

# A NUMERICAL-EXPERIMENTAL COMPARATIVE ANALYSIS OF WATER IMPACT BEHAVIOUR OF SKIN PANELS UNDER DIFFERENT IMPACT CONDITIONS

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**Abstract:** Water impacts of helicopters are likely to have tragic consequences for the passengers of helicopters. Most of the passive safety devices to improve helicopter crashworthiness are designed for ground impact.

When impacting a solid or a soft surface, impact loads are rather different and therefore energy absorption devices developed for ground impact are not effective during a water impact.

In order to collect reliable data for numerical model validation, water impact drop tests were carried out on a rigid Aluminium alloy thick plate endowed with specifically designed pressure transducers. The tests aimed to measure impact decelerations and pressures considering different impact conditions, i.e. different velocities, impact angles and test article masses.

Afterwards, a numerical model of the tests was realized. The fluid region was modelled with *Smoothed Particle Hydrodynamics* (SPH) method. Applications of SPH method to water impact analysis are not common because of SPH model instability in unbounded regions and difficulties related to finding the correct number of SPH particles.

The conditions for SPH model stability were investigated and the accuracy of the overall model was evaluated referring to the data collected in the tests.

Eventually, a close experimental-numerical correlation was obtained and guidelines for further investigations were drawn.

## INTRODUCTION

Water impact is a topic that is increasingly gathering interest in crashworthiness design of helicopters. Accordingly with recent statistics [1], the 11% of civil aircraft accidents involves water impact and the percentage rises to the 20% for military aircraft accidents.

When considering only US civil helicopter accidents [2], over the 40% of the accidents involves ground or water impact and over 10% is fatal for the helicopter occupants. Remarkable progresses in crashworthiness design have been achieved recently even if most of the passive safety devices have been developed considering ground impacts [3].

The structural response and the loads transferred during a water impact and a ground impact are rather different. Therefore, it is not unusual that energy absorption devices developed for ground impact are not effective during a water impact.

Impact loads during a water impact are not as high as during a ground impact, but the impact duration is longer, the distribution of the forces is different and involves parts of the structure that are not designed to carry impact loads [4].

Furthermore, during a ground impact, load transfer depends only on the structure behaviour whilst, when impacting a fluid surface, load transfer depends both on structural response and on fluid-structure interaction.

During a ground impact (Figure 1), the subfloor structure of a helicopter (frames and spars) absorbs the impact energy by progressively deforming and guaranteeing smooth deceleration profiles whilst the skin panels are not involved. On the contrary, during a water landing (Figure 1), the water pressure insists on the skin panels which are not meant to carry such a large load and hence skin panels collapse. Consequently, the load transfer mechanism from skin panels to spar fails, the loads are no longer redistributed on the spars and the subfloor structure is not more capable to absorb the impact energy. The failure of skin panels leads to two potentially critical consequences: reduction of the energy absorption capability of subfloor (as the load path changes) and water inrush into the subfloor with consequent various types of malfunctioning (such as cabin flood and reduction of helicopter floating time). Fluid-structure interaction is a complicated event to model and its numerical analysis is extremely difficult to investigate. Therefore experimental water impact tests are mandatory. Nevertheless, water impact tests are often not repeatable, expensive and difficult to perform. Researches aiming at deepening the knowledge of the event are fundamental to develop efficient numerical tools to reduce the number of tests and to design high efficiency water impact worthy structures.

