PAPER Nr.: 52

SOME APPROACHES FOR IMPROVING THE ACCURACY OF NONUNIFORM ROTOR BLADE DYNAMIC INTERNAL FORCE CALCULATION

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AUGUST 28–31, 1984–THE HAGUE, THE NETHERLANDS

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ABSTRACT

In this paper some approaches are presented for improving the accuracy of dynamic internal force calculation of nonuniform rotor blade with discontinuous stiffness and mass distribution. They are 1) the method using high-order finite elements, for which a family of nonuniform rotating beam conforming elements is developed, 2) the dynamic stiffness method, in which the internal forces of blade are determined directly from the nodal displacements by means of the dynamic stiffness matrixes of the finite elements, and 3) mixed-finite-element method, in which the method of weighted residuals is used. As an example, the blade flapwise bending vibration has been analyzed. Bending moments and some other numerical results are presented for a blade which has discontinuous bending stiffness and mass distribution along the spanwise direction. The results show that the approaches presented in this paper are effective.

NOTATION

e —— distance from center of rotation to blade root

EJ——bending stiffness

F—— nodal force for the mixed-element-method

H—— coefficient matrix of the Hermitian polynomial

K—— stiffness matrix

$\mathbf{K}_e$—— elastic stiffness matrix

$\mathbf{K}_g$ ——centrifugal stiffness matrix

l —— length of finite element

m—— mass per unit length

M—— bending moment

$\mathbf{M}$—— mass matrix

$\mathbf{N}$—— displacement shape function

$\mathbf{N_F}$—— force shape function

p—— applied load per unit length

$\mathbf{P}$—— external nodal force

$\mathbf{q}$—— amplitude of nodal displacement

r—— blade radial coordinate

R—— rotor radius

$\mathbf{S}$—— nodal force for the displacement method

$\mathbf{S_o}$—— amplitude of nodal force

T—— centrifugal force

$\mathbf{U}$—— nodal displacement

W—— lateral displacement normal to the plane of rotation

x—— element coordinate

$\mathbf{X}$—— row matrix $[1\ x\ x^2 \cdots\cdots x^{2n-1}]$

α, β——see Eq.(4-6), (4-7), (4-8)

Ω—— angular velocity of rotation

ω—— frequency of vibration

$(\dot{\ })$—— $\frac{\partial}{\partial t}$

$(\ )'$—— $\frac{\partial}{\partial r}$

$(\ )^e$——element matrix

Matrices and column vectors are denoted by bold symbols.

1. Introduction

Accurate prediction of rotor blade stresses or internal forces, bending moments, torsion moments, is one of the most difficult analytical problems of helicopter technology. This is due to the importance of nonlinear, unsteady, three-dimensional, compressible aerodynamics, and the complexity of the structural dynamic characteristics of nonuniform rotor blades. In order to improve the accuracy of the prediction, of course, it is the most important to improve the methods of aerodynamic and blade motion response calculations. However, the significance of the accuracy of calculating blade internal forces or stresses must not be underestimated for a certain accuracy of blade motion response calculation. There were some examples of dynamic component redesign during the helicopter development as a result of inaccurate blade elastic moment calculation. And this problem have been presented and discussed.[1][2] During developing a composite main rotor blade for the y-2 helicopter, the elastic moment calculation problem was also presented. Therefore, some approaches were explored to improve the accuracy of nonuniform rotor blade dynamic internal force calculation.

A rotor blade is generally a nonuniform rotating beam with discontinuities in stiffness and mass distribution. For such a structure, the conventional Rayleigh-Ritz method is not suitable, but the finite element method is a very good approach to calculate dynamic internal forces. The finite element method has been used widely for rotor dynamics analysis, including aeroelastic analysis.[3][4] In those analysis, however, conventional-beam-elements are generally used. Generally speaking, acceptable modal frequencies, modeshapes and displacement response for a variety of rotor dynamics problems can be obtained by using this element. The derivative which determines the dynamic stress,however, is almost always unacceptable. And the internal forces which are determined by the derivatives, as a rule, do not satisfy the equilibrium conditions at the nodes and the boundary conditions. With the purpose to overcome these disadvantages, we presented the following approaches in this paper.

