CLOSED FORM SOLUTIONS FOR ROTATING NON-HOMOGENEOUS TIMOSHENKO BEAMS

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Abstract

In this paper, the governing equations for free vibration of a non-homogeneous rotating Timoshenko beam, having uniform cross-section, is studied using an inverse problem approach, for both cantilever and pinned-free boundary conditions. The bending displacement and the rotation due to bending are assumed to be simple polynomials which satisfy all four boundary conditions. It is found that for certain polynomial variations of the material mass density, elastic modulus and shear modulus, along the length of the beam, the assumed polynomials serve as simple closed form solutions to the coupled second order governing differential equations with variable coefficients. It is found that there are an infinite number of analytical polynomial functions possible for material mass density, shear modulus and elastic modulus distributions, which share the same frequency and mode shape for a particular mode. The derived results are intended to serve as benchmark solutions for testing approximate or numerical methods used for the vibration analysis of rotating non-homogeneous Timoshenko beams.

1. INTRODUCTION

Rotating elastic beams serve as important mathematical models for a wide range of mechanical structures like helicopter rotor blades, turbine blades, propellers, satellite booms etc. Gas and steam turbine blades are short and rigid and can be modeled as Timoshenko beams. Beams with variable properties are mostly used in order to optimize the distribution of strength and weight, and also sometimes to satisfy certain functional requirements. Rotating Euler-Bernoulli beams only consider centrifugal force in addition to the inertial and elastic forces for vibration analysis. The secondary effects such as shear deformation and rotary inertia have a small effect on lower modes but have considerable effect on higher modes. Hence for accurate prediction of higher modes, the Timoshenko beam model is employed.

The governing equation of a rotating non-homogeneous Timoshenko beam consists of two coupled differential equations, which does not yield any closed-form solutions unlike uniform non-rotating Timoshenko beams. Hence, approximate and numerical methods have been developed by researchers to investigate the vibration problem of rotating Timoshenko beams like Finite Element Method [1–6], power series solution [7, 8], dynamic stiffness method [9], differential transform method [10] and differential quadrature method [11, 12]. However, due to the complicated mathematical structure of the coupled rotating Timoshenko beam governing equations, the *forward problem* of finding the mode shapes and frequencies given the beam, becomes quite challenging.

An idea for obtaining closed form solutions for different classes of non-homogeneous Euler-Bernoulli beams was proposed by Elishakoff & Candan [13]. They assume a simple polynomial which satisfies the boundary conditions as the mode shape and then solve the *inverse problem* of finding the elastic modulus and material mass density variations along the length of the beam. In this research, we extend the idea proposed by Elishakoff and his co-workers to a rotating non-homogeneous Timoshenko beam. We show that for a given frequency, mode shape and uniform rotation speed, positive analytical polynomial functions exist for the density, shear modulus and elastic modulus variations which serves as a simple closed form solution to the coupled governing differential equations. The main objective of the derived polynomial functions is to provide exact closed-form solutions for validating numerical or approximate methods which are routinely developed for the vibration study of non-homogeneous rotating Timoshenko beams.

2. PROBLEM FORMULATION

Considering harmonic vibration, the dynamics of a non-homogeneous rotating Timoshenko beam is governed by the coupled differential equations given by [9]

$$\rho(x)A\omega^2 W(x) + \frac{\partial}{\partial x}\left[T(x)\frac{\partial W(x)}{\partial x}\right] + \frac{\partial}{\partial x}\left[kAG(x)\left(\frac{\partial W(x)}{\partial x} - \phi(x)\right)\right] = 0 \tag{1}$$

$$\rho(x)I\omega^2\phi(x) + \rho(x)I\Omega^2\phi(x) + \frac{\partial}{\partial x}\left[E(x)I\frac{\partial \phi(x)}{\partial x}\right] + kAG(x)\left[\frac{\partial W(x)}{\partial x} - \phi(x)\right] = 0 \tag{2}$$

where $T(x) = \int_x^L \rho(x)A\Omega^2 x dx$ is the centrifugal force term, $E(x)$ is the elastic modulus, $G(x)$ is the shear modulus, $\rho(x)$ is the material density, $W(x)$ is the bending displacement, $\phi(x)$ is the rotation due to bending, A is the uniform cross-section, I is the area moment of inertia, k is the shear correction factor, Ω is the rotation speed, and L is the length of the beam. The density, shear modulus and elastic modulus are now assumed to be simple polynomial functions of the form

$$\rho(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \tag{3}$$

$$G(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 \tag{4}$$

$$E(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 \tag{5}$$

The choice of the order of the polynomials will be explained later. We now assume a simple polynomial as the prescribed mode shape $W(x)$ and rotation due to bending $\phi(x)$, of the form

$$W(x) = \sum_{i=0}^{4} d_i\left(\frac{x}{L}\right)^i, \ \phi(x) = \sum_{i=0}^{3} e_i\left(\frac{x}{L}\right)^i \tag{6}$$

