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FINITE ELEMENT ANALYSIS AND MULTIBODY DYNAMICS ISSUES IN ROTORCRAFT DYNAMIC ANALYSIS

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Abstract

There is general agreement that the development of effective rotorcraft analysis software will require the use of modern computational mechanics methodologies, especially finite element analysis and multibody dynamics. This paper examines the analysis of rotorcraft dynamics from the perspective of these methodologies. First, a general discussion of rotorcraft analysis and modeling is presented. Then, a hierarchy of rotorcraft analyses is presented, ranging from simple to complex kinematics, where it is shown that in comprehensive rotorcraft software, finite element analysis must be augmented by multibody dynamics in order to properly analyze large motions of rotorcraft components. Finally, a review of multibody dynamics is presented to further familiarize the rotorcraft community with this technology.

1. Introduction

The development of analytical methods to predict rotorcraft aeromechanical characteristics is a challenge that has absorbed the continuous attention of rotorcraft engineers and researchers. Numerous specialized analyses have been developed to treat the various rotorcraft disciplines, such as aerodynamics, dynamics, flight control, propulsion, etc. A number of analyses have addressed multiple disciplines in order to predict interdisciplinary problems such as rotor loads and vibration. Because of the complexity of the rotorcraft aeroelastic phenomena and the highly coupled nature of the physical system, significant efforts in recent years have been devoted to comprehensive rotorcraft analyses. One of the most prominent of these is the Second Generation Comprehensive Helicopter Analysis System (2GCHAS), Ref. [1]. The term "comprehensive" is used here to mean that the analysis is broadly interdisciplinary, and treats aerodynamics, dynamics, flight controls, and propulsion in order to predict a wide range of rotorcraft problems, including performance, loads and vibrations, aeroelastic stability, stability and control, aerodynamics, and acoustics.

Until recently, comprehensive rotorcraft analyses were undertaken with so-called "first generation" codes (e.g., C60, C81, REXOR, CAMRAD) which are formulated around fixed structural models that provide no modeling flexibilty beyond letting the user input a limited set of model parameters. Morever, the usefulness of these codes in analyzing dynamic phenomena is further limited by the absence of an adequate large motion maneuver capability, and by the use of obsolescent theory for rotor blade analysis (see Ref. [3]). In recent years, government and industry have embarked on several major projects (i.e., RDYNE, COPTER, DYSCO) with the aim of improving rotorcraft analysis capabilities. For the most part, these new codes offer improved rotor blade analyses based on second order or geometrically exact kinematics (see Refs. [4], [5], [6], [7], [8]), and some additional flexibility in modeling fuselage components, but none of these codes can be regarded as having a true, general purpose modeling capability, and none has a rigorous large motion maneuver capability.

The 2GCHAS project has attempted to remedy the deficiencies of earlier codes by formulating the structural problem using classical finite element technology. The finite element method is essential to developing general purpose modeling capability, but this technology is largely restricted to small motion response analyses by virtue of fundamental assump-

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tions made in the finite element derivations. Recently, geometrically exact finite elements have been developed that can accomodate arbitrarily large rotations, but the technology of these elements has not yet fully matured, and in rotorcraft applications, these elements have mostly been applied to improving steady flight rotor blade analyses. Consequently, 2GCHAS, which is essentially an enhanced finite element code, is inherently prevented from analyzing arbitrary, large motion maneuver response of rotorcraft. As will be seen later, a crucial part of this limitation is the inability to handle large motion that is *arbitrary*. If the large motion is prescribed, the problem may be analyzed using a straightforward extension of the usual finite element method, which explains the ability of 2GCHAS and other second generation codes to account for steady-state rotor spin.

Along with finite element analysis, multibody dynamics is a computational mechanics methodolgy that can be useful in comprehensive rotorcraft analysis. This method pertains to the analysis of the large motion response of systems of interconnected bodies, which are generally assumed to be rigid. Clearly, multibody dynamics could be used to simulate the large motion response of a rotorcraft modeled as coupled rigid bodies, but it is inapplicable to deformable body analysis, so it alone cannot support the development of comprehensive rotorcraft software. Figure 1 illustrates the different features of finite element and multibody dynamics methods. Techniques are available that combine multibody and finite element analyses, and one application of these techniques employed in 2GCHAS software is the use of inertial stiffness and damping matrices to account for the effects of frame motion. While broader applications of these techniques are appropriate in some problems, these techniques are often quite costly at the most general level, and it seems clear that not all rotorcraft analyses require such generality.

In view of the above remarks, it is important that rotorcraft analysis requirements be fully understood in light of computational mechanics methods before software implementing these methods is designed. The first part of this paper is a broad overview of rotorcraft analysis tasks and the essential components of rotorcraft models. The accuracy of various assumptions made in analysis and modeling will be assessed within the context of comprehensive rotorcraft analysis goals. The second part of this paper is a discussion of rotorcraft analyses methods from the perspective of finite element and multibody dynamics methods. A hierarchy of increasingly complex analyses is described and their usefulness and restrictions in the context of comprehensive rotorcraft analysis is discussed. It will be seen that the most general type of comprehensive analysis requires involves the analysis of large motion dynamics of flexible bodies, and combines aspects multibody and finite element analyses. The last section of the paper presents a brief review of the multibody dynamics literature to assess the suitability of this technology for rotorcraft analysis applications. This paper is a revised version of Ref. [2].

2. Rotorcraft Analysis and Modeling

Before proceeding further, it will be useful to discuss in general terms some of the specialized aspects of rotorcraft analysis and modeling from the perspective of structural dynamics. This is intended to prepare the way for more detailed analytical treatment of rotorcraft dynamics and better understand the dynamics issues addressed in this paper.

2.1 Rotorcraft Analysis Types

A variety of diferent types of analyses are used to treat rotorcraft problems. These analyses are usually tailored to the specific needs of the desired application; many are adequately served without addressing the full complexity of the rotorcraft system. For example, rotor blade loads analyses commonly assume the rotor hub is fixed to a rigid support and completely ignore any coupling between the rotor and the fuselage system. Such an assumption is generally suitable for calculating rotor blade aerodynamic loads and vehicle performance characteristics. Although usually very inaccurate, rotor hub vibratory loads obtained for the fixed hub condition are sometimes used to estimate fuselage vibration response by applying those isolated rotor loads to an uncoupled fuselage structure.

A more complete special case of the general problem treats rotor-fuselage dynamic coupling for the trimmed flight condition, where the vehicle exhibits steady state periodic responses. Several important simplifications from the general problem may be invoked. For steady state conditions, all vibratory motions of the vehicle may be taken as small (in a general sense). The steady state operating condition is an important one where the designer seeks to obtain accurate predictions of rotor system dynamic loads and the best possible estimates of fuselage vibration. Full rotor fuselage dynamic coupling is an important factor in these properties.

The most general problem involves arbitrary transient motion of the vehicle in flight, including dynamic coupling between the rotor and fuselage subsystems. Such an analysis would be required to accurately predict dynamic loads and vehicle vibrations

during maneuvering flight conditions. These conditions are especially difficult because at the limiting maneuvering conditions of the vehicle, the complex fluid flow problems such as blade stall, vortex interactions, and compressibility effects are strongest. In addition, nonlinear structural deformations are the largest, and velocities and accelerations are high and changing rapidly. Such maneuver conditions are especially important because the maximum structural stress and vibrations significantly influence the operational military capabilities of the vehicle. Therefore, it is important that designers be able to predict rotorcraft characteristics as accurately as possible. Full treatment of blade flexibility, rotor-fuselage-drive train dynamic coupling, large scale vehicle rigid body motions, and rotor rotational speed variations must be addressed. In summary, the most difficult and demanding problems for rotorcraft dynamic analysis occur during maneuvering flight and this application is primarily responsible for the fundamental dynamics issues addressed in this paper.

2.2 Rotorcraft Modeling

The complexity of rotorcraft systems leads to a wide range of anlaytical methods tailored to the physical characteristics and behavior of the different components of the system. A brief discussion of kinematics concepts used to encompass the behavior of these components and a description of the basic components will be presented.

2.2.1 Kinematic Concepts

Rotorcraft elastic motions are generally measured with respect to frames. Different frames may be used for different portions of the rotorcraft structure and they may or may not be attached directly to a point on the structure, the choice made according to the requirements of the analysis. The frame motion generally represents motion of the rigid body degrees-of-freedom of the structure, and this motion may be large or small, and prescribed or nonprescribed, depending on the requirements of the particular dynamic analysis.

Rotorcraft elastic deformations may also be small or large in rotorcraft. Except for rotor blades, most rotorcraft elastic deformations are small. Rotor blade deformations are at least moderately large, and must be treated as finite rotations. Formulations for very large deformations (small strain) have also been developed, but the moderate deformation analyses are usually adequate.

2.2.2 Rotorcraft Components

A comprehensive rotorcraft analysis must be applicable to a wide range of different types of vehicles including single or tandem rotor helicopters, compound helicopters, and tiltrotors, to name a few common examples. Similarly, a variety of rotors such as articulated, hingeless, and bearingless rotor types, all having distinctly different physical characteristics, must be treated. Other rotorcraft components present additional modeling requirements for dynamic analysis. The purpose of this section is to describe the physical features and unique physical properties of the key rotorcraft components and then discuss modeling and dynamics issues that pertain to them.