Firstly, using the high-order finite elements is suggested. For that, a family of nonuniform rotating beam conforming elements is developed. Secondly, the dynamic stiffness method is used. In this method the internal forces of blades can be calculated directly from the nodal displacements by means of the dynamic stiffness matrices. The above two approaches are based on the displacement method. The third approach is using a mixed-finite-element method, in which the basic unknown parameters are not only the displacments but also the forces of nodal points. The formulae of the mixed-finite-element method for the rotor dynamics analysis are derived by using the method of weighted residuals, and a solving process is presented.

The problem of determining free vibration, response and stability characteristics of rotor is complex, especially when flapwise and chordwise bending and torsion are considered. Therefore, only the blade flapwise bending vibration is analyzed in this paper, thus, the main idea of these approaches can be expounded simply and clearly. And, for the same reason, only the numerical results of the natural frequencies, and the modeshapes for the displacement and the bending moment of blade flapwise bending vibration are presented in this paper. These approaches, however, can be applided to some more complex problems.

2. A Family of Nonuniform Rotating Beam Conforming Elements

In order to improve the accuracy of the analysis using the finite element method based on assuming shape functions of the elements, it is an effective approach to increase the order of the shape functions. For example, we may use the 5th, 7th or still higher order polynomial instead of the 3rd order. For that, there are various combination of nodes and/or nodal parameters. For a beam element, for instance, we may increase the number of degrees of freedom at two extreme nodes[5], or increase nodes within the element[6]. The former is not suitable for rotor blade with discontinuously varying properties, but a very good result can be obtained if the latter is used[6]. The analysis in Ref.6 is only for nonrotating beams. In the present work, the analysis is developed for rotating beams, and a family of nonuniform rotating beam conforming elements is presented.

A beam element rotating at constant angular speed Ω about an axis o–o is considered. The bending motion is described by W (Fig.1). The beam is assumed to be inextensional and the bending motion is purely out of plane (flapping).

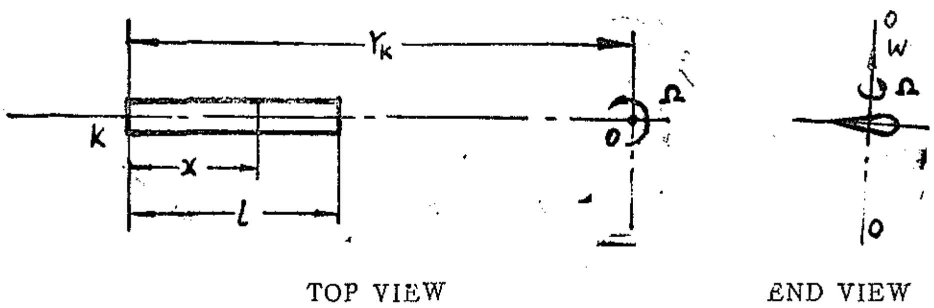

Fig.1 Geometry of the kth beam element

It is assumed that n ($\geqslant 2$) is the number of the nodes on the element. There is one node at each end of the element, and other n–2 nodes (if n>2) are within the element. The displacement W and slope $\frac{dW}{dx}$ at every node are used as nodal paramenters. The displacement function can be expressed as

$$W(x)=\sum_{i=1}^{n}\left[H_{0i}(x)\,W(x_i)+H_{1i}(x)\frac{dW(x_i)}{dx}\right] \qquad (2-1)$$

where $W(x_i)$ and $\frac{dW(x_i)}{dx}$ are the displacement and slope at the node i , respectively. $H_{ji}(x)$ is Hermitian polynomial ($j=0,1, i=1,2,\cdots n$).

Obviously, W(x) is an arbitrary odd power, 2n-1, polynomial. Equation (2-1) can be written in matrix form as

$$W(x)=\mathbf{XHU}^e \tag{2-2}$$

where

$$\mathbf{X}=[1 \;\; x \;\; x^2 \cdots\cdots x^{2n-1}]$$

$$\mathbf{U}^e=\left[W_1 \;\; \frac{dW_1}{dx} \cdots\cdots W_n \;\; \frac{dW_n}{dx}\right]^T$$

$\mathbf{H}$=coefficient matrix of the Hermitian polynomial

n is a variable number, so W (x) is a power series of a variable number of terms. Using these shape functions, we can develop a family of conforming elements.