Since $\phi(x)$ represents the rotation due to bending it is assumed to have the same polynomial order as the derivative of $W(x)$, and hence it's order is 3. The constants d_i's and e_i's can be determined using the cantilever and pinned-free boundary conditions, respectively, along with the normalization conditions. Once the expressions for the constants d_i's and e_i's are known, we can get the assumed polynomial expressions for the bending displacement $W(x)$ and the rotation due to bending $\phi(x)$. Using these polynomials we will try to determine the mass density, shear modulus and elastic modulus variations such that the assumed polynomials serve as fundamental closed-form solutions to the coupled governing differential equations, given by Eqns. (1) & (2). The following sections present the detailed formulations for both the cantilever and pinned-free boundary conditions, respectively.

2.1. Cantilever beam

The boundary conditions for a rotating cantilever Timoshenko beam is given by

$$W(0) = 0, \ \phi(0) = 0, \quad \phi'(L) = 0, \ W'(L) - \phi(L) = 0 \tag{7}$$

Putting Eqn. (6) into the cantilever boundary conditions, given by Eqn. (7), along with the conditions of normalization, given by $W(L) = 1$ & $\phi(L) = 1$, we can solve for the constants d_i's and e_i's, given by

$$d_0 = 0, d_3 = -3d_1 - 2d_2 - L + 4, d_4 = 2d_1 + d_2 + L - 3, e_0 = 0, e_2 = 3 - 2e_1, e_3 = e_1 - 2 \tag{8}$$

Thus, the polynomial expressions for the assumed bending displacement $W(x)$ and rotation due to bending $\phi(x)$ are given by

$$W(x) = \frac{x^4(2d_1 + d_2 + L - 3)}{L^4} + \frac{x^3(-3d_1 - 2d_2 - L + 4)}{L^3} + \frac{d_1x}{L} + \frac{d_2x^2}{L^2} \tag{9}$$

$$\phi(x) = \frac{(e_1 - 2)x^3}{L^3} + \frac{(3 - 2e_1)x^2}{L^2} + \frac{e_1x}{L} \tag{10}$$

Putting Eqns. (3), (4), (5), (9) and (10) into the coupled governing differential equations, given by Eqns. (1) & (2), we will get two polynomial equations in x, each with the highest term of x^8. For these two polynomial equations to be satisfied for all values of x ($0 \leq x \leq L$), the coefficient of the various powers of x must be set to zero, thus yielding a set of 18 linear homogeneous equations in 19 unknowns $(a_0, \ldots, a_4, b_0, \ldots, b_5, c_0, \ldots, c_7)$.

At this point it is to be noted that the order of the assumed polynomials for the mass density, shear modulus and elastic modulus, given by Eqns. (3), (4) & (5), respectively, were chosen in a manner such that the number of unknowns (19) should be greater than or equal to the number of equations (18) in order to obtain an analytical solution. Solving this set of 18 linear homogeneous equations we can solve for the unknowns a_i's, b_i's and c_i's in terms of one of the unknowns c_7. If the total mass of the beam is constrained, the constant c_7 can be determined using the following equation.

$$\int_0^L \rho(x)A dx = M \tag{11}$$

where M is the total mass of the rotating Timoshenko beam. Thus, the variations of the mass density $\rho(x)$, shear modulus $G(x)$ and elastic modulus $E(x)$ can now be determined using Eqns.(3), (4) & (5), respectively. The derived expressions are simple polynomials in x whose coefficients are dependent on the beam length L, natural frequency ω, rotation speed Ω, and the arbitrary constants d_1, d_2 & e_1.