1. Fuselage. For the purpose of comprehensive dynamic analysis, rotorcraft fuselages present few special issues, especially in comparison with other parts of the rotorcraft systems. The material properties are usually taken to be linear and elastic deformations are small and may be treated with a suitable conventional linear finite element analysis, e.g., NASTRAN. When approximations are appropriate, simplified modal representations may be used. It is to be emphasized, however, that the analysis of complex fuselage structures cannot be regarded is routine, in view of the sensitivity of vibration response to such subtlties as large cutouts, concentrated masses, secondary structures and attachments, isolator subsystems, and nonlinear structural damping.

2. Rotor Blade. Rotor blade structural dynamics is a central concern of modern rotorcraft analysis. The large centrifugal forces of long slender rotating beams limit elastic deformations, but in general, and particularly for cantilever (hingeless and bearingless) blades, these deformations are sufficiently large to cause nonlinear kinematic couplings between bending and torsional motions, which are important in blade aeroelastic phenomena. Accurate analyses of these phenomena therefore require that blade deformations be treated as moderately large, which means that terms of at least second order must be considered when describing the kinematics of blade rotations. In sum, the elastic rotor blade will undergo moderate elastic deformations with respect to a reference frame, while the rotor blade frame will undergo very large motions.

3. Rotor Blade Articulation. Articulated and semi- articulated rotor blades have flap and lead-lag hinges or flap hinges respectively to accommodate rotor blade motions in flight and relieve blade root stresses. The rigid body rotation of the blades about these hinges may be large and in general cannot be treated as a small rotation.

4. Rotor Blade Feathering. Virtually all rotor systems employ cyclic and/or collective feathering to

control the rotor forces acting on the vehicle. Feathering is a rigid body pitch rotation about the blade root feather hinge (or torsion of a bearingless rotor blade flexbeam) produced by mechanical action of the swashplate and pushrod links connected to the blade pitch horns. The applied feathering control must be regarded as a finite rotation, but the feathering contributed by elastic deformation of the pushrods may be regarded as small perturbations. A key issue for structural analysis is the large periodic rotation of blade section principal elastic axes that occurs when cyclic pitch is applied to the rotor blades.

5. Rotor-Body Coupling. One of the most important rotorcraft dynamics analysis issues involves coupling between the rotor(s) and the fuselage. Since the equations for the two major subsystems, the rotor and the fuselage, are usually associated with different reference frames, rotor-body coupling basically deals with transformation of variables between the two reference systems, a nonrotating frame associated with the fuselage and a rotating frame associated with the rotor. This topic will be dealt with more explicitly in the analytical treatment that is presented below. In general, the rotor frame experiences large rotation motion, while the fuselage frame may or may not experience large rotation, depending on whether a steady state or a maneuvering flight condition exists. Other aspects of rotor-body coupling involve multiple rotor- fuselage coupling, rotor-to-rotor coupling, and the distinctions between coupling of rotor-fuselage pitch and roll moments and coupling of the rotor-fuselage shaft torque moment. In general, multiple rotor-fuselage coupling is similar to coupling of a single rotor. Rotor-to-rotor coupling and coupling of the rotor-fuselage shaft torque moment will be addressed further under the discussion of the rotor-drive train coupling.

6. Rotor-Drive Train Coupling. Coupling of the rotor(s) and drive train can be considered as part of the general subject of rotor-body coupling. A typical rotorcraft will generally consist of a fuselage and two rotor systems. The rotors are connected via a drive shaft system and are also connected to the propulsion system that provides drive torque to overcome blade aerodynamic drag. There are two questions of interest: 1) relating the rotorshaft spin degree-of-freedom to the fuselage degrees-of-freedom and 2) modeling the internal degrees-of-fredom of the rotor-drive train system. This discussion may be aided by considering a simple example. In the first case, a rigid rotor fixed to a rigid shaft is coupled to rigid fuselage in such a way that the rotor shaft spins freely in the fuselage. In this case the six rigid body degrees-of-freedom of each subsystem reduce to a total of seven for the coupled system. The seven consist of the six rigid body fuselage degrees-of-freedom and the rotor-shaft spin degree-of-freedom. The other five rotor degrees-of-freedom are constrained out by coupling with the fuselage. This example essentially represents the autorotation flight condition. Addition of a second rotor, engine, transmission, and a flexible drive shaft produces a complete drive train dynamic system that will include internal torsional elastic degrees-of-freedom. The rotor-drive train will retain the seventh rigid body shaft degree-of-freedom of the coupled rotor-body system until further constraint is applied to the system. This constraint is the interface shaft torque between the rotor-drive train and fuselage. In realistic rotorcraft systems, this constraint is provided by the engine RPM governor control system. The consequences for rotorcraft dynamics analysis is that under some conditions the rotor shaft spin degree-of-freedom may be analogous to a seventh rigid body vehicle degree-of-freedom and under other conditions it may be analogous to an internal elastic degree of freedom. These conditions will determine whether the rotor frame motion is large or small.

7. Swashplate. The swashplate transfers flight control actuator motions from the fuselage in the fixed system – collective, lateral cyclic, and longitudinal cyclic pitch – to rotor blade feather motion in the rotating system. The swashplate also mechanically transforms cyclic pitch into individual blade pitch motions. It generally comprises structural members that rotate with respect to each other and connect to both the fixed system flight control actuators and the blade pitch control push rods. In its simplest sense, it may be represented by simple kinematic equations relating fixed system rotor-pitch variables to rotating system blade pitch variables. A more elaborate representation could model the mass and stiffness properties of the swashplate components, including azimuthal variations in stiffness. An accurate representation of the swashplate must also account for the reaction forces transmitted by actuator links and pushrods between the fixed and rotating systems.

8. Lag Dampers. Articulated rotors and many hingeless and bearingless rotor systems incorporate mechanical dampers, both hydraulic and elastomeric, to provide blade lag damping in the rotor system. These dampers are strongly nonlinear, especially the hydraulic type. Generally this is not an issue from the point of view of structural dynamics analysis; such components must be modeled empirically or represented as a force element having numerically defined properties, and the issue becomes one of most efficiently treating such a representation in the numerical solution process.

It should be noted that not all of the analytical re-

quirements for modeling these rotorcraft components are addressed in the material that follows. However, the above discussion is intended to survey sources of dynamics analysis issues in a broad sense and provide perspective for the material that follows.

3. Small Deformation Analyses for Elastic Rotorcraft

This section develops the equations for several generic dynamic analyses that are useful for rotorcraft dynamics. The analyses are developed in progressive fashion by using increasingly general kinematic assumptions to development sets of governing equations. The progression culminates in the most general problem of a rotorcraft experiencing small elastic deformations and arbitrarily large rigid body motions. The analytical derivations do not include all steps; they are meant to illustrate the basic approach and relevant dynamics issues. The progressive development yields specific equations suitable for the range of rotorcraft analysis types described earlier, and clearly reveals the applicability of finite element analysis and multibody dynamics in the various stages of rotorcraft analysis.

3.1 Small Motions Relative to an Inertial Frame

This section treats the dynamics of an elastic body undergoing small deformations relative to an inertial frame. Kinematically speaking, these are the simplest problems, and they can be analyzed using the well-known, general purpose finite element codes for solid mechanics applications. For the purposes of this discussion, "small deformations" include the possibility of moderately large deformations of rotor blades. In the finite element method, the body being analyzed is subdivided into subregions called elements that are connected at points called nodes. The displacement within an element interior is interpolated from nodal displacements in a manner that depends on the structural behavior modeled by the element (e.g., beam, plate, shell, etc.). Thus, the displacement of a generic point in the body may be expressed as:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_l + \mathbf{u}_{nl} \\ \mathbf{u}_l &= [N(\mathbf{r}_0)]\{q\} \\ \mathbf{u}_{nl} &= \mathbf{u}_{nl}(\mathbf{r}_0, \{q\}) \end{aligned} \tag{1}$$

where $\mathbf{u}$ denotes the displacement of a generic point whose position within the undeformed structure is $\mathbf{r}_0$, $\mathbf{u}_l$ and $\mathbf{u}_{nl}$ denote the linear and nonlinear contributions to the displacement, $\{q\}$ is a vector of nodal degrees-of-freedom, and $[N(\mathbf{r}_0)]$ is a matrix of interpolation functions. Introducing equation (1) into the principle of virtual work, leads to the following set of equations (Ref. [9])

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F_{ext}\} + \{F_{nl}\} \tag{2}$$

where $[M]$, $[C]$, and $[K]$ are the familiar mass, damping and stiffness matrices with:

$$\begin{aligned} [M] &= \int_V \rho [N]^T [N] dV \\ [K] &= \int_V [B_l]^T [D][B_l] dV \end{aligned}$$

where $[B_l]$ represents the linear part of the strain-displacement equations applied to $[N]$, $[D]$ represents the stress-strain constitutive relations, which are assumed linear, and $\{F_{ext}\}$ are the consistent nodal loads contributed by external forces, including aerodynamic and gravitational forces. The damping matrix, $[C]$, contains only structural effects, and is usually determined empirically. The $\{F_{nl}\}$ loads arise from the u_{nl} term in equation (1), and represents geometric stiffness. In rotorcraft applications, geometric stiffness is mostly significant in rotor blade behavior, and is responsible for phenomena such as bending-torsion coupling and centrifugal stiffening. In all the equations presented herein, geometric nonlinearities will be retained on the right-hand side, but it is often necessary to put geometric nonlinearities on the left-hand side when solving these equations, in order to obtain stable and robust solution algorithms.

