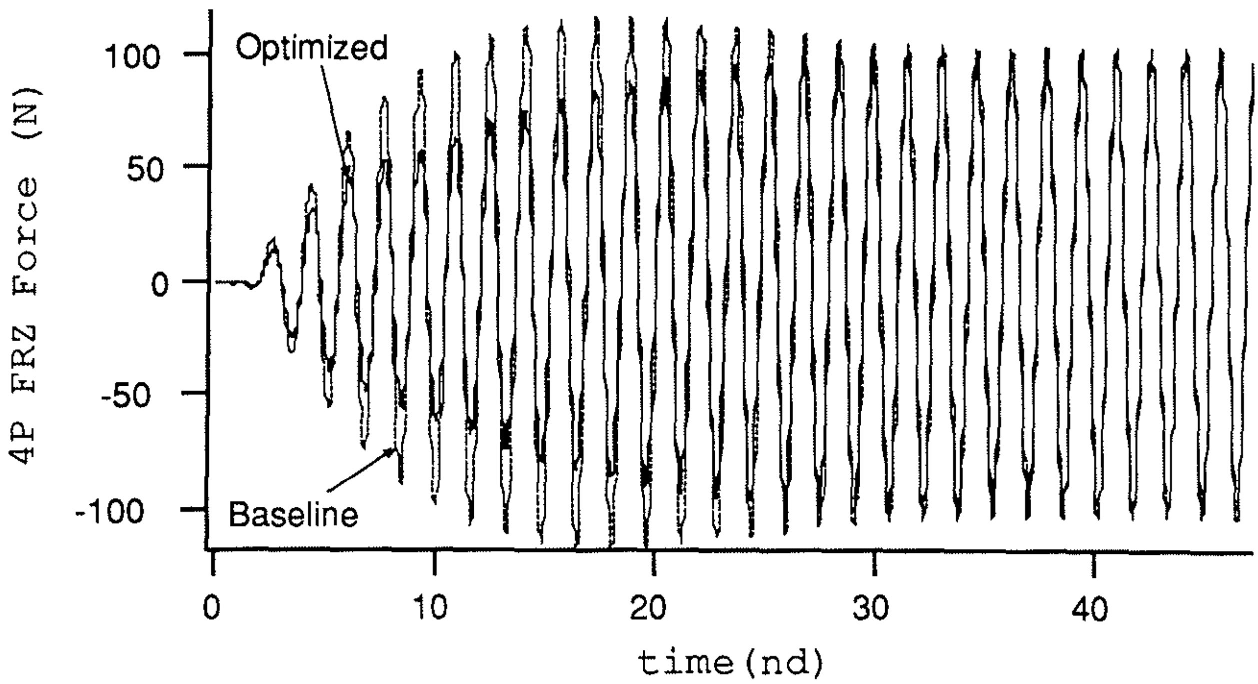

Figure 6: 4P Frz Force versus time

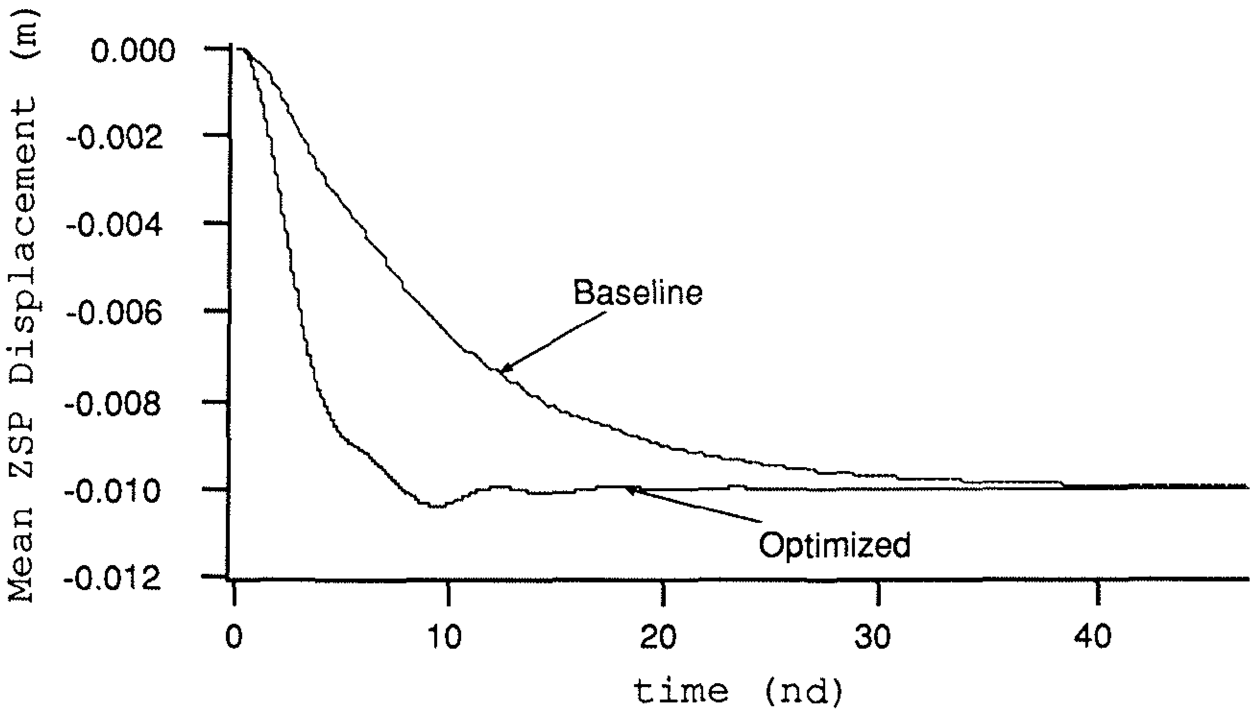

Figure 7: Mean Zsp Displacement versus time

Appendix A - Control System Realization

Independent of the type of control law methodology, control systems are generally described by a set of scalar block diagrams, however, the block diagram structure details are application specific. Thus, for modeling purposes it is desirable to allow the block diagram structure to be general and specified through the input data deck. The methodology promoted here assumes the control system is comprised of many filters arranged in an arbitrary fashion. Each filter is a multiple input and single output filter given in polynomial form. The inputs to each filter can consist of pilot stick inputs, outputs of individual filters, plant states, and derivatives of plant states. The basic control system data for each filter consists of the order of the filter, the numerator and denominator coefficients of the filter, the number of inputs to the filter, the input identifiers for the filter, and the gain value of each input to the filter. After the baseline control system data, the number of control system parameters that are to be varied is input along with the input identifier and filter number of the corresponding variable control system parameter. With this minimum set of input data a fully coupled state space control system model may be realized with the variable control system parameters separated into a constant diagonal matrix.