As an example, we take a rotating cantilever Timoshenko beam with properties shown in Table 1, and having a rectangular cross-section. For the arbitrary constants d_1, d_2 & e_1, we chose the values -10, -8 & 2, respectively. The values are chosen in a manner such that the final variations of the mass density, shear modulus and elastic modulus do not become negative at any point spanning the length of the beam. Thus, the final expressions for the bending displacement $W(x)$ and rotation due to bending $\phi(x)$ are given by

$$W(x) = \frac{(L-31)x^4}{L^4} + \frac{(50-L)x^3}{L^3} - \frac{8x^2}{L^2} - \frac{10x}{L} \tag{12}$$

$$\phi(x) = \frac{2x}{L} - \frac{x^2}{L^2} \tag{13}$$

The assumed mode shape variation is shown in Fig. 1. Since it turns out that the mode shape has an internal node, it represents the second elastic mode of a cantilever beam. And hence, the assumed frequency would be the second natural frequency.

Using the method discussed in this section, we have derived the mass density, shear modulus and elastic modulus variations of a rotating cantilever AFG Timoshenko beam, whose second mode frequency $\omega = 400$ rad/s and uniform rotation speed $\Omega = 360$ RPM, having a total mass of 10 kg, as follows

$$\rho(x) = 1830.41x^2 - 1830.28x + 485.427 \tag{14}$$

$$G(x) = -1.1771\times10^7x^4 + 2.20456\times10^7x^3 \\ -7.29698\times10^6x^2 - 3.85667\times10^6x + 2.00582\times10^6 \tag{15}$$

$$E(x) = 1.14951\times10^8x^7 - 2.9312\times10^8x^6 \\ +1.78914\times10^8x^5 + 3.2666\times10^7x^4 - 3.1175\times10^6x^3 \\ -1.93357\times10^7x^2 + 2.06448\times10^7x + 7.58609\times10^6 \tag{16}$$

Thus, if we have a rotating cantilever Timoshenko beam whose mass density, shear modulus and elastic modulus variations are given by Eqns. (14), (15) & (16), having property values shown in Table 1, and having a uniform rotating speed of 360 RPM, then it's second mode shape will be given by Eqn. (12) and will have a second mode frequency of 400 rad/s. Figs. 2, 3 & 4 shows the variations of the density, shear modulus and elastic modulus, respectively, for different values of the total mass M of the beam.

In order to show that the derived expressions of mass density, shear modulus and elastic modulus, given by Figs. 2, 3 & 4, respectively, do indeed satisfy the coupled governing differential equations, given by Eqns. (1) & (2), we substitute them back into the differential equations and calculated the residues, R_{c1} & R_{c2}, respectively. Along with the derived expressions of the beam properties, we also put the assumed mode shape (Eqn. (12)), rotation due to bending (Eqn. (13)), frequency ($\omega = 400$ rad/s) and uniform rotating speed ($\Omega = 360$ RPM) to calculate the residues. A plot of the residues are shown in Fig. 5, from which we can conclude that the residues are zero for all points along the length of the beam, meaning all the expressions exactly satisfy the coupled governing differential equations. Thus, reinstating the fact that for certain variations of the density, shear modulus and elastic modulus, the assumed mode shapes and frequency represents closed-form solutions to the coupled governing differential equations of a rotating non-homogeneous cantilever Timoshenko beam.

Table 1: **Properties of the example rotating axially functionally graded (AFG) Timoshenko beam**

Properties	Values
Length(L)	1 m
Area (A)	0.0554256 m^2
Moment of Inertia (I)	0.0354724 m^4
Shear correction factor (k)	5/6

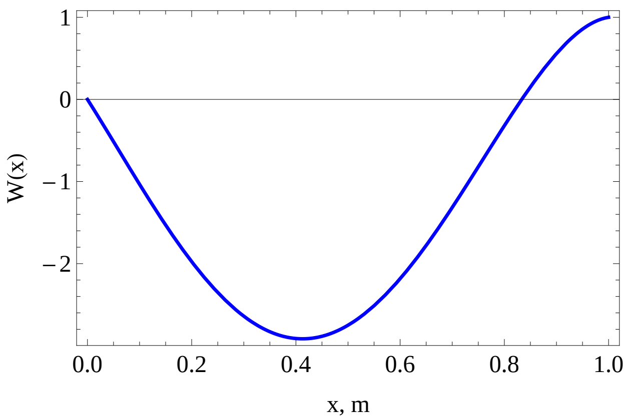

Figure 1: **The assumed mode shape $W(x)$, for a rotating cantilever Timoshenko beam second mode**