The degrees-of-freedom correspond to the motion of node points. In what follows, it is assumed that the degrees-of-freedom in the vector $\{q\}$ are arranged according to node; i.e.,

$$\{q\}^T = \{\{q_1\}^T, \{q_2\}^T, \ldots, \{q_n\}^T\} \tag{3}$$

where n is the number of nodes and $\{q_i\}$ are the degrees-of-freedom for node i. It is further assumed that the degrees-of-freedom for each node are arranged as follows:

$$\{q_i\}^T = \{u_{i1}, u_{i2}, u_{i3}, \theta_{i1}, \theta_{i2}, \theta_{i3}\} \tag{4}$$

where the u's are the displacement degrees-of-freedom, and the θ's are the rotation degrees-of-freedom. For simplicity, it is assumed that all nodal quantities within a given body are referred to the same coordinate system.

A small motion, deformable body analysis may be used for detailed stress analysis; typically the dynamic terms ($[M]\{\ddot{q}\} + [C]\{\dot{q}\}$) are determined from a separate analysis of a relatively coarse structural model, and then placed on the right-hand side of

equation (2). The most common applications of this analysis to rotorcraft dynamics are fuselage vibration response, and eigenmode analysis. If a structrual component is linear, that component may be represented in an analysis by a reduced set of eigenmodes, but it is well-known that representing a nonlinear component (e.g., a rotor blade) with a reduced set of linear eigenmodes can lead to serious errors. A linear eigenmode analysis can also provide essential data for the design of rotorcraft components; for example, it may be necessary to know if the natural frequencies of a component lie within certain bounds in order to satisfy vibration control requirements.

3.2 Small Motions Relative to a Prescribed Moving Frame

The analysis just considered is applicable only if the body undergoes small motions in an inertial frame. The deformations of a helicopter rotor blade are small only withn a rotating frame, which is noninertial, and the effects of frame rotation on dynamic response of the blade are profound and must considered in a dynamic analysis. This section expands the scope of the previous section to motions of bodies relative to frames undergoing prescribed motion. This analysis, although not sufficient for a completely general rotorcraft analysis, has many important applications and represents the type of analysis performed by most rotorcraft codes.

3.2.1 Isolated Bodies

The development here considers the general case where frames have translational motion and nonzero angular acceleration. This type of analysis is most commonly applied to fixed hub rotor blade response, in which case the prescribed frame motion is the constant speed angular rotation of the rotor frame.

A finite element formulation of this analysis will now be presented. Referring to Figure 2, the motion of a generic point on a deformable body may be represented as:

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{r}_0 + \mathbf{u} \tag{5}$$

where $\mathbf{R}_0$ is the position vector from the origin of the inertial frame to the moving frame's origin. The terms $\mathbf{r}_0$, $\mathbf{u}$, and $\{q\}$ have the same meanings as in equation (1), but all these quantities are relative to the moving frame. The motion of the frame is characterized by its angular velocity ($\vec{\omega}$), and its translational velocity ($\vec{V}_0$), both of which are prescribed. A kinematic issue that must be resolved is how the actual body is constrained to move relative to the prescribed frame section. It will become apparent that how this is done depends in part on the application. Since the principal application of this analysis is to fixed hub response, it is assumed that the frame is rigidly attached to the body, but, this assumption is not appropriate for other analyses involving prescribed frame motion.

Utilizing equation (5) in the principle of virtual work leads to the following equations of motion (Refs. [11], [13]):

$$[M]\{\ddot{q}\} + [C']\{\dot{q}\} + [K']\{q\} = \{F\} \tag{6}$$

where:

$$\begin{aligned}
[C'] &= [C] + 2\int_V \rho[N]^T[\tilde{\omega}][N]dV \\
&= [C] + [C_I] \qquad (7) \\
[K'] &= [K] + \int_V [N]^T([\tilde{\omega}][\tilde{\omega}] + [\dot{\tilde{\omega}}])[N]dV \\
&\equiv [K] + [K_I] \qquad (8) \\
\{F\} &= \{F_{nl}\} + \{F_{ext}\} - \\
&\quad -[P][\tilde{\omega}]\{V_0\} - [H]\{\dot{\omega}\} - \\
&\quad -(\int_V \rho[M]^T[\tilde{\omega}][\tilde{\omega}]dV)\{r_0\} - \\
&\quad -[P]\{\dot{V}_0\} \qquad (9) \\
[P] &= \int_V \rho[N]^T dV \qquad (10) \\
[H] &= -\int_V \rho[N]^T[\tilde{r}_0]dV \qquad (11)
\end{aligned}$$

and $\{\omega\} = \{\omega_1, \omega_2, \omega_3\}^T$ and $\{V_0\} = \{V_{01}, V_{02}, V_{03}\}^T$ denote the components of the angular and translational velocities of the frame. A tilde over a vector signifies a skew-symmetric matrix containing the vector components; e.g.

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{12}$$

As before, $\{F_{nl}\}$ contains geometric stiffness effects, but as will be seen shortly, the presence of nonlinear constraints can contribute terms to $\{F_{nl}\}$.

It may be seen that equation (6) differs from equation (5) by inertial contributions to the damping and stiffness matrices ($[C_I]$ and $[K_I]$), and by frame-induced contributions to the forcing functions on the right-hand side of the equation. $[C_I]$ represents Coriolis acceleration; it is antisymmetric and therefore does not result in any energy dissipation. $[K_I]$ contains a symmetric part, which comes from centripetal acceleration, and an antisymmetric part, which comes from angular acceleration. If the aerodynamic forces are included in the analysis, then the nonlinearities in these forces in $\{q\}$ and $\{\dot{q}\}$ will also appear in $\{F_{nl}\}$.

In view of the structure of $[C_I]$ and $[K_I]$, and it would appear that equation (6) is of the same general form as equation (2), and that equation (6) may therefore be assembled and solved using the same, well-known techniques employed in the small deformation finite element codes. This is largely the case, and this fact underlies the design philosophy for the first level release of 2GCHAS. However, the geometric nonlinearities that are important in rotor blade response require special handling, which will now be explained.

Geometrically nonlinear phenomena that are crucial in the rotor blade response include stiffening effects of centrifugal forces, the effects of blade foreshortening on Coriolis forces, and coupled bending-torsion effects. It is well known (Ref. [4]) that in order to account for these phenomena, the analysis of rotor blade beam kinematics must assume that elastic rotations are finite and at least "moderately large." The need to treat finite rotations for rotor blades significantly impacts the assembly process as well as the formulation of rotor blade finite elements. Finite rotations are not vector quantities, and this often leads to nonlinear constraints, as opposed to the linear constraints that are assumed by most assembly processors. For example, when rigidly joining nodes that are parameterized with different orientations angles, as when joining rotor blades at skew angles, it is impossible to select a single set of parameters to which one can linearly relate the orientation angles of each element. The simplest solution to this problem is to use alternative rotational degrees-of-freedom, such as incremental rotations, that simplify the transformation process, but if the orientation angles are retained, the assembly process must account for the nonlinear relationship between the element degrees-of-freedom at the skewed joint. Nonlinear constraints also arise when coupling blade segments at an articulated joint, and when coupling blade feathering rotations to pitch link motions, but in both these instances, the nonlinearity cannot be removed by an alternative parameterization of the rotational degrees-of-freedom. An assembly method has been proposed (Ref. [12]) for a limited class of nonlinear constraints that uses the nonlinear transformation to eliminate redundant degrees-of-freedom; while this method is conceptually straightforward, the elimination process is involved, and the resulting equations are quite complicated. An alternative is to couple the element rotations using Lagrange multipliers (Ref. [21]) which leads to simpler equations at the expense of additional degrees-of-freedom. Note that both of the latter assembly processes will contribute additional terms to $\{F_{nl}\}$ in equation (6) because the constraint is nonlinear.

The foregoing discussion assumes that the elastic rotations of the rotor blade are relative to the *moving* frame, and that these rotations must therefore include the effects of blade articulation due to the presence of a hinge or pitch bearing. If articulation occurs, an alternative analysis procedure is to treat the articulation as a frame motion and to assume that elastic displacements of the blade are relative to the articulated frame. A difficulty of this procedure is that the articulated frame motions are unknown, and solving for these unknowns requires significant enhancements to the assembly and solution processes. Solving for unknown frame motions is a problem that arises in more general rotorcraft analyses, and will be discussed in more detail later.

3.2.2 Coupled Bodies

In order to consider a complete rotorcraft model, the previous analysis must be expanded to consider multiple bodies, with each body moving in its own prescribed frame. The problem of dynamic response of a coupled rotor-fuselage system of a rotorcraft in trimmed steady state flight is the most common example of coupled body analysis under prescribed frame motion. The identification of the "bodies" is problem dependent, but in rotorcraft applications, the bodies are most often the fuselage and the rotors, and a frame with prescribed motion is assigned to each body. A discussion of linear rotor-fuselage coupling is given in Ref. [14].