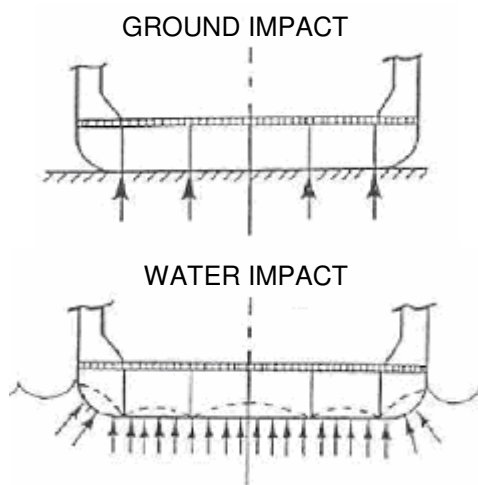


Figure 1: Loads distribution on an aircraft subfloor [4]

The research carried out at the Laboratory for Safety in Transports (LAST) of Politecnico di Milano, was focused on fluid-structure interaction between water and an Aluminium alloy thick plate.

In detail, the research consisted of two phases: an experimental phase and a numerical phase. In the experimental phase, an intense test campaign was carried out and impact decelerations and pressures were acquired. Several water impact drop tests were carried out using a 400x400 mm sides and 15.0 mm thick Aluminium alloy plate. Since its thickness, the plate was considered a rigid body. The plate was installed on a dedicated test frame and the pressure measurements were obtained realizing specific pressure transducers directly on the plate. Drop tests were performed also considering different impact conditions, i.e. different velocities, impact angles and test article masses. The data were also compared to the ones acquired in a previous research on a deformable panel, to highlight its energy absorption capability.

The tests aimed at collecting reliable experimental data to develop and validate numerical models. In the numerical phase, the tests were reproduced adopting the Finite Element (FE) approach to model the test article and a meshless formulation to model the fluid region: the Smoothed Particle Hydrodynamics method (SPH). Numerical-experimental correlation was considered referring to the horizontal tests. In view of that, assets and drawbacks of SPH fluid modelling were discussed and *rules of thumb* were drawn. Finally, findings and guidelines for further investigations and to study more complex events were obtained.

## 1. EXPERIMENTAL WATER IMPACT DROP TESTS

The intense test campaign carried out in the first part of the research consisted in performing water impact drop tests on a Aluminium alloy thick panel.

A solid test frame was built to investigate the impact behaviour of the panel. During the tests impact decelerations and pressures were acquired.

Besides, high velocity movies of the tests were recorded to evaluate the impact dynamics of the event.

### 1.1. The specimen

The specimen (Figure 2) was a flat 400x400 mm Al 6082 panel. The thickness of the panel was 15.0 mm and the mass 4.5 Kg. Typical aeronautic Aluminium alloy skin panels are usually thinner, about 1 mm, but the aim of these tests was to measure the impact pressure and acceleration on a rigid body and comparing with the ones measured during the tests on a deformable panel of the same material [5].

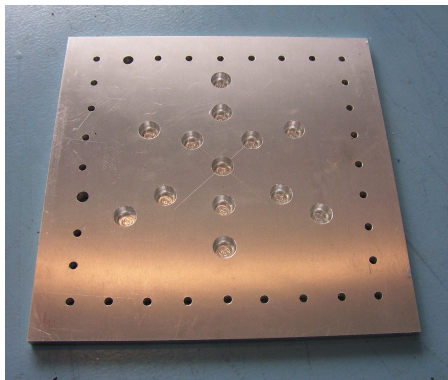


Figure 2: The Aluminium alloy panel

### 1.2. Test article

The test article (Figure 3) consisted of a massive base frame, four lateral flat Aluminium alloy panels and L-shaped corner stiffeners. The base frame, in particular, was a 400x400 mm, 40-mm height Al 6082-Ta16 plate machined to have a square hole of 320x320 mm. The Aluminium alloy panel was bolted on the base frame so that the actual impact region was 320x320 mm.

The test article was properly sealed to avoid water inrush. The global dimensions of the test article were 400x400x500mm and the mass was 16 Kg.

Most of the weight of the test article was due to the frame (massive and little deformable) so that the centre of mass was located at the bottom of the test frame. The lateral panels and the stiffeners (introduced to avoid sinking and to guide the test article during the fall) were rather stiff but lighter than the frame. The Aluminium frame allowed to test panels of different materials and thicknesses and to focus the analysis only on the panel behaviour. Water impact drop tests to evaluate the behaviour of skin panels to improve aircraft water crashworthiness are still quite rare and in this way this research is pioneering.