2.2. Pinned-free beam

The boundary conditions of a rotating pinned-free Timoshenko beam is given by

$$W(0) = 0,\ \phi'(0) = 0, \\ \phi'(L) = 0,\ W'(L) - \phi(L) = 0 \tag{17}$$

Putting Eqn. (6) into the pinned-free boundary conditions, given by Eqn. (17), along with the conditions

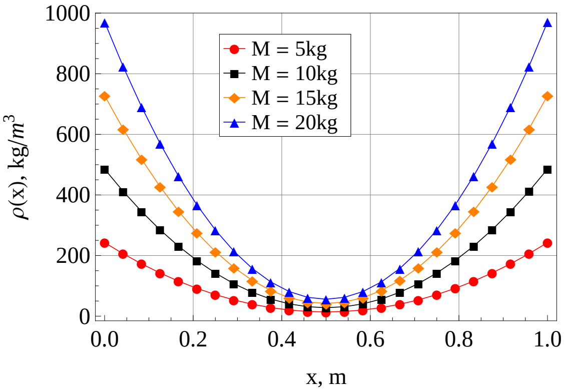

Figure 2: **Density variations for a rotating cantilever Timoshenko beam, for different values of the total mass, whose second mode shape is given by Fig. 1**

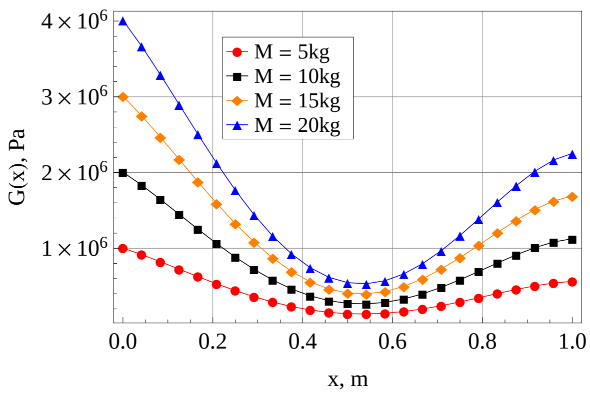

Figure 3: **Shear modulus variations for a rotating cantilever Timoshenko beam, for different values of the total mass, whose second mode shape is given by Fig. 1**

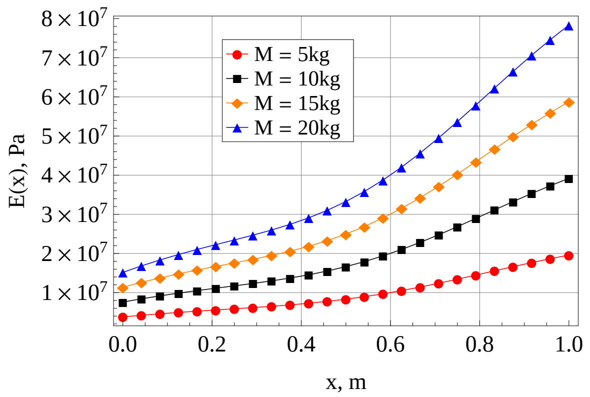

Figure 4: **Elastic modulus variations for a rotating cantilever Timoshenko beam, for different values of the total mass, whose second mode shape is given by Fig. 1**