Conceptually, this analysis can be separated into two tasks: first, the equations of the separate bodies are formulated, and then the separate sets of equations are coupled to reflect the joining of the bodies. Both these tasks will now be discussed.

The equations of each body may be formulated using equation (6), but as mentioned earlier, a question that must first be resolved is how bodies are coupled to their respective frames. In the context of the analysis assumptions, the fuselage frame motion is the flight path of the rotorcraft, and the rotor frame motion translates with the fuselage frame and rotates relative to it in some prescribed fashion. The combined rotor and fuselage frames can therefore be thought of as a fictitious rigid rotorcraft that is used for formulating the dynamic response analysis of the vehicle. Observe that the frames are really fictitious rigid bodies that are assumed to define, to within an elastic perturbation, the motion of the flexible body. The exact motion of any point on a body, however, cannot be prescribed because it is an unknown that must be solved for. Inasmuch as the frame motions are entirely prescribed, the bodies must not be constrained relative to their frames if the rotorcraft re-

sponse problem is to be properly posed.

The equations of the separate bodies must be solved subject to the constraint that the responses of the bodies are the same at the point where the bodies are joined. The imposition of this constraint on the equations of motion is the well-known "rotor-body coupling problem." The two most commonly used rotor-body coupling methods will now be briefly discussed. In the analysis that follows, it is assumed for simplicity that the vehicle consists only of one rotor and a fuselage.

The "fully coupled" method for rotor-body coupling involves eliminating redundant degrees-of-freedom so that the rotor and fuselage equations are combined into one unified set. Recalling the earlier discussion of rotor blade kinematics, rotations of the rotor blade relative to its frame must usually be regarded as finite, but the rotations at the hub relative to the rotating frame are small enough to justify their being treated using the small angle approximation. As a consequence, rotor-body coupling may be accomplished with a linear transformation. Formally, the process proceeds as follows. First, partition the rotor and fuselage degrees-of-freedom according to:

$$\begin{aligned}
\{q_r\} &= \{q_{rh}, q_{rr}\}^T \\
&= \{\{q_{rh}\}^T, \{q_1\}^T, \{q_2\}^T, \ldots\} \\
&= \{u_{rh1}, u_{rh2}, u_{rh3}, \theta_{rh1}, \theta_{rh2}, \theta_{rh3}, \ldots\} \\
\{q_f\} &= \{q_{fh}, q_{ff}\}^T \\
&= \{\{q_{fh}\}^T, \{q_1\}^T, \{q_2\}^T, \ldots\} \\
&= \{u_{fh1}, u_{fh2}, u_{fh3}, \theta_{fh1}, \theta_{fh2}, \theta_{fh3}, \ldots\}
\end{aligned}$$

where the subscripts rh and fh, respectively may be interpreted as identifiers of nodes on the rotor and fuselage where the bodies are joined. The subscripts rr and ff refer, respectively, to nodes on rotor and fuselage that are not at the attachment points. Specializing equation (6) to the rotor and fuselage bodies and collecting the equations leads to:

$$[\bar{M}]\{\ddot{\bar{q}}\} + [\bar{C}]\{\dot{\bar{q}}\} + [\bar{K}]\{\bar{q}\} = \{\bar{F}\} \qquad (13)$$

where:

$$\begin{aligned}
[\bar{M}] &= \begin{bmatrix} M_f & 0 \\ 0 & M_r \end{bmatrix} \\
[\bar{C}] &= \begin{bmatrix} C'_f & 0 \\ 0 & C'_r \end{bmatrix} \\
[\bar{K}] &= \begin{bmatrix} K'_f & 0 \\ 0 & K'_r \end{bmatrix} \\
\{\bar{F}\} &= \begin{Bmatrix} F_f \\ F_r \end{Bmatrix} \\
\{\bar{q}\} &= \begin{Bmatrix} q_f \\ q_r \end{Bmatrix}
\end{aligned}$$

Note that equation (13) applies to bodies that have been aggregated, but not yet coupled.

Figure 3 shows the geometry of the undeformed and deformed locations of the fuselage and rotor in their respective frames. It is assumed that the coordinate systems are oriented so that the 3-axes of the fuselage and rotor frames are parallel to the spin axis, which means that vector components parallel to these axes are the same in both coordinate systems. Vector components along the 1 and 2 axes in the different frames are related by a simple rigid body rotation, and therefore:

$$\begin{aligned}
\{q_{rh}\} &= \begin{bmatrix} T_{fr} & 0 \\ 0 & T_{fr} \end{bmatrix} \{q_{fh}\} \\
&= [T_h]\{q_{fh}\} \qquad (14)
\end{aligned}$$

where:

$$[T_{fr}] = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (15)$$

where:

$$\Psi = \Psi(t) \qquad (16)$$

is the rotor azimuth angle. In steady flight conditions, $\Psi = \Omega t$ where Ω is the rotor rotational speed, but it will be assumed here that $\Psi(t)$ is a general function of t.

Because of the constraint, there are redundant degrees-of-freedom, which must be eliminated. Let the retained system degrees-of-freedom be:

$$\{q_{sys}\}^T = \begin{Bmatrix} q_{fh} \\ q_{ff} \\ q_{rr} \end{Bmatrix} \qquad (17)$$

In view of equations (14) and (17), we have:

$$\begin{aligned}
\{q_f\} &= \begin{bmatrix} [I] & 0 & 0 \\ 0 & [I] & 0 \end{bmatrix} \{q_{sys}\} \\
&\equiv [T_{fs}]\{q_{sys}\} \qquad (18) \\
\{q_r\} &= \begin{bmatrix} [T_h] & 0 & 0 \\ 0 & 0 & [I] \end{bmatrix} \{q_{sys}\} \\
&\equiv [T_{rs}]\{q_{sys}\} \qquad (19)
\end{aligned}$$

or:

$$\begin{aligned}
\begin{Bmatrix} q_f \\ q_r \end{Bmatrix} &= \begin{bmatrix} T_{fs} & 0 \\ 0 & T_{rs} \end{bmatrix} \{q_{sys}\} \\
&\equiv [T_{sys}]\{q_{sys}\} \qquad (20)
\end{aligned}$$

Substituting equation (20) into equation (6) and premultiplying by $[T_{sys}]^T$ yields:

$$[M_{sys}]\{\ddot{q}_{sys}\} + [C_{sys}]\{\dot{q}_{sys}\} + [\bar{K}]\{\bar{q}\} = \{F'_{sys}\} \qquad (21)$$

where:

$$\begin{aligned}
[M_{sys}] &= [T_{sys}]^T[\bar{M}][T_{sys}] \\
[C_{sys}] &= [T_{sys}]^T(2[\bar{M}][\dot{T}_{sys}] + [\bar{C}][T_{sys}]) \\
[K_{sys}] &= [T_{sys}]^T([\bar{M}][\ddot{T}_{sys}] + [\bar{C}][\dot{T}_{sys}] \\
\{F_{sys}\} &= [T_{sys}]^T\{\bar{F}\} \qquad (22)
\end{aligned}$$

Equation (21) contitutes the complete set of governing equations for the rotorcraft.

Since the transformations between the rotor and fuselage interface degrees-of-freedom are time dependent, the coefficient matrices of the governing equations contain terms that are time varying, but if the frame motions are prescribed, the time varying terms are known explicitly, and conventional finite element techniques may be applied to assembing the coefficient matrices and then solving the governing equations.

Another method for rotor-body coupling is the "force-balance" method, which treats the rotor-fuselage interface forces as auxiliary variables, and then solves separately for the rotor and fuselage motions until the interface forces converge. If the rotor motions, fuselage motions, and the interface forces are solved for jointly, the force balance method can be thought of as an application of the classical Lagrange multiplier method for enforcing constraints, and the Lagrange multipliers may be interpreted as interface forces. In what follows, the Lagrange multiplier method is used to derive the force-balance equations.

Regarding the rotor and fuselage as unconstrained bodies, it follows from equation (6) that the virtual work of internal and external forces is:

$$\delta W_{uc} = \{\delta\bar{q}\}([\bar{M}]\{\ddot{\bar{q}}\} + [\bar{C}]\{\dot{\bar{q}}\} + [\bar{K}]\{\bar{q}\} - \{\bar{F}\}) \quad (23)$$

By virtue of the constraint relations (equation (20)), the virtual displacements of the rotor and fuselage are related by:

$$\{\delta\bar{q}\}^T \begin{bmatrix} I \\ 0 \\ -[T_{fr}]^T \\ 0 \end{bmatrix} \equiv \{\delta\bar{q}\}^T[A] = 0 \qquad (24)$$

The presence of these constraints leads to constraint forces, and the virtual work of these forces must be considered in the system equations. It can be shown (see Ref. [21]) that the virtual work of the constraint forces is:

$$\delta W_{cf} = \{\delta\bar{q}\}^T[A]\{\lambda\} \qquad (25)$$

The principle of virtual work for the constrained system is then:

$$\delta W = \delta W_{uc} + \delta W_{cf} = 0 \qquad (26)$$

where the virtual displacements can now be regarded as independent quantities. The complete system equations are obtained by appending the constraint equation to the virtual work equations, which gives:

$$\begin{bmatrix} \bar{M} & 0 \\ 0 & 0 \end{bmatrix}\{\ddot{\bar{q}}\} + \begin{bmatrix} \bar{C} & 0 \\ 0 & 0 \end{bmatrix}\{\dot{\bar{q}}\} + \begin{bmatrix} \bar{K} & A^T \\ A & 0 \end{bmatrix}\{\bar{q}\} = \begin{Bmatrix} \bar{F} \\ 0 \end{Bmatrix} \qquad (27)$$

where:

$$\{\bar{q}\} = \begin{Bmatrix} \bar{q} \\ \lambda \end{Bmatrix} \qquad (28)$$

It can be shown from equation (27) that $-\{\lambda\}$ are the forces exerted by the fuselage on the rotor degrees-of-freedom, while $[T_{fr}]^T\{\lambda\}$ are the forces exerted by the rotor on the fuselage degrees-of-freedom.