Each of the *ncsblk* control system filters is given in polynomial form, as shown in Figure 8. In Figure 8, $od(k)$ is the order of the kth filter, and $N_{k,i}$ and $D_{k,i}$ are the ith numerator and denominator polynomial coefficients of the kth filter, respectively. Depending on the input data, these parameters can be fixed or variable.

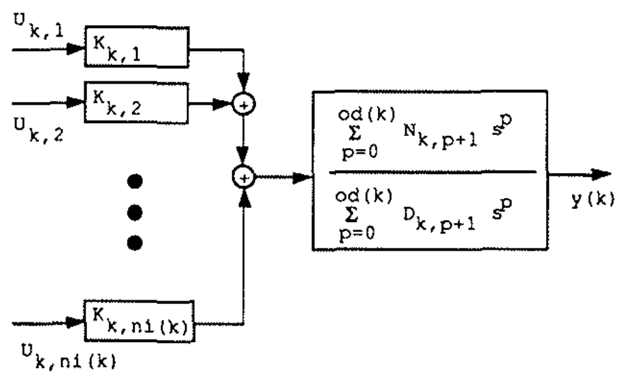

Figure 8: kth Control System Filter

Initially all the parameters of a filter are assumed to be variable. A filter of order $od(k)$ has $2od(k)+3$ inputs and outputs and the corresponding system matrices are populated with 0's and 1's only. This initial individual filter realization can be expressed as,

$$\dot{x}_s = A_s x_s + B_{s1} u_e + B_{s2} u_{iv} + B_{s3} u_{if} \tag{34}$$

$$y_e = C_{s1} x_s + D_{s11} u_e + D_{s12} u_{iv} + D_{s13} u_{if} \tag{35}$$

$$y_{iv} = C_{s2} x_s + D_{s21} u_e + D_{s22} u_{iv} + D_{s23} u_{if} \tag{36}$$

$$y_{if} = C_{s3} x_s + D_{s31} u_e + D_{s32} u_{iv} + D_{s33} u_{if} \tag{37}$$

where x_s is the state vector of the $od(k)th$ filter, u_e and y_e are the external input and output of the filter, u_{iv} and y_{iv} are the internal variable parameter inputs