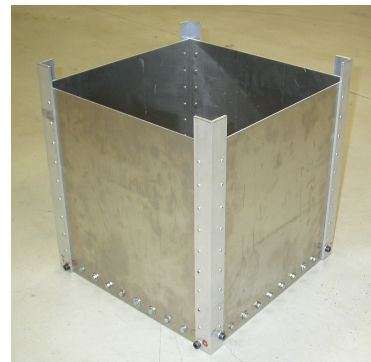


Figure 3: The Aluminium alloy test frame

### 1.3. Test facility

The dimensions of the specimen allowed performing the drop tests using the indoor facilities of LAST. A 3,000 tons bridge crane was used as hoisting system and a 1.5-m diameter and 1.4-m depth PVC round pool was used as water basin. The test article was hanged to a quick-release system and four steel cables were used to guide the article during the fall and to maintain the impact incidence within acceptable limits (i.e. smaller than 3 deg). The test facility is shown in Figure 6.

### 1.4. Measuring instruments and data acquisition system

Impact decelerations and pressures are quantities of paramount interest in designing structures safe in water landing and hence were measured.

### Accelerometers

Eight mono-axial ENTRAN D-0-500 accelerometers were used to measure impact decelerations. The accelerometers were fixed on the midpoints of the base frame sides (Figure 5).

Since the needing to perform non zero incidence tests, four accelerometers were used to measure the vertical accelerations and the others were used to measure the lateral accelerations.

The number and the pattern of the accelerometers allowed a sufficient redundancy of the measurements and the possibility to evaluate the impact incidence of the test article.

### Pressure transducers

The commercially-available pressure transducers are not meant to measure rapidly varying pressures in severe impact conditions. Therefore, a pressure transducer was developed to match the severe tests requirements. The idea was to evaluate the pressure measuring the deformation of a plate placed on the impacting surface of the specimen.

Accordingly, HBM MK/G 15/350 diaphragm strain gages were installed on blind holes made on the base plate (Figure 4) applying the same principle used in [6]. The sensitivity and the range of linearity of the transducers depend on the thickness of the base plate. The peak the pressure transducers had to catch was estimated referring to both analytical solutions for simple geometry rigid body water impact [7].

It was concluded that the impact pressure should have been order 4 bar (i.e. roughly 4.0 MPa) and occurs during a 2-m drop-height test. Moving from these considerations, the thickness of the transducers was decided referring both to analytical formulae [8] and FE analyses.

The characteristics of the transducers are listed in Table 1. Every single transducer was carefully calibrated applying a known pressure on the support and measuring the output tension of the gages.

The transducers showed a linear behaviour in the range of pressures considered. Figure 5 shows the transducer configuration that allows to measure pressure on 5 and 10 positions respectively along the plate radius and circumference.

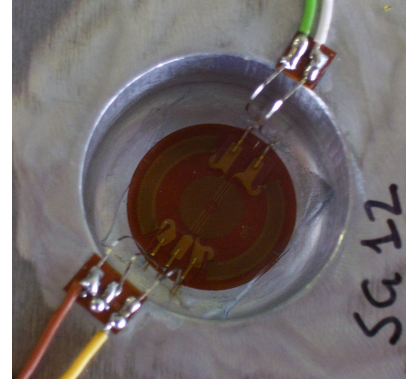


Figure 4: The installed pressure transducer

Inner diameter	20	mm
Outer diameter	25	mm
Thickness	0.8	mm
Sensitivity	0.75	mV/bar