In order to improve the accuracy, the rotating beam element in which the cross-sectional dimensions or mechanical properties may vary along its length is considered. It is assumed that the variations of mass m(x) and bending stiffness EJ(x) of the element can be expressed by

$$m(x)=m_k\left(1+\alpha_1\frac{x}{l}+\alpha_2\frac{x^2}{l^2}\right) \tag{2—3}$$

$$EJ(x)=EJ_k\left(1+\beta_1\frac{x}{l}+\beta_2\frac{x^2}{l^2}+\beta_3\frac{x^3}{l^3}+\beta_4\frac{x^4}{l^4}\right) \tag{2—4}$$

where m_k and EJ_k refer to the values at the left end of the element, i.e. node k, l is the length of the element, x is the local co-ordinate runing from 0 to l in the element, and α_i (i=1, 2), β_j (j=1,2,3,4) are the coefficients depending on the structural properties.

The mass matrix $\mathbf{M}^e$, the elastic stiffness matrix $\mathbf{K}^e_e$, and the centrifugal stiffness matrix $\mathbf{K}^e_g$ of the nonuniform rotating beam conforming element can be obtained

$$\left.\begin{aligned}
\mathbf{M}^e&=m_k\mathbf{H}^T\left(\int_0^l\left(1+\alpha_1\frac{x}{l}+\alpha_2\frac{x^2}{l^2}\right)\mathbf{X}^T\mathbf{X}dx\right)\mathbf{H}\\
\mathbf{K}^e_e&=EJ_k\mathbf{H}^T\left(\int_0^l\left(1+\beta_1\frac{x}{l}+\beta_2\frac{x^2}{l^2}+\beta_3\frac{x^3}{l^3}+\beta^4\frac{x^4}{l^4}\right)\frac{d^2\mathbf{X}^T}{dx^2}\;\;\frac{d^2\mathbf{X}}{dx^2}dx\right)\mathbf{H}\\
\mathbf{K}^e_g&=\Omega^2\mathbf{H}^T\left(\int_0^l T_x\frac{d\mathbf{X}^T}{dx}\;\frac{d\mathbf{X}}{dx}dx\right)\mathbf{H}
\end{aligned}\right\} \tag{2—5}$$

where $\Omega^2\cdot T_x$ represents the centrifugal force acting on the section x within the element

$$T_x=T_k+m_kr_kx+\frac{1}{2}m_k\left(\alpha_1\frac{r_k}{l}-1\right)x^2$$

$$+\frac{1}{3}m_k\left(\alpha_2\frac{r_k}{l^2}-\alpha_1\frac{1}{l}\right)x^3-\frac{1}{4}m_k\frac{\alpha_2}{l^2}x^4 \tag{2—6}$$

in which r_h is the distance from the left end of the element to the center of rotation, and

$$\overline{T}_h = T_h/\Omega^2$$

T_h is the centrifugal force acting on the left end section of the element.

The matrices $\mathbf{M}^e, \mathbf{K}_e^e$ and $\mathbf{K}_g^e$ of the elements with displacement functions based on the 3rd, 5th and 7th order polynomial respectively are presented in Ref.6 and 7, and are not given in the present paper due to lack of space.

3. The Dynamic Stiffness Method for Internal Force Calculation

For a undamped vibration, the equations of motion for each element are

$$\mathbf{M}^e\ddot{\mathbf{U}}^e + \mathbf{K}^e\mathbf{U}^e = \mathbf{S}^e(t) \qquad (3\text{—}1)$$

And for harmonic vibration, the equations can be written as

$$(-\omega^2\mathbf{M}^e + \mathbf{K}^e)\mathbf{q} = \mathbf{S}_o^e \qquad (3\text{—}2)$$

where the vectors $\mathbf{q}$ and $\mathbf{S}_o^e$ are the amplitudes of the nodal displacement $\mathbf{U}^e$ and the nodal forces $\mathbf{S}^e$ respectively, ω is the circular frequency of vibration. Obviously, after the $\mathbf{q}$ (or also ω) is obtained, the nodal forces $\mathbf{S}_o^e$ (and $\mathbf{S}^e$) can be easily got from (3—2).

It is interesting and important to note that the nodal forces of the two extreme nodes are just equal to internal forces on the end sections of the element for beam and bar types of elements, and that Eq. (3—2) is similar to the force-displacement relationship in static analysis, so matrix $\mathbf{D} = (-\omega^2\mathbf{M}^e + \mathbf{K}^e)$ is defined as dynamic stiffness matrix. Therefore the internal forces of the blade (and the beam, bar types of structures) can be calculated directly by Eq. (3—2) from the nodal displacement $\mathbf{q}$ and the dynamic stiffness matrix $\mathbf{D}$. This approach is called the dynamic stiffness method. This approach is only suitable for harmonic vibration because of using Eq. (3—2).