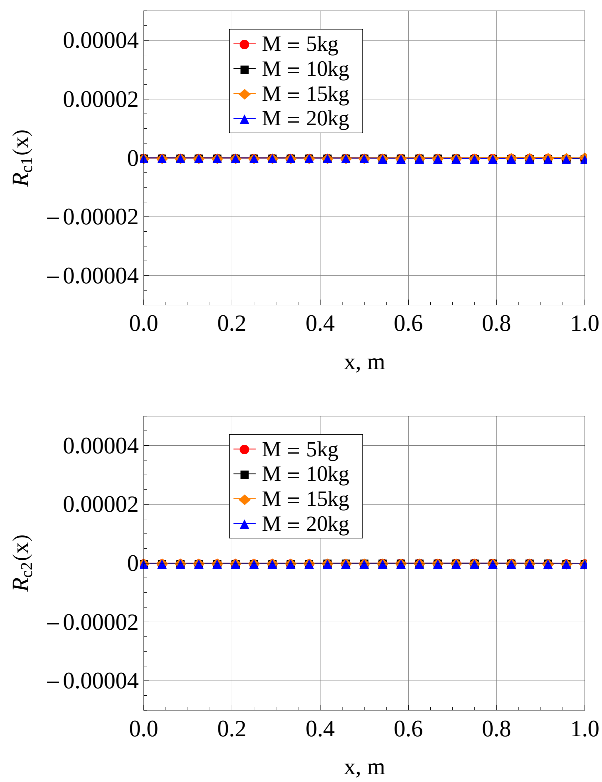

Figure 5: **Plot of the residues, R_{c1} & R_{c2}, after putting the derived expressions of mass density, shear modulus and elastic modulus, given by Figs. 2, 3 & 4, respectively, of a rotating cantilever Timoshenko beam into the governing differential equations**

of normalization, given by $W(L) = 1$ & $\phi(L) = 1$, we can solve for the constants d_i's and e_i's, given by

$$d_0 = 0, d_3 = -3d_1 - 2d_2 - L + 4, \quad d_4 = 2d_1 + d_2 + L - 3, e_0 = 1 - \frac{e_2}{3}, \quad e_1 = 0, e3 = -\frac{1}{3}(2e_2) \tag{18}$$

Thus, the polynomial expressions for the assumed bending displacement $W(x)$ and rotation due to bending $\phi(x)$ are given by

$$W(x) = \frac{x^4(2d_1 + d_2 + L - 3)}{L^4} + \frac{x^3(-3d_1 - 2d_2 - L + 4)}{L^3} + \frac{d_1 x}{L} + \frac{d_2 x^2}{L^2} \tag{19}$$

$$\phi(x) = -\frac{2e_2 x^3}{3L^3} + \frac{e_2 x^2}{L^2} - \frac{e_2}{3} + 1 \tag{20}$$

Putting Eqns. (3), (4), (5), (19) and (20) into the coupled governing differential equations, given by Eqns. (1) & (2), we will get two polynomial equations in x, each with the highest term of x^8. For these two polynomial equations to be satisfied for all values of

x ($0 \le x \le L$), the coefficient of the various powers of x must be set to zero, thus yielding a set of 18 linear homogeneous equations in 19 unknowns $(a_0, \ldots, a_4, b_0, \ldots, b_5, c_0, \ldots, c_7)$. Solving this set of 18 linear homogeneous equations we can solve for the unknowns a_i's, b_i's and c_i's in terms of one of the unknowns c_7. If the total mass of the beam is constrained, the constant c_7 can be determined using Eqn. (11). Thus, the variations of the mass density $\rho(x)$, shear modulus $G(x)$ and elastic modulus $E(x)$ can now be determined using Eqns.(3), (4) & (5), respectively. The derived expressions are simple polynomials in x whose coefficients are dependent on the beam length L, natural frequency ω, rotation speed Ω, and the arbitrary constants d_1, d_2 & e_2.

As an example, we take a rotating pinned-free Timoshenko beam with properties shown in Table 1, and having a rectangular cross-section. For the arbitrary constants d_1, d_2 & e_2, we chose the values -10, -6 & 8, respectively. Once again, the values are chosen in a manner such that the final variations of the mass density, shear modulus and elastic modulus do not become negative at any point spanning the length of the beam. Thus, the final expressions for the bending displacement $W(x)$ and rotation due to bending $\phi(x)$ are given by

(21)
$$W(x) = \frac{(L-29)x^4}{L^4} + \frac{(46-L)x^3}{L^3} - \frac{6x^2}{L^2} - \frac{10x}{L}$$

(22)
$$\phi(x) = -\frac{16x^3}{3L^3} + \frac{8x^2}{L^2} - \frac{5}{3}$$

The assumed mode shape variation is shown in Fig. 6. Since it turns out that the mode shape has an internal node, it represents the first elastic mode of a pinned-free beam. And hence, the assumed frequency would be the fundamental natural frequency.