It is readily shown that elimination of the constraint forces from equation (27) (Ref. [15]) leads to the fully coupled method. The force-balance method, with the Lagrange multipliers retained as variables, is advantageously used in certain solution algorithms for trim.

3.3 Small Motions Relative to Arbitrarily Moving Frames

The discussion thus far has assumed that the large rigid body motions of the rotorcraft are prescribed, but there are many applications where this is not the case; for example, it has already been mentioned that blade articulation may be analyzed by assigning a nonprescribed frame to the articulated blade. Arbitrary frame motions are also needed in analyzing rotorcraft phenomena such as large motion maneuver response to arbitrary pilot controls, and autorotation where large rotor speed changes occur.

3.3.1 Isolated Bodies

To illustrate the application of arbitrary frame motion in rotorcraft analysis, it shall be assumed, as before, that the bodies are the fuselage and rotors. Since the frame motions are nonprescribed, they can absorb rigid body motion, and therefore constraints must be applied to relate the bodies and their frames. In what follows, it is assumed that the frame is rigidly attached to a point on the body. If the kinematics of each body is expressed using equation (5), then equation (6) with the frame motions as unknowns, remains valid:

$$\begin{aligned}
[M^*]\{\ddot{q}^*\} \quad + \quad & [C'^*]\{\dot{q}^*\} + [K'^*]\{q^*\} + [H^*]\{\dot{\omega}\} + \\
& [P^*]\{\dot{V}_0\} = \{F_q^*\} \qquad (29)
\end{aligned}$$

where:

$$\{F_q^*\} = \{F_{nl}^*\} + \{F_{ext}^*\} - [P^*][\tilde{\omega}] -$$

$$-(\int_V \rho[N^*]^T[\tilde{\omega}][\tilde{\omega}]dV)\{r_0\} \quad (30)$$

Starred quantities are obtained from unstarred quantities by zeroing rows and columns corresponding to degrees-of-freedom at the node where the frame is attached. This modification is necessary because the motion of a frame nodes is embodied entirely in the frame degrees-of-freedom, and the elastic deformations at these nodes must therefore be zero. In order to compute the motions of the frames, additional equations are required, which are described next.

The frame equations of motion may be obtained from the principle of virtual work by imparting virtual displacements to frame degrees-of-freedom. Suitable virtual displacements are infinitesimal displacements along frame coordinate axes for the translational degrees-of-freedom, and infinitesimal rotations about frame coordinate axes for the rotational degrees of freedom. Note that the equations obtained from these virtual displacements correspond to translational and rotational equilibrium equations for the entire body. The translational equilibrium equations are:

$$[P^*]^T\{\ddot{q}^*\} + [G^*]^T\{\dot{\omega}\} + M\{\dot{V}\} = \{F_V^*\} \quad (31)$$

where:

$$[G^*]^T = -\int_V \rho\widetilde{[(r_0 + [N^*]\{q\})]}dV \quad (32)$$

$$\{F_V^*\} = \{f_{ext}^*\} - M[\tilde{\omega}]\{V_0\} - 2[\tilde{\omega}][P^*]^T\{\dot{q}^*\} - [\tilde{\omega}][\tilde{\omega}]\int_V \rho(\{r_0\} + [N^*]\{q^*\})dV \quad (33)$$

in which $\{f_{ext}^*\}$ is the resultant external translational force vector. and M is the mass of the body. The second set of frame equations may be obtained by considering moment equilibrium of the entire body:

$$[H^*]^T\{\ddot{q}\} + [I_\omega]\{\dot{\omega}\} + [G^*]\{V_0\} = \{M_\omega\} \quad (34)$$

where:

$$\{M_\omega\} = \{M_{ext}\} - [G][\tilde{\omega}]\{V_0\} - -\{M_{cor}\} - \{M_{cent}\} \quad (35)$$

$$[I_\omega] = -\int_V \rho[\tilde{r}_0]^2 dV \quad (36)$$

$$[H^*]^T = \int_V \rho[\tilde{r}_0][N^*]dV \quad (37)$$

$$\{M_{cor}\} = 2(\int_V \rho[\tilde{r}][\tilde{\omega}]dV)\{\dot{q}^*\} \quad (38)$$

$$\{M_{cent}\} = \int_V \rho\widetilde{[r_0 + [N^*]\{q^*\}]}[\tilde{\omega}]^2(\{r_0\} + [N^*]\{q^*\})dV \quad (39)$$

in which M is the mass of the body and $[I_\omega]$ is the inertia matrix of the undeformed body.

To aid in interpreting the frame equations physically, consider what happens when the elastic deformations are ignored. After converting to vector notation, the frame equations become:

$$M\dot{\vec{V}}_0 + M(\dot{\vec{\omega}} \times \vec{r}_{CG} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{CG})) = \vec{F}_{ext} \quad (40)$$

$$\underline{I}_\omega\dot{\vec{\omega}} + M\vec{r}_{CG} \times (\dot{\vec{V}}_0 + \vec{\omega} \times \vec{V}_0) = \vec{M}_{ext} \quad (41)$$

where $\vec{r}_{CG}$ is the position vector of the center of mass of the undeformed body. Equations (40) and (41) are recognized as the force and moment equilibrium equations of a rigid body having the inertial attributes of the undeformed body. The terms that have been dropped in deriving these equations may be thought of as the effects of elastic deformations on the attributes of the body when it is regarded as rigid.

By defining the column matrices:

$$\{q_G\} = \begin{Bmatrix} q \\ d \\ \phi \end{Bmatrix}, \{\dot{q}_G\} = \begin{Bmatrix} \dot{q} \\ V \\ \omega \end{Bmatrix} \quad (42)$$

the equations of motion may be written in the compact form:

$$[M_G]\{\ddot{q}_G\} + [C_G]\{\dot{q}_G\} + [K_G]\{q_G\} = \{F_G\} \quad (43)$$

where:

$$[M_G] = \begin{bmatrix} M^* & P^* & H^* \\ P^{*T} & MI & G^* \\ H^{*T} & G^{*T} & I_\omega \end{bmatrix} \quad (44)$$

$$[C_G] = \begin{bmatrix} C'^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (45)$$

$$[K_G] = \begin{bmatrix} K'^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (46)$$

$$\{F_G\} = \begin{Bmatrix} F_q'^* \\ F_V'^* \\ M_\omega^* \end{Bmatrix} \quad (47)$$

Equation (43) is the complete set of equations for a single body. This equation is partially formulated using intrinsic coordinates (i.e., the frame translational and angular velocities), which lead to relatively simple equations, but do not describe the how the frames are positioned or oriented in space. Positional and orientational data is necessary not only to determine the vehicle's location, but to compute orientation dependent external loads such as gravity. Consequently, it is necessary to augment the equations of motion with additional equations that relate the intrinsic coordinates of the frames to their Lagrangian coordinates. Reference [20] gives a discussion of systems of Lagrangian coordinates and how they may be related to intrinsic coordinates.

3.3.2 Coupled Bodies

Figure 4 illustrates the geometry of the coupled bodies. The figure shows the coupled system undeformed at time t_0, then at a later time t when both frames have undergone rigid body rotations, where the deformed system is shown in comparison with the undeformed system. In contrast to the case of prescribed frame motions, the frames are attached to their bodies, and are able to move relative to each other. It is assumed that the 3-axes of the rotor and fuselage frame are initially aligned along the rotor spin axis, and that the rotation of the rotor disk plane relative to the fuselage frame is small. Before proceeding, it is first necessary to clearly identify the states of the bodies. For the fuselage, the states may be partitioned analogously to the case of prescribed frame motion:

$$\{q_{Gf}\} = \begin{Bmatrix} q_{fh} \\ q_{ff} \\ d_f \\ \phi_f \end{Bmatrix} \tag{48}$$