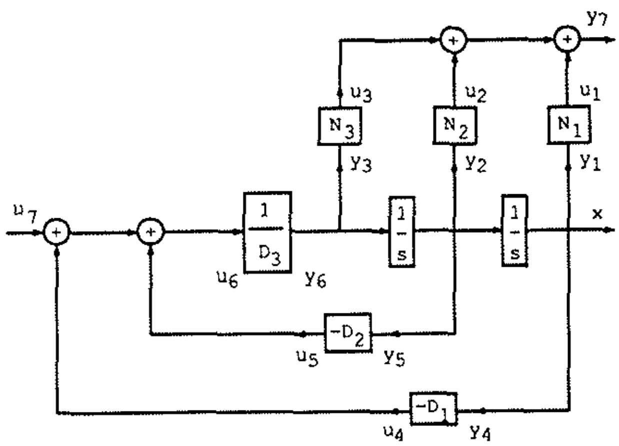

Figure 9: 2nd Order System Block Diagram

and outputs of the filter, and u_{if} and y_{if} are the internal fixed parameter inputs and outputs. The internal parameter inputs and outputs are related by,

$$u_{iv} = K_{iv} y_{iv} \tag{38}$$

$$u_{if} = K_{if} y_{if} \tag{39}$$

As an example, consider an arbitrary second order filter,

$$H(s) = \frac{N_3 s^2 + N_2 s + N_1}{D_3 s^2 + D_2 s + D_1} \tag{40}$$

where the block diagram for the system is given in Figure 9. Using Figure 9 as a guide and assuming that all six filter parameters are variable, the filter is initially realized as,

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} \tag{41}$$

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&1 \\ 1&1&1&0&0&0&0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} \tag{42}$$

If the parameters N_2 and D_1 are variable, then K_{iv} would be given by,

$$\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{bmatrix} N_2 & 0 \\ 0 & -D_1 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_4 \end{Bmatrix} \tag{43}$$

while K_{if} would be given by,

$$\begin{Bmatrix} u_1 \\ u_3 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & 0 \\ 0 & N_3 & 0 & 0 \\ 0 & 0 & -D_2 & 0 \\ 0 & 0 & 0 & 1/D_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_3 \\ y_5 \\ y_6 \end{Bmatrix} \tag{44}$$

Subsequent to initially realizing the filter, the parameters of the filter which are fixed are eliminated with the substitution $u_{if} = K_{if} y_{if}$. Thus each filter is then written as,

$$\dot{x}_s = \tilde{A}_s x_s + \tilde{B}_e u_e + \tilde{B}_v u_{iv} \tag{45}$$

$$y_e = \tilde{C}_e x_s + \tilde{D}_{11} u_e + \tilde{D}_{12} u_{iv} \tag{46}$$

$$y_{iv} = \tilde{C}_v x_s + \tilde{D}_{21} u_e + \tilde{D}_{22} u_{iv} \tag{47}$$

where,

$$\tilde{A}_s = A_s + B_{s3} K_{if} (I - D_{s33} K_{if})^{-1} C_{s3} \tag{48}$$

$$\tilde{B}_e = B_{s1} + B_{s3} K_{if} (I - D_{s33} K_{if})^{-1} D_{s31} \tag{49}$$

$$\tilde{B}_v = B_{s2} + B_{s3} K_{if} (I - D_{s33} K_{if})^{-1} D_{s32} \tag{50}$$

$$\tilde{C}_e = C_{s1} + D_{s13} K_{if} (I - D_{s33} K_{if})^{-1} C_{s3} \tag{51}$$

$$\tilde{C}_v = C_{s2} + D_{s23} K_{if} (I - D_{s33} K_{if})^{-1} C_{s3} \tag{52}$$

$$\tilde{D}_{11} = D_{s11} + D_{s13} K_{if} (I - D_{s33} K_{if})^{-1} D_{s31} \tag{53}$$

$$\tilde{D}_{12} = D_{s12} + D_{s13} K_{if} (I - D_{s33} K_{if})^{-1} D_{s32} \tag{54}$$

$$\tilde{D}_{21} = D_{s21} + D_{s23} K_{if} (I - D_{s33} K_{if})^{-1} D_{s31} \tag{55}$$

$$\tilde{D}_{22} = D_{s22} + D_{s23} K_{if} (I - D_{s33} K_{if})^{-1} D_{s32} \tag{56}$$