Using the method discussed in this section, we have derived the mass density, shear modulus and elastic modulus variations of a rotating pinned-free Timoshenko beam, whose fundamental frequency $\omega = 400$ rad/s and uniform rotation speed $\Omega = 360$ RPM, having a total mass of 10 kg, as follows

(23)
$$\rho(x) = 758.234x^3 + 596.011x^2 \\ -1053.89x + 319.14$$

(24)
$$G(x) = -4.50531 \times 10^6 x^5 - 942073.x^4 \\ +1.39976 \times 10^7 x^3 - 6.08629 \times 10^6 x^2 \\ -2.50959 \times 10^6 x + 1.55701 \times 10^6$$

(25)
$$E(x) = 4.3454 \times 10^6 x^7 - 452846.x^6 \\ -2.44018 \times 10^7 x^5 + 1.40324 \times 10^7 x^4 \\ +2.41648 \times 10^7 x^3 - 1.53312 \times 10^7 x^2 \\ -2.529 \times 10^6 x + 6.42216 \times 10^6$$

Thus, if we have a rotating pinned-free Timoshenko beam whose mass density, shear modulus and elastic

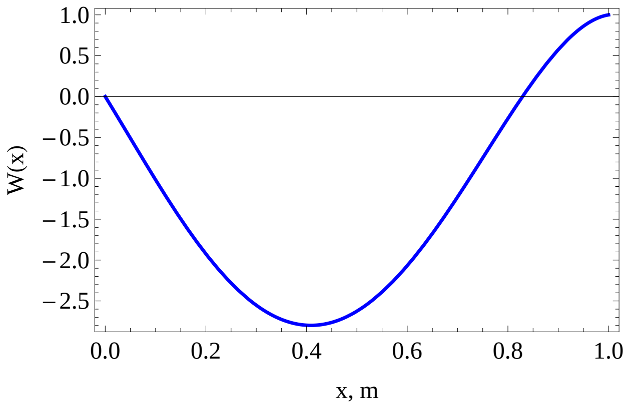

Figure 6: **The assumed mode shape $W(x)$, for a rotating pinned-free Timoshenko beam first mode**

modulus variations are given by Eqns. (23), (24) & (25), having property values shown in Table 1, and having a uniform rotating speed of 360 RPM, then it's fundamental mode shape will be given by Eqn. (21) and will have a fundamental frequency of 400 rad/s. Figs. 7, 8 & 9 shows the variations of the density, shear modulus and elastic modulus, respectively, for different values of the total mass M of the beam.

Once again, in order to show that the derived expressions of mass density, shear modulus and elastic modulus, given by Figs. 7, 8 & 9, respectively, do indeed satisfy the coupled governing differential equations, given by Eqns. (1) & (2), we substitute them back into the differential equations and calculated the residues, R_{p1} & R_{p2}, respectively. Along with the derived expressions of the beam properties, we also put the assumed mode shape (Eqn. (21)), rotation due to bending (Eqn. (22)), frequency ($\omega = 400$ rad/s) and uniform rotating speed ($\Omega = 360$ RPM) to calculate the residues. A plot of the residues are shown in Fig. 10, from which we can conclude that the residues are zero for all points along the length of the beam, meaning that all the expressions exactly satisfy the coupled governing differential equations. Thus, reinstating the fact that for certain variations of the density, shear modulus and elastic modulus, the assumed mode shapes and frequency represents closed-form solutions to the coupled governing differential equations of a rotating non-homogeneous pinned-free Timoshenko beam.

3. CONCLUSION

In this paper, we have shown that there exists a certain class of rotating non-homogeneous Timoshenko beam, having cantilever and pinned-free boundary conditions, which has a closed form polynomial solution to its coupled governing differential equations. We assume a simple polynomial for the bending dis-

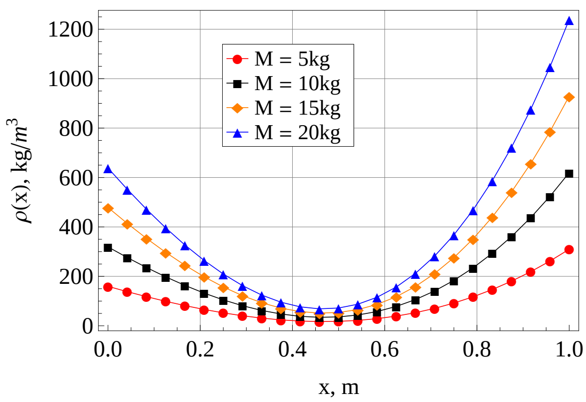

Figure 7: **Density variations for a rotating pinned-free Timoshenko beam, for different values of the total mass, whose first mode shape is given by Fig. 6**