For the rotor, it shall prove convenient to assume that its frame is attached to the rotor-fuselage interface node. Since there is no elastic deformation at the frame node, there can be no degrees-of-freedom $\{q_{rh}\}$. Consequently, the rotor states are partitioned as follows:

$$\{q_{Gr}\} = \begin{Bmatrix} q_{rr} \\ d_r \\ \phi_r \end{Bmatrix} \tag{49}$$

Lagrange multipliers are used to couple the bodies. The process of coupling the bodies largely parallels what was done earlier in the case of prescribed frames. The first step in the coupling process is collecting the uncoupled equations and expressing them in the form:

$$[\bar{M}_G]\{\ddot{\bar{q}}_G\} + [\bar{C}_G]\{\dot{\bar{q}}_G\} + [\bar{K}_G]\{\bar{q}_G\} = \{\bar{F}_G\} \tag{50}$$

where:

$$\begin{aligned} [\bar{M}_G] &= \begin{bmatrix} M_{Gf} & 0 \\ 0 & M_{Gr} \end{bmatrix} \\ [\bar{C}_G] &= \begin{bmatrix} C_{Gf} & 0 \\ 0 & C_{Gr} \end{bmatrix} \\ [\bar{K}_G] &= \begin{bmatrix} K_{Gf} & 0 \\ 0 & K_{Gr} \end{bmatrix} \\ \{\bar{F}_G\} &= \begin{Bmatrix} F_{Gf} \\ F_{Gr} \end{Bmatrix} \\ \{\bar{q}_G\} &= \begin{Bmatrix} q_{Gf} \\ q_{Gr} \end{Bmatrix} \end{aligned}$$

The virtual work of the unconstrained system is therefore:

$$\delta W_{uc} = \{\delta\bar{q}_G\}^T([\bar{M}]\{\ddot{\bar{q}}\} + [\bar{C}]\{\dot{\bar{q}}\} + [\bar{K}]\{\bar{q}\} - \{\bar{F}\}) \tag{51}$$

where:

$$\{\delta\bar{q}\} = \begin{Bmatrix} \delta q_{Gf} \\ \delta q_{Gr} \end{Bmatrix} \tag{52}$$

$$\{\delta q_G^T\} = \{\delta q^T, \delta r_1, \delta r_2, \delta r_3, \delta\psi_1, \delta\psi_2, \delta\psi_3\} \tag{53}$$

where the δr_i and the $\delta\psi_i$ are virtual displacements and rotations in the directions of the frame coordinate axes.

The next step is to develop constraint relations for the coupled bodies. Two cases will be considered. In the first case, the rotor azimuth angle rotates in a prescribed manner relative to the fuselage, and in the second case, there is no *kinematic* constraint on the azimuth angle. Note that the second case includes the possibility that there is an azimuth-dependent torque acting between the rotor shaft and the fuselage. When the rotor rotation is prescribed, it can be shown that the translational and angular velocities of the coupled bodies are related by the following equations:

$$\begin{Bmatrix} V_{r1} \\ V_{r2} \\ V_{r3} \end{Bmatrix} = [T_{fr}^{\#}] \left(\begin{Bmatrix} \dot{u}_{fh1} \\ \dot{u}_{fh2} \\ \dot{u}_{fh3} \end{Bmatrix} + \begin{Bmatrix} V_{f1} \\ V_{f2} \\ V_{f3} \end{Bmatrix} \right) \tag{54}$$

$$\begin{Bmatrix} \omega_{r1} \\ \omega_{r2} \\ \omega_{r3} \end{Bmatrix} = [T_{fr}^{\#}] \left(\begin{Bmatrix} \dot{\theta}_{fh1} \\ \dot{\theta}_{fh2} \\ \dot{\theta}_{fh3} \end{Bmatrix} + \begin{Bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{Bmatrix} \right) \tag{55}$$

where:

$$[T_{fr}^{\#}] = [T_{fr}]([I] - [\tilde{\theta}_{fh}]) \tag{56}$$

It immediately follows from equations (54) and (55) that the virtual displacements of the two bodies are related by:

$$\begin{Bmatrix} \delta r_{r1} \\ \delta r_{r2} \\ \delta r_{r3} \end{Bmatrix} = [T_{fr}^{\#}] \left(\begin{Bmatrix} \delta u_{fh1} \\ \delta u_{fh2} \\ \delta u_{fh3} \end{Bmatrix} + \begin{Bmatrix} \delta R_{f1} \\ \delta R_{f2} \\ \delta R_{f3} \end{Bmatrix} \right) \tag{57}$$

$$\begin{Bmatrix} \delta\psi_{r1} \\ \delta\psi_{r2} \\ \delta\psi_{r3} \end{Bmatrix} = [T_{fr}^{\#}] \left(\begin{Bmatrix} \delta\theta_{fh1} \\ \delta\theta_{fh2} \\ \delta\theta_{fh3} \end{Bmatrix} + \begin{Bmatrix} \delta\psi_{f1} \\ \delta\psi_{f2} \\ \delta\psi_{f3} \end{Bmatrix} \right) \tag{58}$$

Equations (57) and (58) may be expressed in matrix form as:

$$[A^{\#}] \begin{Bmatrix} \delta q_{fh} \\ \delta q_{ff} \\ \delta R_f \\ \delta\psi_f \\ \delta q_{rr} \\ \delta R_r \\ \delta\psi_r \end{Bmatrix} = 0 \tag{59}$$

where:

$$[A^{\#}]^T = \begin{bmatrix} [T^{\#}_{fr}]^T & 0 \\ 0 & [T^{\#}_{fr}]^T \\ [T^{\#}_{fr}]^T & 0 \\ 0 & [T^{\#}_{fr}]^T \\ 0 & 0 \\ -[I] & 0 \\ 0 & -[I] \end{bmatrix} \tag{60}$$

The constraints give rise to constraint forces, whose virtual work is:

$$\delta W_{cf} = \{\delta \bar{q}_G\}[A^{\#}]^T\{\lambda\} \tag{61}$$

where $\{\lambda\}$ is a vector of Lagrange multipliers The total virtual work is the sum of the virtual work of the uncoupled bodies and the virtual work of the constraint forces. Since the virtual displacements are arbitrary, the equations of motion of the constrained system are obtained by setting the coefficients of the virtual displacements to zero. The complete system equations are obtained by appending the constraint equations to the equilibrium equations, which gives:

$$\begin{bmatrix} \bar{M}_G & 0 \\ 0 & 0 \end{bmatrix}\{\ddot{\bar{q}}_G\} + \begin{bmatrix} \bar{C}_G & 0 \\ A^{\#} & 0 \end{bmatrix}\{\dot{\bar{q}}_G\} + \begin{bmatrix} \bar{K}_G & A^{\#T} \\ 0 & 0 \end{bmatrix}\{\bar{q}_G\} = \begin{Bmatrix} \bar{F}_G \\ 0 \end{Bmatrix} \tag{62}$$

where:

$$\{\bar{q}_G\} = \begin{Bmatrix} \bar{q}_G \\ \lambda \end{Bmatrix} \tag{63}$$

For the case of variable rotor speed, equations (53) and (56) also apply to the translational constraints, but the rotational constraints must be modified to reflect the fact that the rotor angular velocity parallel to the spin axis frame is not constrained to the fuselage. This effectively reduces the number of rotational constraints from three to two. The constraint on rotational motion is then:

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \end{Bmatrix} = [T^{\#}_{fr\psi}]\left(\begin{Bmatrix} \dot{\theta}_{fh1} \\ \dot{\theta}_{fh2} \\ \dot{\theta}_{fh3} \end{Bmatrix} + \begin{Bmatrix} \omega_{f1} \\ \omega_{f2} \\ \omega_{f3} \end{Bmatrix}\right) \tag{64}$$

where

$$[T^{\#}_{fr\psi}] = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \end{bmatrix}([I] - [\tilde{\theta}_{fh}]) \tag{65}$$

The azimuth angle (Ψ) is a variable, and is related to the system degrees-of-freedom by:

$$\dot{\Psi} = \omega_{r3} - [\theta_{fh2}(\dot{\theta}_{fh1} + \omega_{f1}) - \theta_{fh1}(\dot{\omega}_{fh2} + \omega_{f2})] - \omega_{f3} - \dot{\theta}_{fh3} \tag{66}$$

The complete system equations have the same form as equation (62), but $[A^{\#}]$ must be replaced by:

$$[A^{\#}_{\psi}]^T = \begin{bmatrix} [T^{\#}_{fr}]^T & 0 \\ 0 & [T^{\#}_{fr\psi}]^T \\ [T^{\#}_{fr}]^T & 0 \\ 0 & [T^{\#}_{fr\psi}]^T \\ 0 & 0 \\ -[I] & 0 \\ 0 & -[I_{\psi}]^T \end{bmatrix} \tag{67}$$

where:

$$[I_{\psi}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{68}$$

Equation (62) is the complete set of equations for analyzing the large motion dynamics of elastic bodies, but these equations are fundamentally different from the ones presented earlier for other classes of rotorcraft analyses. The key difference is the presence of frame motion variables, which are not field variables; they apply across an entire body, which means that their "assembly process" is fundamentally different from that of the elastic deformation variables. Large motion analysis is further complicated by the fact that the frame motions and elastic deformations are coupled nonlinearly in the governing equations. Clearly, the treatment of large motion dynamics problems, such as those involving rotorcraft, requires analysis methods that go well beyond traditional small motion finite element methods. The equations developed here could serve as the basis for analyzing large motion rotorcraft dynamics problems, but their main purpose has been to delineate technical issues, and they may not be computationally efficient. Dynamics of bodies undergoing large motion falls within the domain of multibody dynamics, and we must look to that discipline in order to develop definitive methods of comprehensive rotorcraft analysis.