Table 1: Pressure transducer characteristics

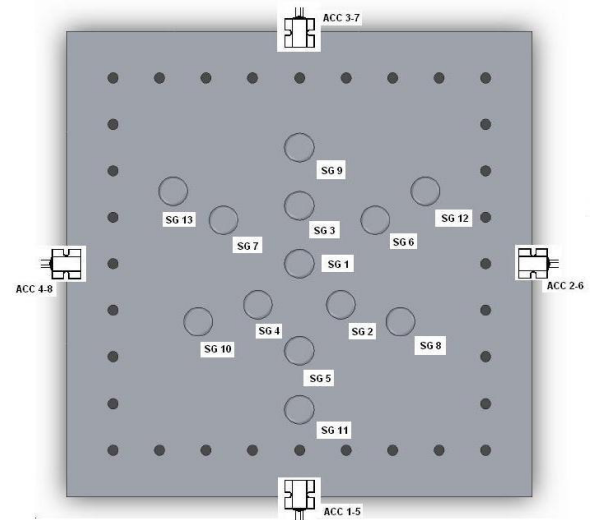


Figure 5: The transducer configurations

### High speed camera

The tests were filmed using a high speed camera to capture the impact dynamics of the event and to have a deep insight in it. The movies were also used to estimate the impact velocity and the incidence of the test article.





Figure 6: The test facility

#### Data acquisition system

The accelerometers and the pressure transducers were connected to two Power-DAQ 14 bit / 16 channels data acquisition systems. Signals were acquired at 10,000 Hz to avoid *aliasing* and to guarantee a reasonable number of *sample points* during the initial phase of the impact when accelerations and pressures have a sudden growth. The value of the sampling rate was also decided in order to evaluate the delay between the accelerometers pulses.

### 1.5. Carried out tests

The water impact tests were carried out releasing the test article from several prescribed heights. The facility used in the tests allows a maximum drop-height of 3.0 m. Nevertheless, to avoid local plasticity or cracks of the pressure transducers, the maximum drop-height was limited to 1.0 m. Measured impact velocities and analytical predictions based on weights drop showed that the influence of the friction of the guides was negligible (smaller than 3%). Carried out tests and measured impact velocities are listed in Table 2. For every height, the tests were repeated at least five times to ensure the accuracy of the measures and to verify the repeatability of the data acquired.

The impact incidence of the test article was evaluated on the basis of both high speed movies (Figure 7) and differences in acquired decelerations (pulse values and time delays). Only the tests with an impact incidence smaller than 3 deg were considered acceptable and therefore the number of tests carried out was larger than the one suggested from Table 2.

To compare the data obtained during the tests with previous measured data on a deformable Aluminium panel [5] installed on the same test frame, some ballasts were applied on the test frame to highlight the mass addition influence on impact decelerations. Table 3 shows the tests performed with two different impact masses, *i.e.* 27 and 30 Kg. Moreover the impact angle effect on pressure and acceleration was evaluated performing several tests with 7°, 15° and 27° test article pitch attitudes, as shown in Table 4.

Test #	Drop Height [m]	Impact velocity [m/s]
1	0.1	1.4
2	0.3	2.4
3	0.5	3.1
4	0.7	3.7
5	1.3	5.0

Table 2: Tests carried out (zero incidence)

Test #	Drop Height [m]	Impact mass [Kg]
1	0.1	25, 27, 31
2	0.5	25, 27, 31
3	0.7	25, 27, 31
4	1.0	25, 27, 31

Table 3: Tests carried out (variable mass)

Test #	Drop Height [m]	Impact angle [deg]
1	0.3	0, 7, 15, 27
2	0.7	0, 7, 15, 27
3	1.0	0, 7, 15, 27

Table 4: Tests carried out (impact angle)

## 1.6. Data collected

Frames from a high-speed movie of a test are shown in Figure 7. The impact deceleration and pressure time histories for three reference drop-heights are respectively plotted in Figure 8 and Figure 9. It is possible to infer the general trend of the measurements: a first peak and the following oscillations due to the base plate vibrations. Figure 10 shows the impact deceleration dependency on the drop height, plotted for all the tests performed and also for the tests on the deformable Aluminium panel (black dashed curve).