If ω and $\mathbf{q}$ are natural frequency and mode shape respectively, the mode internal forces can be obtained by means of the dynamic stiffness method.

It is necessary to remember that $\mathbf{K}^e$ for rotating beam consists of the elastic stiffness matrix $\mathbf{K}_e^e$ and the centrifugal stiffness matrix $\mathbf{K}_g^e$, that is

$$\mathbf{K}^e = \mathbf{K}_e^e + \mathbf{K}_g^e$$

It also should be noted that only the internal forces on the end sections of the element can be obtained by using the dynamic stiffness method (i.e.Eq.(3—2)), and the nodal forces within high-order element are not equal to internal forces.

There are some points which should be emphasized. The internal forces which are determined by the dynamic stiffness method do satisfy the equilibrium conditions at the nodes between neighburing elements and the boundary conditions, and they will converge to the exact solution if the global solution **U** (or also ω for free vibration) converge to the exact solution. The accuracy of the internal forces only depends on the accuracy of the solution **U** (or also ω), and it is not related to the derivatives of the solution **U**, such as $\frac{d^2W}{dx^2}$.

4. The Mixed-Finite-Element Method for the Rotor Dynamic Analysis

In this section, the formulae of the mixed-finite-element method for the rotor dynamic analysis are derived by using the method of weighted residuals,[8] and a solving process is presented. In the mixed-finite-element method, the basic unknown parameters of the problem are not only the displacements but also the forces of nodal points. The advantage of the approach is that both the displacements and the forces with a certain accuracy can be obtained simultaneously. As an example, the blade flapwise bending vibration is still considered.

The differential equation of the blade flapwise bending vibration can be appropriately written as

$$M'' - (TW')' + m\ddot{W} = p \tag{4—1}$$

$$W'' - \frac{1}{EJ}M = 0 \tag{4—2}$$

The boundary conditions are given by

$$\left.\begin{matrix} M=0 \\ M'=0 \end{matrix}\right\} \quad \text{at} \quad r=R$$

and

$$\left.\begin{matrix} W=0 \\ M=0 \end{matrix}\right\} \quad \text{at} \quad r=e \quad \text{(articulated blade)}$$

or

$$\left.\begin{matrix} W=0 \\ W'=0 \end{matrix}\right\} \quad \text{at} \quad r=e \quad \text{(cantilevered blade)}$$

(4—3)

According to the method of weighted residuals,[8] the formulae of the mixed-finite-element method are derived as follows.

The blade (global domain) is divided into a number of elements (subdomains). In the interior of each element the displacements and the bending moments are assumed, respectively, to be of the forms

$$\begin{aligned} W &= \mathbf{N}^e\mathbf{U}^e \\ M &= \mathbf{N}_F^e\mathbf{F}^e \end{aligned} \tag{4—4}$$

where $\mathbf{U}^e$ and $\mathbf{F}^e$ are the nodal displacements and forces respectively. They are independent. $\mathbf{N}^e$ and $\mathbf{N}_F^e$ are corresponding shape functions. Imposing compatibility conditions and equilibrium conditions, the nodal parameters $\mathbf{U}^e$ and $\mathbf{F}^e$ can be combined into a matrix of displacements $\mathbf{U}$ and a matrix of forces $\mathbf{F}$ on the assembled structure respectively. The local approximation, Eq.(4—4), can be extended over the whole domain by defining them as zero outside the particular element with which they are associated. Then, the global approximation can be expressed as

$$\begin{aligned} W &= \mathbf{N}\mathbf{U} \\ M &= \mathbf{N}_F\mathbf{F} \end{aligned} \tag{4—5}$$

where $\mathbf{U}$ and $\mathbf{F}$ are undetermined independent nodal displacements and forces respectively, and $\mathbf{N}$, $\mathbf{N}_F$ are corresponding shape functions.