4. Methods of Multibody Dynamics

The relatively new field of multibody dynamics is the study of the dynamic response of mechanical systems undergoing arbitrarily large motions; until recently, most treatises on analytical dynamics (e.g., Refs. [21], [22]) were concerned with general principles and methologies and were not concerned with applying these methods to the analysis of complex, dynamical systems. By the 1960's, the advent of increasingly sophisticated flight vehicles and mechanical systems, coupled with the need for more refined analyses of these systems stimulated research into improved methods of dynamic analysis. Early work

considered only systems of rigid bodies, but subsequently, rigid body analyses were augmented with finite element methods for analyzing deformable bodies in order to permit the analysis of multibody dynamics of flexible bodies.

This section contains a brief review of the multibody dynamics literature. For brevity, it is largely a qualitative discussion of formulation and solution methods.

4.1 Systems of Rigid Bodies

Formulating the dynamical equations of a mechanical system-whether it contains rigid or flexible bodies-can be thought of as a two step process: first, the equations of motion of each component must be formulated, and then the equations of motion of the separate components must be combined into a set of equations representing the complete system. When a mechanical system consists only of rigid bodies, the equations of each component are the well-known Newton-Euler equations (Refs. [21], [22]) so the real task is assembling the complete set of equations. In general, the methods for assembling the system equations fall into two categories: the first category assumes the system configuration conforms to some generic model, while the second category assumes the system configuration is completely general. Methods in both these categories will now be discussed, starting with the category of generic models.

Initially, investigators in multibody dynamics focused on the analysis of hinge connected systems of rigid bodies having a tree topology (i.e, no closed loops), because this configuration is relatively easy to analyze, and corresponds to many actual flight vehicles and mechanical systems. A key development in this area was the work of Hooker and Margulies (Ref. [23]), who derived the dynamical equations of the system in terms of constraint torques and externally applied forces by applying the Newton-Euler equations to judiciously selected subassemblies. The constraint torques were then eliminated with an additional equation expressing the condition that the torques must be orthogonal to the hinge rotations.

Huston and Passerello (Refs. [24], [25]) employed Kane's equations of motion (Ref. [17]) to analyze dynamical systems with tree topology. The equations were applied to the same subassemblies considered by Hooker and Margulies, but consideration of constraint torques was avoided altogether by the use of relative hinge rotation rates as generalized speeds. The avoidance of extraneous constraint forces, coupled with an efficient, recursive method for computing angular velocities results in an extremely efficient formulation.

Many mechanical systems encountered in practice are not tree topologies, but contain closed loops. A way to analyze such systems, which exploits the hightly efficient methods for solving multibody networks with tree topologies, is to represent a closed loop system as a tree system and append additional constraints that enforce loop closure. The equations of the full dynamical system are then the open tree dynamical equations plus the loop closure constraint equations. Several methods have been proposed for solving the combined system equations. One method adjoins the constraint equations to the dynamical equations using Lagrange multipliers (Ref. [19]), while another solution method reduces the dynamical equations using a transformation obtained from a singular value decomposition of the constraint equations (Ref. [26]). A study comparing these solution methods (Ref. [27]) has shown that their relative efficiencies are highly problem dependent.

The second category of multibody dynamics analysis completely foregoes consideration of tree topologies, or any other generic configuration, and is based on directly formulating the equations of motion of systems of arbitrary configuration. An early example of this approach is the mechanical simulation program ADAMS (Automatic Dynamic Analysis of Mechanical Systems), which developed from the work of Orlandea (Refs. [28], [29]) Since there is no generic model to work from, ADAMS must formulate the dynamical equations of each rigid body in terms of its absolute translational and rotational coordinates, rather than make use of relative coordinates as in the methods based on tree topologies. The equations of each rigid body are formulated using Lagrange's equations instead of Euler's equations, and use generalized momenta as auxiliary variables. A library of mechanical joints embodying a wide variety of body interconnections gives the user considerable flexibility in modeling. Constraint equations representing the interconnections are adjoined to the system equations using Lagrange multipliers. The equations assembled by ADAMS are far from a minimal equation set, but the equations are quite sparse and ADAMS is specially designed to exploit that sparseness.

A very different approach to self-formulating mechanical simulation software is the code SD-EXACT (Ref. [30]). This code, which is based on Kane's equations of motion, employs symbolic manipulation techniques to formulate the dynamical equations of user specified mechanical systems. The equations produced are more complex than those generated by ADAMS, but constitute a minimal solution set.

4.2 Systems of Flexible Bodies

If the bodies comprising a system are quite stiff, it suffices to model the system as a collection of rigid bodies, in which case the methods just described will give a complete view of system dynamic response. In the case of rotorcraft, the bodies are flexible and the motion of these bodies involves substantial coupling between elastic and rigid body response.

Two methods have been proposed for analyzing the multibody dynamics of flexible bodies and their differences are based on how they handle kinematics. The first method was developed by Likins (Refs. [13], [18]) and is embodied in equations (1) - (12) given above. As noted earlier, this method treats the motion of each flexible body as the superposition of a large rigid body motion of a frame attached to the body and some small elastic deformation of the body relative to the frame. Partitioning the response in this manner allows the extensive technology that has been developed for small motion finite element analysis to be fully exploited, but as noted earlier, this method can greatly complicate the process of assembing system equations.

Since equations (1) - (12) are valid only for a single body, some way must be found to couple these equations in order to permit the analysis of multiple bodies. The methods that have been devised for this process parallel the methods given previously for multiple rigid bodies. The method of Huston and Passerrello for systems with tree topologies was extended to systems with flexible bodies by Singh, VanderVoort, and Likins (Ref. [31]). This formulation is the basis for the multibody dynamics codes DISCOS, TREETOPS, and CONTOPS (Ref. [32]), which have been used for analyzing spacecraft dynamics.

A general purpose code for analyzing flexible bodies is DADS (Refs. [33], [34], [35]) DADS may be regarded as an extension of the ADAMS methodology to flexible bodies. Like ADAMS, it employs absolute coordinates for the large motion frame variables, and enforces body coupling using Lagrange multipliers. A somewhat different approach to flexible body analysis is embodied in the code LATDYN (Ref. [36]) which is primarily intended for deployment analysis of large, flexible space structures. LATDYN uses a co-rotational approach which uses the nodal displacements of each element to define the large motion frame for that element. Each element is therefore a "body" and standard finite element connectivity is used to enforce coupling. If additional constraints are needed, the constraint equations are used to eliminate redundant degrees of freedom.

It must be emphasized that the multibody dynamics codes mentioned above generally assume the flexible bodies are linearly elastic, and they do not have elements that can model large elastic deformations. The reason for this is that these codes were formulated for application in the mechanical design and aerospace industries, and geometric stiffness effects are typically ignored in these applications. For example, in mechanical design applications, mechanical components can spin at high speeds, but the components are usually too stiff for the spin to induce geometric stiffening; in spacecraft applications, highly flexible components are present but spin rates are too small to induce geometric stiffening. Although most current multibody dynamics codes cannot accomodate large elastic deformations at the *element* level, some of these codes can model geometric stiffness effects; for example, Wu and Haug (Ref. [37]) have demonstrated that geometric stiffness can be modeled in DADS by linking flexible bodies with large motion mechanical joints. It is not clear if this would be an efficient alternative to large deformation elements for analyzing geometric stiffness effects in rotor blades.

Recently, a new class of methods for multibody dynamics, known as "recursive methods" (Refs. [38], [39]), has been developed that offers dramatic improvements in computational efficiency over most previous methods and is well suited for parallel processing applications. The method is based on mechanical systems with tree topology, but closed loop systems can be handled by use of appropriate constraint equations. Suppose the equations of motion of a multibody system are represented in the form:

$$[M]\{\ddot{q}\} = \{F\} \tag{69}$$

where $[M]$ is the system mass matrix. Recursive methods use recursion to form $\{\ddot{q}\} = [M]^{-1}\{F\}$. Recursion eliminates the need to manipulate large, full matrices, which is a prime contributor to the high computational cost of many multibody dynamics codes.

As noted earlier, the primary reason conventional finite element software cannot analyze multibody dynamics is that most finite elements are not designed to accomodate finite rotations. Recently, geometrically exact finite elements have been developed (Refs. [5], [6], [7], [8]) that are valid for arbitrarily large rotations. Although some of these elements are too computationally expensive to be used in practical simulation software, research is underway to develop more efficient elements, and this approach could lead to the unification of finite element analysis and multibody dynamics into a single discipline.

An important contribution to flexible body dynamics from the rotorcraft community is the code GRASP (Refs. [42],[43]). GRASP is limited in scope to the stability analysis of a hovering rotorcraft, but to achieve that end, it employs a computational for-

mulation that combines geometrically exact finite elements with a novel approach to solving for the hover condition. Unlike virtually all other methodologies previously mentioned here, GRASP uses an implicit solution formulation, which means that explicit equations of motion for the system are never formed. The hover state is calculated by using an iterative solution process to force to zero the generalized forces of all system degrees-of-freedom, which can include frame degrees-of-freedom, and element degrees-of-freedom that are defined within the frames. Generalized forces corresponding to element elastic deformations are assembled using an element-by-element approach. The analysis of frame generalized forces is facilitated by grouping elements in hierarchically organized substructures within which frame motions are defined. Starting with the root substructure, frame virtual displacements are propagated down through the substructure hierarchy, and frame generalized forces are assembled by traversing the hierarchy in the reverse direction. Once the hover solution is obtained, the linearized stiffness matrix is generated by numerical perturbation of the generalized forces, while the mass and damping/gyroscopic matrices are formed from symbolically derived expressions.