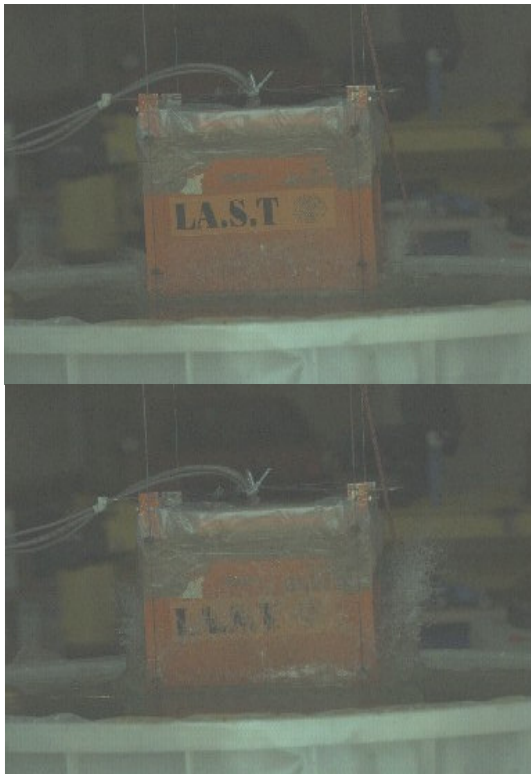


Figure 7: Frames from a high-speed movie

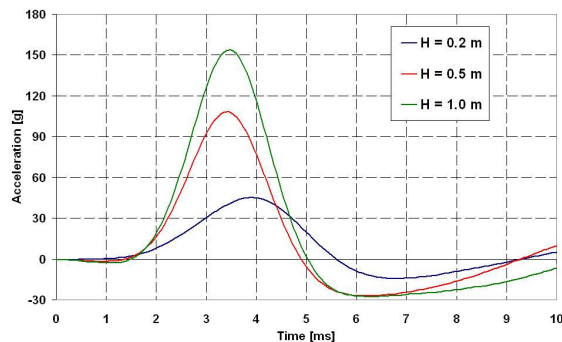


Figure 8: Impact decelerations of the test article

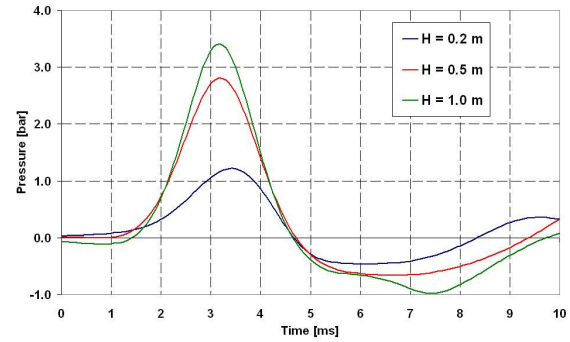


Figure 9: Impact pressure on the Aluminium plate

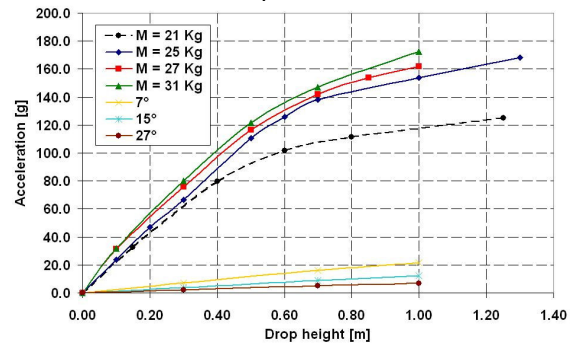


Figure 10: Impact deceleration vs drop height

## 1.7. Discussion

After analysing the obtained results, it was possible to draw some conclusions about the experimental phase of the research.

The repeatability of the measurements was achieved guaranteeing the same initial conditions in terms of impact angle, impact velocity and water conditions during the tests from each drop height.