The individual filters are then assembled into an overall state space system where the filters are uncoupled from one another. The overall initial realization can be written as,

$$\dot{x}_{CS} = A_I x_{CS} + B_{I1} u_E + B_{I2} u_{V1} \tag{57}$$

$$y_E = C_{I1} x_{CS} + D_{I11} u_E + D_{I12} u_{V1} \tag{58}$$

$$y_{V1} = C_{I2} x_{CS} + D_{I21} u_E + D_{I22} u_{V1} \tag{59}$$

where x_{CS} is the state vector for the entire control system and includes the state of all control system filters, u_E and y_E are the vector of all filter inputs and outputs, and u_{V1} and y_{V1} are the input and output feedback vectors. It should be recognized that the filter definitions concatenated into equations 57, 58, and 59 are uncoupled.

The filters are coupled together using the basic filter data, in particular, the input identifiers, $U_{k,j}$, and gain values, $K_{k,j}$, of Figure 8. With this data the following coupling matrices can be directly formed.

$$u_E = K_1 y_E + K_2 \delta + K_3 x_{AC} + K_4 \dot{x}_{AC} + K_5 u_{II} \tag{60}$$

where, u_{II} is the input from a variable filter gain input, and,

$$u_{II} = K_{II} y_{II} \tag{61}$$

$$y_{II} = G_1 y_E + G_2 \delta + G_3 x_{AC} + G_4 \dot{x}_{AC} \tag{62}$$

The final control system model with the variable parameters separated from the plant equations is arrived at by substituting equation 60 into equations 57, 58, and 59. Also, the variable parameters, u_{II}, are appended onto u_{IV} to form the complete vector of variable parameters, u_{OF}. The final equations are,

$$\dot{x}_{CS} = J_1 x_{CS} + J_2 \delta + J_3 x_{AC} + J_4 \dot{x}_{AC} + J_5 u_{OF} \tag{63}$$

$$y_E = Q_1 x_{CS} + Q_2\delta + Q_3 x_{AC} + Q_4 \dot{x}_{AC} + Q_5 u_{OF} \quad (64)$$

$$y_{OF} = L_1 x_{CS} + L_2\delta + L_3 x_{AC} + L_4 \dot{x}_{AC} + L_7 u_{OF} \quad (65)$$

where,

$$J_1 = A_I + B_{I1} N_1 \quad (66)$$

$$J_2 = B_{I1} N_2 \quad (67)$$

$$J_3 = B_{I1} N_3 \quad (68)$$

$$J_4 = B_{I1} N_4 \quad (69)$$

$$J_5 = \begin{bmatrix} B_{I2} + B_{I1} N_5 & B_{I1} N_6 \end{bmatrix} \quad (70)$$