The approximate global solution Eq.(4—5) is substituted into the Eq.(4—1) and (4—2). The shape functions $\mathbf{N}$ are used as weighing for Eq. (4—1), and $\mathbf{N}_F$ for Eq.(4—2) (i. e. the Galerkin method). The weighted residual, obtained through appropriate combination of the weighted differential equation and boundary condition residuals, is integrated by parts. Then, the following equations are obtained

$$-\boldsymbol{\alpha}\mathbf{F}+\mathbf{K}_g\mathbf{U}+\mathbf{M}\ddot{\mathbf{U}}=\mathbf{P} \tag{4—6}$$

$$\boldsymbol{\alpha}^T\mathbf{U}+\boldsymbol{\beta}\mathbf{F}=0 \tag{4—7}$$

where matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{K}_g$, $\mathbf{M}$, $\mathbf{P}$ can be formed from the corresponding element matrices $\boldsymbol{\alpha}^e$, $\boldsymbol{\beta}^e$, $\mathbf{K}_g^e$, $\mathbf{M}^e$, $\mathbf{P}^e$. The assembly of the element matrices into the complete system matrices is similar to the conventional finite element method, when utilizing the direct stiffness approach. And these element matrices are:

$$\begin{aligned} \boldsymbol{\alpha}^e &= \int_0^l \frac{d(\mathbf{N}^e)^T}{dx}\,\frac{d\,\mathbf{N}_F^e}{dx}\,dx \\ \boldsymbol{\beta}^e &= \int_0^l (\mathbf{N}_F^e)^T\,\frac{1}{EJ}\,\mathbf{N}_F^e\,dx \\ \mathbf{K}_g^e &= \int_0^l \frac{d(\mathbf{N}^e)^T}{dx}\,T\,\frac{d\mathbf{N}^e}{dx}\,dx \\ \mathbf{M}^e &= \int_0^l (\mathbf{N}^e)^T\,m\,\mathbf{N}^e\,dx \\ \mathbf{P}^e &= \int_0^l (\mathbf{N}^e)^T\,p\,dx \end{aligned} \tag{4—8}$$

It is clear that $\mathbf{K}_g^e$, $\mathbf{M}^e$ and $\mathbf{P}^e$ are the same element matrices as in the displacement approach. $\mathbf{K}_g^e$ is the centrifugal stiffness matrix, $\mathbf{M}^e$ is the mass matrix, and $\mathbf{P}^e$ is the external loading column matrix.

Solving the equations Eq.(4—6) and Eq.(4—7), all of the unknowns $\mathbf{U}$ and $\mathbf{F}$ are obtained. A solving method is presented as follows.

From Eq.(4—7), $\mathbf{F}$ can be expressed as

$$\mathbf{F}=-\boldsymbol{\beta}^{-1}\boldsymbol{\alpha}^T\mathbf{U} \tag{4—9}$$

Substituting Eq.(4—9) into Eq(4—6)

$$\boldsymbol{\alpha}\boldsymbol{\beta}^{-1}\boldsymbol{\alpha}^T\mathbf{U}+\mathbf{K}_g\mathbf{U}+\mathbf{M}\ddot{\mathbf{U}}=\mathbf{P} \tag{4—10}$$

Let

$$\mathbf{K}_e=\boldsymbol{\alpha}\boldsymbol{\beta}^{-1}\boldsymbol{\alpha}^T \tag{4—11}$$

Rewriting Eq.(4—10) as

$$(\mathbf{K}_e+\mathbf{K}_g)\mathbf{U}+\mathbf{M}\ddot{\mathbf{U}}=\mathbf{P} \tag{4—12}$$

Then, $\mathbf{U}$ can be obtained by solving Eq.(4—12), and $\mathbf{F}$ can be obtained from Eq.(4—9).

For modal parameters determination, let $\mathbf{P}=0$, the equation of motion becomes

$$(\mathbf{K}_e+\mathbf{K}_g)\mathbf{U}+\mathbf{M}\ddot{\mathbf{U}}=\mathbf{0} \tag{4—13}$$

Let $\mathbf{U}=\mathbf{q}e^{i\omega t}$, Eq.(4—13) becomes

$$(\mathbf{K}_e+\mathbf{K}_g-\omega^2\mathbf{M})\mathbf{q}=\mathbf{0} \tag{4—14}$$

The natural frequencies and mode shapes can be obtained from Eq.(4—14), and the mode internal forces can be calculated from Eq.(4—9).

It should be noted that $\mathbf{K}_e$ is a symmetric matrix. And the eigenmodes $\mathbf{q}$ are orthogonal with respect to the stiffness matrix $\mathbf{K}$ $(=\mathbf{K}_e+\mathbf{K}_g)$ and the mass matrix $\mathbf{M}$ respectively.

Obviously, Eq. (4—12) and Eq. (4—14) are the same forms as in the conventional displacement method. Therefore, many approaches and programes used in the displacement method can be also used here. Of course, this is very convenient and wishful.