Unfortunately it was not possible to further increase the drop height without yielding the pressure transducers because the semi-empirical predictions [7] relevantly underestimated the pressure. As a consequence the base plate elastic behaviour had to be taken into account when discussing results of tests with higher impact velocities. The pressure and acceleration trends with respect to the drop height were linear within a range depending on the impact mass, the impact angle and the plate stiffness.

The scattering of the measurements from the linearity was probably due to the deformability of the panel after a certain impact velocity.

Moreover, the mass addition led to an increasing of the acceleration peak and consequently of the pressure value.

This effect is not predicted by the Von Karman-Wagner theory [9] and it is presumably due to the different momentum transfer between the test article and the water during the first impact phase. Indeed, a flat plate does not considerably penetrate the water and a ballast addition increase both the initial momentum and the momentum transferred to the water, in terms of water mass starting to move.

According to [10] the measurements were made dimensionless.

It was noticed that as long as the accelerations and pressures are linear with respect to the drop height, the dimensionless curves are identical whilst, increasing impact velocity, the dimensionless peaks decrease.

Another outstanding effect observed during the tests was the reduction of acceleration and pressure during the inclined drop tests. Also a small impact angle (about  $5^\circ$ ) considerably reduces the loads acting on the impacting body.

Finally, the pressure transducer configuration allows to map the pressure on the plate surface. Even if pressure measurements strictly depend on local water conditions, the suitably designed transducer proved to be effective and a slight reduction of the pressure from the plate centre to the edge was observed. According to the *trapped-air effect* [7] a local pressure increasing was measured by the outer transducers.

Globally, the experimental results were satisfactory and allowed to make some interesting observations on flat plate water impacts. The experimental set-up was numerically reproduced in the second phase of the research.

## 2. NUMERICAL SIMULATIONS

The second phase of the research was devoted to develop and validate a reliable numerical model of the carried out tests.

The Lagrangian FE approach was adopted to model the test article whilst the fluid region was modeled adopting the SPH approach [11].

Despite its known drawbacks, the Eulerian approach is usually preferred in fluid modeling to the Lagrangian FE one because it allows handling severe deformations without significant accuracy reduction. The drawbacks in the use of the Eulerian formulation stimulate researches on different solutions of the problem such as meshless methods based on the Lagrangian approach. SPH is a genuinely meshless method initially introduced in astrophysics [11] and subsequently applied to a number of Continuum Mechanics problems such as events involving fluid-structure interaction or high-velocity impacts. Only zero-incidence tests were performed, hence the double symmetry of the problem was exploited and only a quarter of both the test article and the fluid region was modeled. Proper symmetry constraints were applied both to the Lagrangian model and the water region.

The numerical simulations were performed using LSTC/LS-Dyna [12], a proven non-linear finite element code that implements an effective SPH solver.

### 2.1. FE model of the specimen

The geometry of the base plate was simple and, hence, it was possible to build a rather regular mesh. The plate was modelled using four-node solid elements except for the pressure transducers which were modelled using shell elements. The chosen reference length (about 2.5 mm) was a trade-off between accuracy and CPU-time required by the simulations and strictly depends on the characteristic distance of the SPH particles.

The model consisted of 7389 nodes and 31884 elements. Because of its thickness, the plate was modelled with a rigid material whilst the elastic piecewise linear plasticity material model was adopted for the transducers.

Only tests with a base plate rigid behaviour were numerically reproduced, then the use of rigid material is recommended.

Static and dynamic experimental tests were performed to compute the Aluminium mechanical properties. As a consequence Cowper-Symonds coefficients were introduced to represent the strain-rate dependency.

Bolts were not modelled and the base plate was linked to the test frame using a tied contact [12].

## **2.2. FE model of the test article**

The geometry of the test article was simple and it was possible to build a rather regular mesh. A slightly higher reference length than the one used for the base panel was decided. The riveted and bolted joints were not modelled since it was observed that the benefits of modelling in details the joints were not such to justify the increased model complexity and the required CPU-time. Point masses were introduced in place of rivets and bolts in effort to reproduce the correct mass distribution.