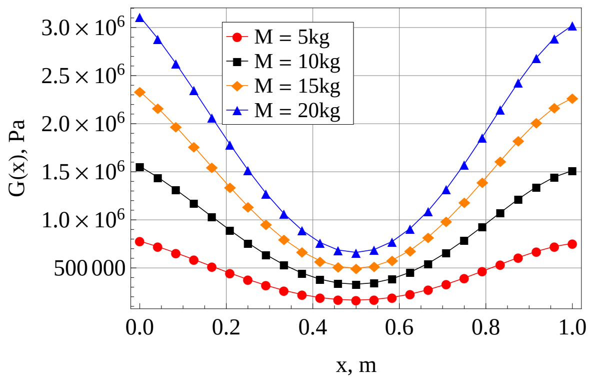

Figure 8: **Shear modulus variations for a rotating pinned-free Timoshenko beam, for different values of the total mass, whose first mode shape is given by Fig. 6**

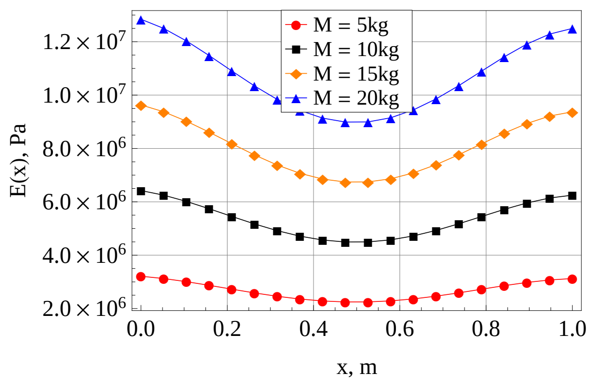

Figure 9: **Elastic modulus variations for a rotating pinned-free Timoshenko beam, for different values of the total mass, whose first mode shape is given by Fig. 6**

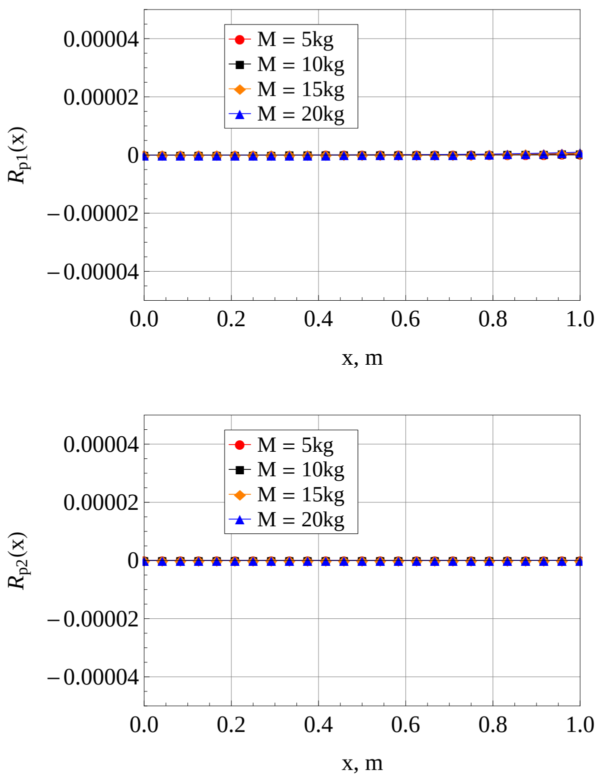

Figure 10: **Plot of the residues, R_{p1} & R_{p2}, after putting the derived expressions of mass density, shear modulus and elastic modulus, given by Figs. 7, 8 & 9, respectively, of a rotating pinned-free Timoshenko beam into the governing differential equations**

placement $W(x)$ and rotation due to bending $\phi(x)$, which satisfies all the given boundary conditions, from which we derived the material mass density $\rho(x)$, shear modulus $G(x)$ and elastic modulus $E(x)$ variations of the beam. The derived properties are simple polynomial functions which depend on the length L of the beam, the rotation speed Ω, the frequency ω and the total mass M of the beam. So essentially, given the length, rotation speed and the frequency of a particular mode (second and first mode for the cantilever and pinned-free boundary conditions, respectively), the mode shape will be given by the assumed polynomial $W(x)$.