As in the displacement method, the element properties matrices depend on the nodal parameters, the shape functions of displacement and force, mass and stiffness distributions within the element. In order to compare with the conventional conforming element, the mixed-finite-elements used in this paper are uniform beam elements, in which the nodal displacements and forces respectively are

$$\begin{aligned}\mathbf{U}^e&=[W_i\ \theta_i\ W_j\ \theta_j]^T\\ \mathbf{F}^e&=[M_i\ Q_i\ M_j\ Q_j]^T\end{aligned} \tag{4—15}$$

Where

$$\theta=\frac{dW}{dx}$$

$$Q=\frac{dM}{dx}$$

And the shape functions are cubic interpolation polynomials

$$\begin{aligned} W &= a_0+a_1x+a_2x^2+a_3x^3 \\ M &= b_0+b_1x+b_2x^2+b_3x^3 \end{aligned} \qquad (4\text{—}16)$$

5. <u>Illustrative Example and Discussion of Results</u>

The approaches presented in previous sections have been used to calculate the natural frequencies, the mode shapes for the displacement and the bending moment for many cases. And they have also been used to calculate the dynamic stresses for the y—2 composite main rotor blade. Some numerical results are presented here only for mode analysis of a nonuniform discontinuous rotating articulated blade.

The blade considered has discontinuous mass and stiffness distributions:

$$m(r)=\begin{cases} 3.75 & 0.07\leqslant r\leqslant 0.5 \\ 0.655-0.026\,r\left(\dfrac{kg-sec^2}{m^2}\right) & 0.5\leqslant r\leqslant 1.25 \quad (m) \\ 0.675-0.042r & 1.25\leqslant r\leqslant 5 \end{cases}$$

$$EJ(r)=\begin{cases} 13000 & 0.07\leqslant r\leqslant 0.5 \\ 5900-4000r \quad (kg-m^2) & 0.5\leqslant r\leqslant 1.25 \quad (m) \\ 1050-120r & 1.25\leqslant r\leqslant 5 \end{cases}$$

Rotor radius $R=5$ m

Flap hinge offset $e=0.07$ m

Angular velocity of rotation $\Omega=37.5$ 1/sec

In table 1 the natural frequencies ω and the bending moments EJW'' for the 3rd and 5th modes are tabulated for four different cases. Here NB3 and NB7 represent the nonuniform rotating beam conforming elements with displacement functions based respectively on the 3rd and 7th order polynomial, and n is the number of the elements. The results show the good convergence of the family of nonuniform rotating beam conforming elements. For a given number of elements, of course, high-order element is superior to low-order element. Even though for a given number of degrees of freedom, high-order element is also superior to low-order element. The fact has been observed by other researchers. Here it should be emphasized that the satisfactory values of the

Table 1 3rd and 5th natural frequencies ω and mode shapes (moment EJW″) (NB3,NB7)

		3rd mode				5th mode			
	r(m)	NB3, n=5	NB7, n=5	NB3, n=15	NB7, n=11	NB3, n=5	NB7, n=5	NB3, n=15	NB7, n=11
EJW″	0.07	−126.892	0.00012	−3.18415	0.00016	−1140.05	−0.02517	−29.6163	−0.02515
	0.50	−1509.11	−1274.47	−1306.96	−1274.48	−10604.7	−8544.49	−8814.37	−8544.52
		−1083.57	−1267.56	−1261.71	−1274.36	−9625.03	−8546.57	−8567.32	−8544.52
	1.00		−1159.07	−1140.24	−1160.28		−4735.65	−4784.14	−4735.29
				−1092.27	−1160.23			−4806.30	−4735.31
	1.25	−779.775	−970.055	−906.604	−976.934	−2053.39	−1907.07	−1958.06	−1904.96
		−1022.98	−977.079	−973.770	−977.098	−877.113	−1904.67	−1871.97	−1904.91
	1.50		−823.422	−821.286	−823.423		652.728	689.119	652.693
				−825.335	−823.422			721.835	652.709
	2.00	−534.128	−503.880	−507.703	−503.897	4216.51	3203.75	3301.84	3203.58
		−604.246	−503.849	−521.641	−503.898	3186.96	3212.00	3621.50	3203.59
	2.50		−42.6584	−58.9787	−42.6600		894.919	1241.21	894.093
				−51.3886	−42.6501			832.745	894.084
	3.00		490.332	484.243	490.323		−2811.08	−2973.26	−2812.66
				500.435	490.323			−3322.79	−2812.69
	3.50	913.910	908.743	922.629	908.734	−4559.55	−2097.39	−2596.16	−2101.40
		1301.60	907.577	945.632	908.733	1465.65	−2100.02	−2280.59	−2101.42
	4.00		959.804	1000.20	959.915		2422.38	2383.58	2419.72
				1009.48	959.915			2972.64	2419.76
	4.50		488.571	532.605	488.819		3119.38	3699.70	3118.92
				461.556	488.830			3149.65	3119.01
	5.00	134.031	−0.334985	−58.8369	0.010993	2584.26	14.8523	−312.671	0.101387
ω(1/sec)		163.1786	162.7428	162.7520	162.7428	405.3395	381.4113	381.7636	381.4112