$$Q_1 = \Gamma M_1 \quad (71)$$

$$Q_2 = \Gamma M_2 \quad (72)$$

$$Q_3 = \Gamma M_3 \quad (73)$$

$$Q_4 = \Gamma M_4 \quad (74)$$

$$Q_5 = \Gamma M_7 \quad (75)$$

$$L_1 = \begin{bmatrix} P_1 \\ E_1 \end{bmatrix} \quad (76)$$

$$L_2 = \begin{bmatrix} P_2 \\ E_2 \end{bmatrix} \quad (77)$$

$$L_3 = \begin{bmatrix} P_3 \\ E_3 \end{bmatrix} \quad (78)$$

$$L_4 = \begin{bmatrix} P_4 \\ E_4 \end{bmatrix} \quad (79)$$

$$L_7 = \begin{bmatrix} P_5 & P_6 \\ E_5 & E_6 \end{bmatrix} \quad (80)$$

$$M_1 = [I - D_{I11} K_1]^{-1} C_{I1} \quad (81)$$

$$M_2 = [I - D_{I11} K_1]^{-1} D_{I11} K_2 \quad (82)$$

$$M_3 = [I - D_{I11} K_1]^{-1} D_{I11} K_3 \quad (83)$$

$$M_4 = [I - D_{I11} K_1]^{-1} D_{I11} K_4 \quad (84)$$

$$M_5 = [I - D_{I11} K_1]^{-1} D_{I11} K_5 \quad (85)$$

$$M_6 = [I - D_{I11} K_1]^{-1} D_{I12} \quad (86)$$

$$E_1 = G_1 M_1 \quad (87)$$

$$E_2 = G_2 + G_1 M_2 \quad (88)$$

$$E_3 = G_3 + G_1 M_3 \quad (89)$$

$$E_4 = G_4 + G_1 M_4 \quad (90)$$

$$E_5 = G_5 + G_1 M_5 \quad (91)$$

$$E_2 = G_1 M_6 \quad (92)$$

$$N_1 = K_1 M_1 \quad (93)$$

$$N_2 = K_2 + K_1 M_2 \quad (94)$$

$$N_3 = K_3 + K_1 M_3 \quad (95)$$

$$N_4 = K_4 + K_1 M_4 \quad (96)$$

$$N_5 = K_1 M_5 \quad (97)$$

$$N_6 = K_5 + K_1 M_6 \quad (98)$$

$$P_1 = C_{I2} + D_{I21} N_1 \quad (99)$$

$$P_2 = D_{I21} N_3 \quad (100)$$

$$P_3 = D_{I21} N_4 \quad (101)$$

$$P_4 = D_{I21} N_2 \quad (102)$$

$$P_5 = D_{I22} + D_{I21} N_5 \quad (103)$$

$$P_6 = D_{I21} N_6 \quad (104)$$

The matrix Γ restricts the control system output vector to consist of only plant inputs and not all filter outputs.

Appendix B - Output Feedback System Matrices

$$H_1 = [I - B_{AC} Q_4]^{-1} (A_{AC} + B_{AC} Q_3) \quad (105)$$

$$H_2 = [I - B_{AC} Q_4]^{-1} B_{AC} Q_1 \quad (106)$$

$$H_3 = [I - B_{AC} Q_4]^{-1} B_{AC} Q_2 \quad (107)$$

$$H_4 = [I - B_{AC} Q_4]^{-1} B_{AC} Q_5 \quad (108)$$

$$F_1 = J_3 + J_4 H_1 \quad (109)$$

$$F_2 = J_1 + J_4 H_2 \quad (110)$$

$$F_3 = J_2 + J_4 H_3 \quad (111)$$

$$F_4 = J_5 + J_4 H_4 \quad (112)$$

$$R_1 = Q_3 + Q_4 H_1 \quad (113)$$

$$R_2 = Q_1 + Q_4 H_2 \quad (114)$$

$$R_3 = Q_2 + Q_4 H_3 \quad (115)$$

$$R_4 = Q_4 H_4 \quad (116)$$

$$S_1 = L_3 + L_4 H_1 \quad (117)$$

$$S_2 = L_1 + L_4 H_2 \quad (118)$$

$$S_3 = L_2 + L_4 H_3 \quad (119)$$

$$S_4 = L_7 + L_4 H_4 \quad (120)$$

$$T_1 = C_{AC} + D_{AC} R_1 \quad (121)$$

$$T_2 = D_{AC} R_2 \quad (122)$$

$$T_3 = D_{AC} R_3 \quad (123)$$

$$T_4 = D_{AC} R_4 \quad (124)$$

Appendix C - Closed Loop System Matrices

$$A_{CL} = \begin{bmatrix} H_1 + H_4\nu_1 & H_2 + H_4\nu_2 \\ F_1 + F_4\nu_1 & F_2 + F_4\nu_2 \end{bmatrix} \quad (125)$$

$$B_{CL} = \begin{bmatrix} H_3 + H_4\nu_3 \\ F_3 + F_4\nu_3 \end{bmatrix} \quad (126)$$

$$C_{CL1} = \begin{bmatrix} T_1 + T_4\nu_1 & T_2 + T_4\nu_2 \end{bmatrix} \quad (127)$$

$$C_{CL2} = \begin{bmatrix} R_1 + R_4\nu_1 & R_2 + R_4\nu_2 \end{bmatrix} \quad (128)$$

$$D_{CL1} = T_3 + T_4\nu_3 \quad (129)$$

$$D_{CL2} = R_3 + R_4\nu_3 \quad (130)$$