5. Concluding Remarks

The development of effective comprehensive rotorcraft analysis software will require a capability that combines large motion analysis with general modeling capability. Although finite element analysis can handle general purpose modeling within the context of small motion analysis, multibody dynamics will be needed to analyze large motions. A formulation for the large motion dynamic analysis of rotorcraft has been presented in order to fully detail the inadequacies of finite element analysis alone in the analysis of rotorcraft dynamics. Although the formulation presented may be used in an actual analysis, the multibody dynamics literature contains far more efficient and more flexible methodologies. Applying these methodologies to rotorcraft dynamics must be a high priority goal for rotorcraft analysts.

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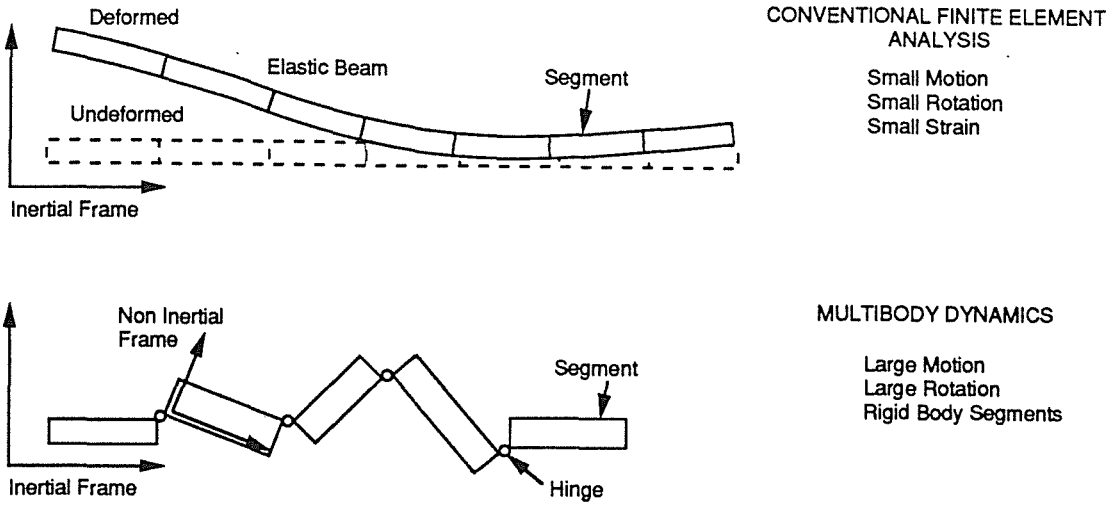


Fig. 1. - Illustration of finite element and multibody dynamics methods.

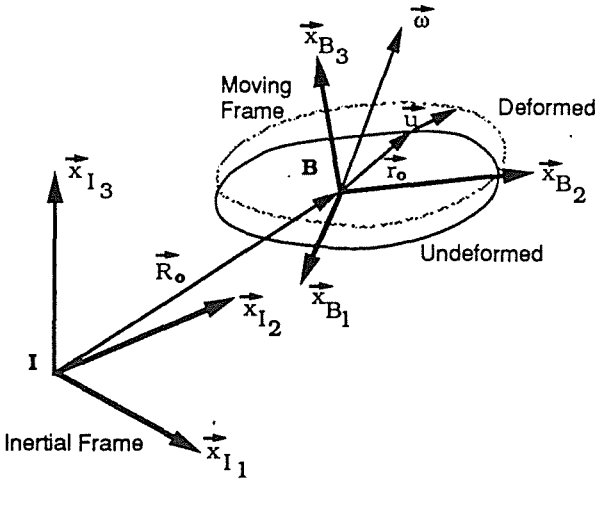


Fig. 2. - Geometry of frames for small motions of a defromable body.

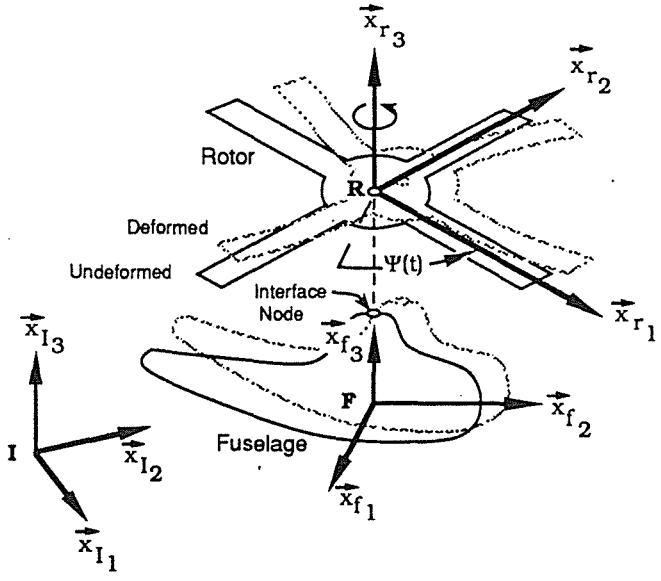


Fig. 3. - Coupling of rotor and fuselage relative to frames with prescribed motion.

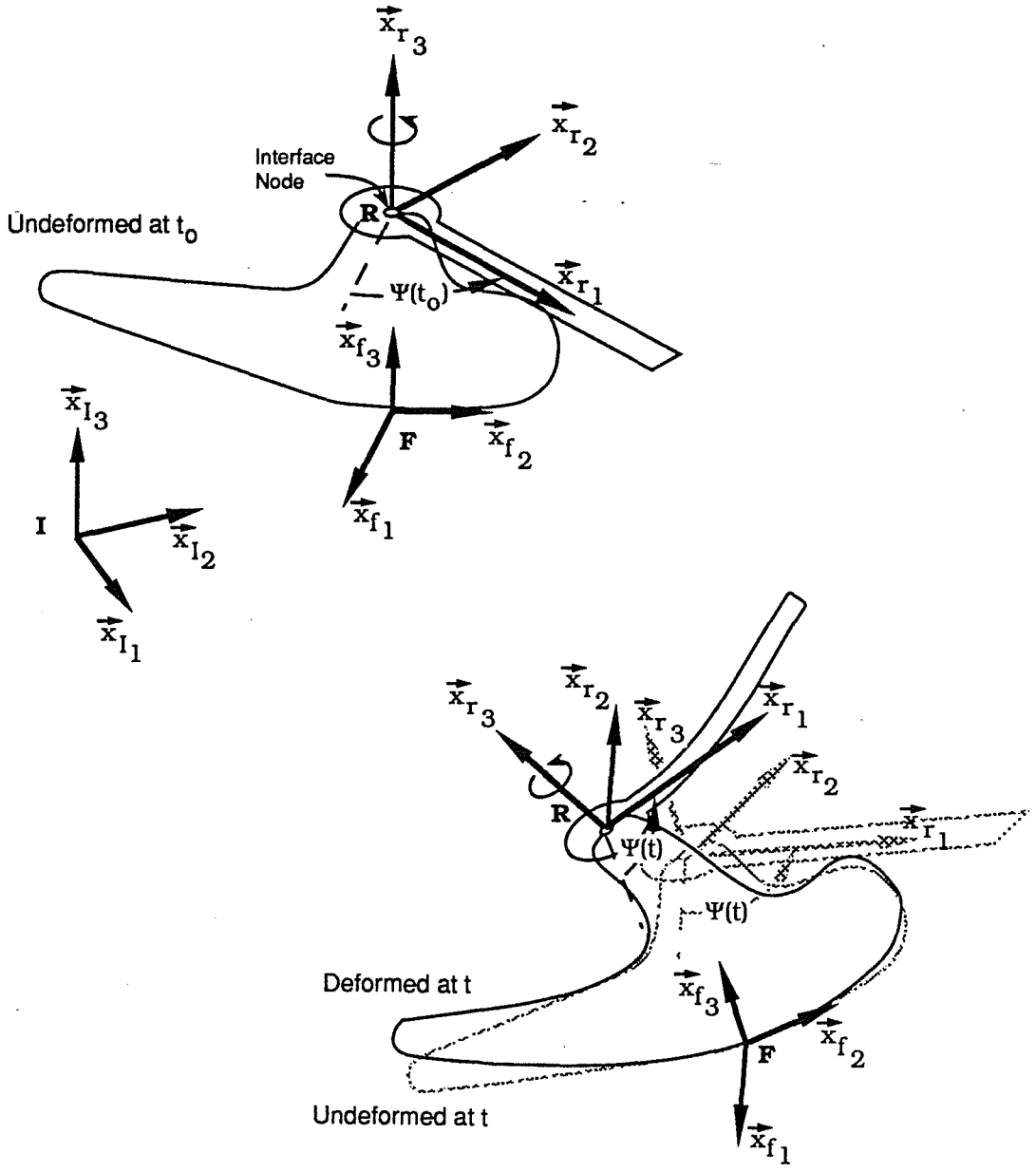


Fig. 4. - Coupling of rotor and fuselage relative to frame with arbitrary motion.