moment EJW'' can be obtained by using only a few high-order elements. This is showed by the values of EJW'' of neighbouring elements at the same node points. So for improving the accuracy of dynamic internal forces, high-order-element is very effective.

In table 2 the displacements W, slopes W', moments EJW'' and M for the fifth mode shape are tabulated for both NB3 and NB7. The number n of the elements is 11. Here the displacements W are normalized for deflection of unity at the tip. M represent the moment calculated by dynamic stiffness

Tablc 2 5th mode shape (W—displacement, M—bending moment)
11—elements (NB3, NB7)

r(m)	W		W'		EJW''		M	
	NB3	NB7	NB3	NB7	NB3	NB7	NB3	NB7
0.07	0.00000	0.00000	−0.80378	−0.80551	−1062.60	−0.02515	0.00000	0.00000
0.50	0.31662	0.31734	−0.61899	−0.62054	−10110.5 −8704.78	−8544.52 −8544.52	−8533.4 −8533.4	−8544.4 −8544.4
1.00	0.33274	0.33367	0.58325	0.58249	−4896.28 −4803.03	−4735.29 −4735.31	−4727.3 −4727.3	−4735.2 −4735.2
1.25	0.11158	0.11253	1.17142	1.17146	−1959.68 −1873.91	−1904.96 −1904.91	−1900.6 −1900.6	−1904.9 −1904.9
1.50	−0.21648	−0.21565	1.33372	1.33421	681.844 982.633	652.693 652.709	650.27 650.27	652.69 652.69
2.00	−0.60553	−0.60550	−0.05081	−0.04812	3571.02 3617.26	3203.58 3203.59	3189.3 3189.3	3203.5 3203.5
2.50	−0.13885	−0.13966	−1.58202	−1.57996	1244.29 835.193	894.093 894.084	883.10 883.10	894.07 894.07
3.00	0.56545	0.56492	−0.78335	−0.78517	−2972.70 −3322.73	−2812.66 −2812.69	−2805.3 −2805.3	−2812.6 −2812.6
3.50	0.38399	0.38428	1.45158	1.44868	−2598.23 −2282.70	−2101.40 −2101.42	−2087.8 −2087.8	−2101.3 −2101.3
4.00	−0.46955	−0.46925	1.31222	1.31255	2383.05 2972.51	2419.72 2419.76	2416.5 2416.5	2419.7 2419.7
4.50	−0.38872	−0.38791	−1.80570	−1.80424	3700.93 3150.72	3118.92 3119.01	3115.1 3115.1	3118.9 3118.9
5.00	1.00000	1.00000	−3.17643	−3.17541	−312.745	0.101387	0.00000	0.00000

method. The results show that the dynamic stiffness method is very simple and effective for internal force calculation, especially, when the low-order elements are used. Here we only emphatically point out the following facts: 1) The internal forces calculated by the method always satisfy the equilibrium conditions and the boundary conditions. As has been stated, however, for the conventional compatible element method the internal forces determined by the derivatives (such as EJW'') generally do not satisfy the above conditions. Therefore, the dynamic stiffness method can simply overcome the disadvantage of the conventional compatible element method. 2) The accuracy of the internal forces calculated by the dynamic stiffness method only depends on the accuracy of the solution **U** (or also ω) of the whole structure, and it is not related to the derivatives of the solution **U** (such as W''). In table 2, the values of EJW'' for NB3 are not usable, but the values of M are quite accurate. The similar results are obtained in many other calculations. 3) Both the internal force calculated by the dynamic stiffness method and that by the derivative of the displacement converge to the same value, but, generally, the former is higher accurate (for the same **U**).