It should be noted, that while assuming the polynomial variations for the elastic modulus $E(x)$ and shear modulus $G(x)$, both the functions have been varied independently, without making any attempt to constrain the physical limits of the Poisson's ratio. Thus, a very large variation for the Poisson's ratio (of the order of 10) was observed over the length of the beam. The only mathematical constraint that we put on the derived variations of the material properties is that they should all be positive throughout the length of the beam due to their physical nature. There are a few reports in the literature which shows the the possibility of existence of Poisson's ratio much greater than 1 for special kind of materials like anisotropic polyurethane foam [14], articular cartilage [15] and elastomer matrix laminates [16]. But the present paper only projects the existence of an infinite number of analytical polynomial functions for the material properties of a rotating non-homogeneous Timoshenko beam, such that the coupled governing differential equations will have an exact solution in the form of the assumed mode shapes and frequency. The derived results are strictly intended to serve as analytical test functions for the verification of different approximate or numerical methods which are routinely developed for the free vibration analysis of the rotating non-homogeneous Timoshenko beams.

Copyright Statement

References

[1] A. Bazoune, Y. A. Khulief, A finite beam element for vibration analysis of rotating tapered Timoshenko beams, Journal of Sound and Vibration 156 (1) (1992) 141–164.

[2] W. L. Cleghorn, B. Tabarrok, Finite element formulation of a tapered Timoshenko beam for free lateral vibration analysis, Journal of Sound and Vibration 152 (3) (1992) 461–470.

[3] S. S. Rao, R. S. Gupta, Finite element vibration analysis of rotating Timoshenko beams, Journal of Sound and Vibration 242 (1) (2001) 103–124.

[4] B. Yardimoglu, A novel finite element model for vibration analysis of rotating tapered Timoshenko beam of equal strength, Finite Elements in Analysis and Design 46 (10) (2010) 838–842.

[5] A. S. Vinod Kumar, R. Ganguli, Violin string shape functions for finite element analysis of rotating Timoshenko beams, Finite Elements in Analysis and Design 47 (9) (2011) 1091–1103.

[6] S. Mohanty, R. Dash, T. Rout, Free vibration of a functionally graded rotating timoshenko beam using fem, Advances in Structural Engineering 16 (2) (2013) 405–418.

[7] H. Du, M. K. Lim, K. M. Liew, A power series solution for vibration of a rotating Timoshenko beam, Journal of Sound and Vibration 175 (4) (1994) 505–523.

[8] S. C. Lin, K. M. Hsiao, Vibration analysis of a rotating Timoshenko beam, Journal of Sound and Vibration 240 (2) (2001) 303–322.

[9] J. R. Banerjee, Dynamic stiffness formulation and free vibration analysis of centrifugally stiffened Timoshenko beams, Journal of Sound and Vibration 247 (1) (2001) 97–115.

[10] O. O. Ozgumus, M. O. Kaya, Vibration analysis of a rotating tapered Timoshenko beam using DTM, Meccanica 45 (1) (2010) 33–42.

[11] S. Rajasekaran, Free vibration of centrifugally stiffened axially functionally graded tapered Timoshenko beams using differential transformation and quadrature methods, Applied Mathematical Modelling 37 (6) (2013) 4440–4463.

[12] D. V. Bambill, C. A. Rossit, R. E. Rossi, D. H. Felix, A. R. Ratazzi, Transverse free vibration of non uniform rotating Timoshenko beams with elastically clamped boundary conditions, Meccanica 48 (6) (2013) 1289–1311.

[13] I. Elishakoff, S. Candan, Apparently first closed-form solution for vibrating: inhomogeneous beams, International Journal of Solids and Structures 38 (19) (2001) 3411–3441.

[14] T. Lee, R. S. Lakes, Anisotropic polyurethane foam with poisson'sratio greater than 1, Journal of Materials Science 32 (9) (1997) 2397–2401.

[15] D. M. Elliott, D. A. Narmoneva, L. A. Setton, Direct measurement of the Poisson's ratio of human patella cartilage in tension., Journal of Biomechanical Engineering 124 (2) (2002) 223–228.

[16] L. D. Peel, Exploration of high and negative Poisson's ratio elastomer-matrix laminates, Physica Status Solidi B Basic Research 244 (3) (2007) 988–1003.