The accelerometers were modelled using specific accelerometer elements that allow to accurately measure the accelerations in local axis.

Overall, 8471 elements were used to model the test article: 6720 eight-node solid elements for the base frame, 1714 four-node shell elements for the lateral panels and the stiffeners, 37 point masses and 4 dedicated discrete elements type accelerometer [12].

The elastic piecewise linear plasticity material model was adopted. The test article was placed over the fluid surface and the initial velocity equal to the one measured during the tests from 0.5 m was imposed to them.

## **2.3. SPH model of the fluid region**

The water basin in the tests was a 1.5-m diameter *pool*.

In effort to limit required CPU-time and memory allocation and to avoid rigid motion of the water, the dimensions of the fluid region in the numerical simulations were smaller than the actual ones: the fluid region was modelled as a 600 mm-side cubic box. The SPH model consisted of 36000 particles. The accuracy of the SPH model depends on regularity of the particles layout; hence a uniform layout was created. The distance between the particles was 22 mm. The fluid-structure interaction was reproduced using a node to surface contact, based on penalty method, between the SPH particles and the test article. The boundary conditions were imposed using a special treatment implemented in LSTC/LS-Dyna [12]. A set of ghost particles was automatically created by reflecting the particles closest to the boundaries.

## **2.4. Numerical-experimental correlation**

Numerical results were compared with experimental evidence referring both to the impact dynamics captured by the high-speed movies and the acquired impact decelerations and pressures.

### *Impact dynamics*

The behaviour of both the test article and the fluid region is similar to the one captured in the high-speed movie (Figure 11). The SPH model described accurately the behaviour of the fluid in terms of water mass motion and also showed clearly the spreading of the water particles.

### *Impact decelerations*

The impact deceleration of the test article was accurately reproduced: both the peak value and the event duration were close to the experimental measurements (Figure 12). In particular the peak value is slightly higher and the peak duration underestimated but globally the correlation is over 90%.



## Impact pressures

The impact pressure on the base plate was accurately reproduced: both the peak value and the event duration were close to the experimental measurements (Figure 13). In particular the peak value is slightly lower and the peak duration overestimated but globally the correlation is over 80%.

## Required CPU-time

CPU-time is central for any design-by-analysis procedure. The first 30 ms of the event were simulated using an Intel Core 2 Quad CPU, 2.40 GHz – 6 GB RAM PC. The same simulation was run five times and the average required CPU-time was 25 hours. Even if the CPU-time seems very high, it is comparable to the one needed for a simulation performed using the ALE approach to model the fluid region.

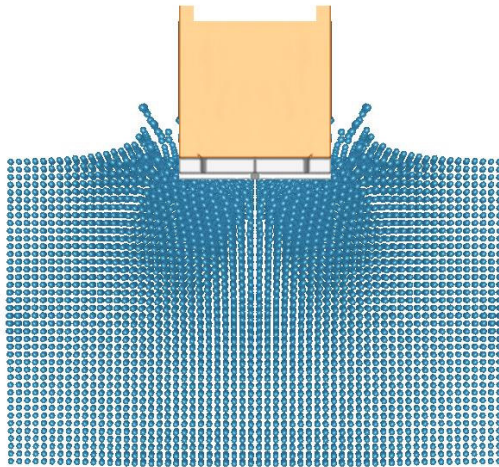


Figure 11: A frame from the numerical simulation

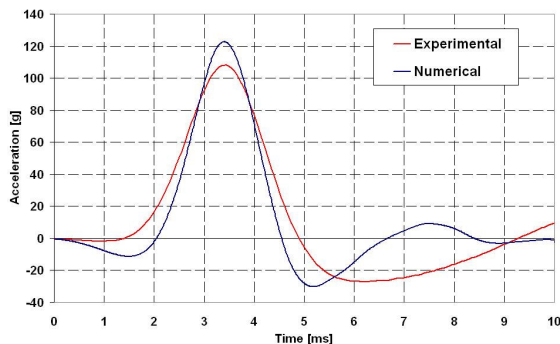


Figure 12: Numerical- experimental correlation (Acceleration)