The natural frequencies ω for 11 elements model for three cases are tabulated in table 3, and the displacements W, the bending moments EJW'' or M for the same cases are presented in table 4. Here UB3 and UB7 represent the uniform rotating beam conforming elements with displacement functions based respectively on the 3rd and 7th order polynomial, and MB3 represents the mixed-element with cubic shape functions. The results show that both the displacements and the internal forces with satisfactory accuracy can be obtained simultaneously by using the mixed-finite-element method. In comparison with the results calculated by UB7, it is shown that the frequencies and moments by MB3 are superior to ones by UB3. The moments M by MB3 are quite accurate, but the moments EJW'' by UB3 are not usable. Therefore, as far as the result is concerned, the mixed-finite-element method is superior to the conventional displacement approach.

Table 3 Natural frequencies ω 11—elements (UB3, UB7, MB3)

I	ω (1/sec)		
	UB3	UB7	MB3
1	37.9258	37.9258	37.9258
2	92.7397	92.7393	92.7393
3	162.8095	162.7993	162.7995
4	257.9381	257.8513	257.8551
5	380.8590	380.4189	380.4489
6	530.6355	529.1757	529.3037
7	710.7011	707.0672	707.4632
8	928.9973	920.7491	921.8397
9	1190.0589	1171.9148	1174.6700
10	1514.6115	1457.5622	1467.2122

I—order of frequency

Table 4　5th mode shape (W—displacement, M—bending moment)
11—elements (UB3, UB7, MB3)

r(m)	W			EJW''		M
	UB3	UB7	MB3	UB3	UB7	MB3
0.07	0.00000	0.00000	0.00000	−1102.97	−0.025886	0.00000
0.50	0.33053	0.33123	0.33155	−10292.5 −8915.00	−8666.59 −8666.56	−8678.34
1.00	0.32550	0.32664	0.32677	−5109.04 −4799.59	−4804.93 −4804.93	−4811.29
1.25	0.09975	0.10085	0.10070	−1929.62 −1881.48	−1931.95 −1931.94	−1933.64
1.50	−0.22637	−0.22549	−0.22622	708.314 1027.13	661.422 661.447	662.446
2.00	−0.60836	−0.60859	−0.61019	3614.82 3574.94	3202.35 3202.34	3209.65
2.50	−0.13477	−0.13594	−0.13622	1199.40 759.386	876.422 876.406	881.083
3.00	0.56976	0.56927	0.57100	−3042.22 −3311.37	−2819.71 −2819.73	−2826.63
3.50	0.38411	0.38498	0.38625	−2596.55 −2198.40	−2101.66 −2101.67	−2111.34
4.00	−0.47275	−0.47207	−0.47377	2462.80 2981.38	2417.08 2417.14	2425.02
4.50	−0.39177	−0.39163	−0.39336	3737.23 3093.02	3130.58 3130.64	3153.44
5.00	1.0000	1.0000	1.00000	−379.923	0.087723	0.00000

6. Conclusion

The three approaches presented in this paper have been shown to be very effective for improving the accuracy of nonuniform rotor blade dynamic internal force calculation. Not only the accuracy of the displacement but also the accuracy of the derivative which determines the internal force are improved by using the high-order-element. The dynamic stiffness method can improve the accuracy of the dynamic internal force calculation for a certain accuracy of displacements. Both the displacements and the internal forces with satisfactory accuracy can be obtained simultaneously by using the mixed-finite-element method. A concrete analysis of concrete conditions must be made to determine which method should be used. For example, the high-order element may be used for the structure which can be represented by using less elements. If it is necessary to represent the structure using more elements because of more discontinuous points of the structure properties or orther reasons, using the low-order elements is still suitable. The internal forces can be calculated simultaneously by the dynamic stiffness method and the conventional approach (snch as EJW'') when the conforming elements are used. However, generally, when the high-order elements are used, EJW'' should be made acceplable, and when the low-order elements are used, the internal forces should be obtained from the dynamic stiffness method. For the case in which using the low-order shape function is desired and the dynamic stiffness method is not usable, the mixed-finite-element method should be used. Obviously, for the conventional low-order-element computer programs which have been used, the dynamic stiffness method is the most convenient and effective for improving the internal force calculation if it is usable.

It should be emphasized that although the analysis and example considered in the present paper have been limited to flapwise bending vibration and the numerical results are from free vibration, the approaches described herein can be extended to the more complex problems such as coupled bending, torsion vibration, etc.

7. Acknowledgement

The author is grateful to Professor Zhu Demao and Mr. Gao Deping for their helps in completing the English manuscript of this paper.

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