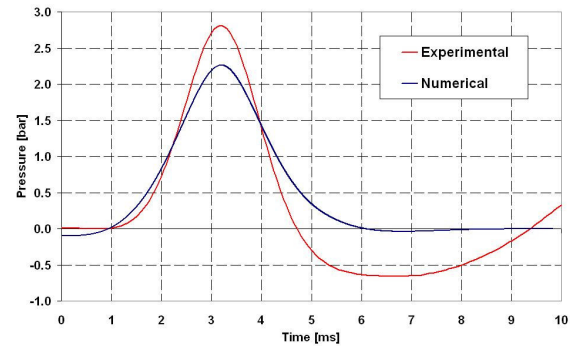


Figure 13: Numerical- experimental correlation (Pressure)

## 2.5. Discussion

The SPH model provided a common-sense description of the impact of the test article. With regard to the event dynamics, the behaviour both of the test article and the fluid was alike to the ones recorded in the high-speed movies of the tests.

Comparing numerical results and experimental data, a close correlation for both the impact decelerations and pressures was obtained. In particular the peak values and the time-profile of the curves were well reproduced. The correlation on acceleration is slightly more accurate with respect to the one on the pressure because acceleration is a global quantity whilst pressure deeply depends on local phenomena. Hence, improving the numerical computation of pressure implies refining the SPH layout with a consequent, and not convenient, increasing in computational cost.

Furthermore, the required CPU time is acceptable – especially when compared with the ones of other approaches such as the Eulerian or ALE formulations and referring to the accuracy of the solution.

In view of these results, the SPH method proved to be an effective approach to analyze the water impact event and the fluid-structure interaction.

The adoption of the SPH approach for the analysis of events featuring structures characterized by more complex geometries and mechanical behaviour is straightforward. In particular, some basic rules of thumb arose.

An accurate description of the materials behaviour, a proper contact algorithm and a rather uniform mesh are necessary for accuracy and stability of the results.

In view of the obtained results, the scenario for future developments of the research may be outlined.

Considering the lack of analytical prediction accuracy, a deeper theoretical investigation of the flat plate water impact has to be done to achieve the capability of predicting the loads acting on the structure. Moreover, a comparison with water region models based on the customary approaches (i.e. Lagrangian FE and the ALE approaches) seems recommendable to further highlight pros and cons of the SPH method.

In addition, a new challenging approach calls for further investigations: the Element Free Galerkin (EFG) Method. EFG [13] is a meshless method introduced to study crack propagation and hence used for soft-body impact analysis. Applications of this method to water impact are rare but the results promising.

Future simulations will be also carried out to investigate the water impact behaviour of deformable structures featuring structural breakdowns and water inrush.

## CONCLUSIONS

Water impacts of helicopters are rather likely to turn into a tragic event. In view of that, it is crucial to develop numerical tools to design safer helicopter structures.

The outcomes of a research carried out at the Laboratory for Safety in Transports (LAST), Politecnico di Milano is here presented. The research consisted of two phases: an experimental phase and a numerical phase. In the experimental phase, water impact drop tests were carried out and impact decelerations and pressures acting on a Aluminium alloy panel were acquired. The tests aimed at collecting reliable data to develop and validate numerical models focusing on impact dynamic and fluid-structure interaction.

The dynamics of the event was captured using a high-speed camera.

In the numerical phase, the tests were reproduced adopting SPH method to model the fluid region. A close numerical-experimental correlation was obtained proving the accuracy of SPH method in analysing water impact.

Application of SPH method to study more complex water impact events is likely to be an effective alternative to the customary Eulerian/ALE approach.

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