

INFLOW BELOW THE ROTOR DISK FOR SKEWED FLOW BY THE FINITE-STATE, ADJOINT METHOD

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Abstract

The inflow below the rotor disk is derived in a mathematically rigorous way from the potential flow equations and the Peters-Morrilo inflow model. With information from the co-states of the adjoint inflow equations (along with the normal states), one can calculate the velocity in the hemisphere below the rotor disk plane (the regions both within and outside the wake are included) without losing accuracy or convergence rate. The new methodology enables one to calculate the velocity below the rotor disk with the Finite-state method, and the only cost is the addition of the uncoupled co-states. Numerical results compared with exact solutions for the z-component in skewed angle flow are showed in order to illustrate the effectiveness of the new method.

Nomenclature			
$\bar{a}_m^n$:	real part for cosine induced inflow expansion coefficients	x_0, y_0 :	the Cartesian coordinates of the intersection point of rotor plane and the free-stream line, passing through a point $v, \eta, \bar{\psi}$
$\bar{b}_m^n$:	imaginary part for cosine induced inflow expansion coefficients	ξ :	non-dimensional coordinate along free-stream line, positive downstream, below disk
$[D]$:	damping matrix	τ_n^m :	cosine part of pressure coefficients
$[M]$:	mass matrix	ω :	frequency
m, r :	harmonic numbers	Φ_n^m :	cosine part of pressure potentials
n, j :	polynomial numbers	Ψ_n^m :	cosine part of velocity potentials
P :	pressure divided by $V_\infty^2 \rho$ (non-dimensional)	$v, \eta, \bar{\psi}$:	ellipsoidal coordinates
$\bar{P}_m^n(v)$:	normalized Legendre function of first kind	$v_0, \eta_0, \bar{\psi}_0$:	the ellipsoidal coordinates of the intersection point of rotor plane and the free-stream line
$\bar{Q}_m^n(i\eta)$:	normalized Legendre function of second kind	$v_0^*, \eta_0^*, \bar{\psi}_0^*$:	the ellipsoidal coordinates of the centro-symmetric point of the intersection point of rotor plane and the free-stream line
R :	rotor radius	$v^*, \eta^*, \bar{\psi}^*$:	the ellipsoidal coordinates of the centro-symmetric point of the current position
t :	reduced time	ρ :	air density kg/ m ³
$\bar{u}, \bar{w}$:	real and imaginary parts of velocity vector	$\bar{\psi}_0$:	azimuth angle on opposite side of rotor, which is equivalent to $\bar{\psi}_0 + \pi$
V_∞ :	free-stream velocity		
$\bar{v}$:	vector of velocity components		
$v_r, v_z, v_{\bar{\psi}}$:	radial, axial, and azimuthal induced velocity components of $\bar{v}$		
x, y, z :	non-dimensional rotor disk coordinates by R		

I. Introduction

There are many methods that can be used to calculate the aerodynamics of rotor systems, such as vortex-lattice methods (VLM) and computational fluid dynamics (CFD). Compared with those approaches, finite-state methods requires less computation time, which assures that they can be used in real-time rotor wake analyses. Moreover, the finite-state method gives physical insight into the dynamic behavior, so it attracts much attention both in the literature and industry.

In the early 1980s, Pitt^[1] developed a linear, unsteady theory that relates the transient rotor loads (thrust, roll moment and pitch moment) to the overall transient response of the rotor induced flow field (uniform flow, a side-to-side gradient, and a fore-to-aft gradient), which is known as the Pitt-Peters model. The theory was verified experimentally.^[2]

However, that model gives only the crudest wake description (uniform plus two gradients) and lacks higher-harmonic terms in the flow field. Cheng-Jian He^[3] extended the concept to form a theory for the all harmonics and radial distributions of the vertical component of flow at the rotor disk.

The He model has been validated against wind-tunnel data.^[4] As a mature dynamic inflow model, the Peters-He model is widely used in many production codes among which are: 1.) the simulation code FLIGHTLAB (Advanced Rotorcraft Technology), 2.) the comprehensive code COPTER (Bell Helicopter), and 3.) codes of ONERA-DFVLR (European Community).^[6]

However, the limitations of the He model are also obvious. The model can only be used to compute the vertical component of induced flow at the rotor disk and is unable to calculate the other components of flow at the disk. With the exception of steady flow, it cannot be used to be used to compute flow from the rotor disk, which is necessary in many applications such as ground effect and multiple rotors.

Wen-Ming Cao attempted to calculate the flow off the rotor disk as well as on the rotor disk.^[5] His attempts failed, but proved that a second set of wake states are needed for calculating the flow off the rotor. What these states would be remained unknown.

Morillo^[6] addressed these issues and showed that the extra states could be found rigorously by including the mass source terms in the expansions. Morillo wrote a generalized velocity potential (valid above the plane of the rotor disk) and expanded that potential in terms of Legendre functions. This was the same set of functions used by Pitt and He—

who only considered odd functions and used them only for the normal component of velocity. By including both odd and even functions—and by treating them as velocity potentials—Morillo was able to apply the Galerkin approach to obtain a closed-form set of equations for all three components of the velocity everywhere in the upper hemisphere including the rotor disk.

The Morillo model gave excellent agreement with a class of closed-form solutions for step response and frequency response, but convergence was slow due to ill-conditioned matrices arising from the absence of certain singular functions. Researchers^{[7], [8], [9]} tried to overcome this issue and were finally able to incorporate the elusive singular functions into a complete dynamic inflow model for all components of flow in the upper hemisphere with good convergence. The stability and effects of the completed dynamic model on aerodynamics inflow have been studied by eigen-analysis.^[10]

Recently, Fei extended Morillo's model and found a rigorous solution for the flow below the plane of the rotor by the finite-state method. His approach is general but has been validated only for step response and frequency response in axial flow. The validation of the new formula for skewed inflow still needs to be verified, which is the motivation for this present work.

The rest of the paper is organized as follows. The exact solution for a simple harmonic excitation is introduced in Section II. The derivation of rigorous solution for skewed flow will be developed in Section III. Numerical results for different skewed angle compared with exact solution are shown in Section IV to illustrate the proposed results and finally the conclusion of the paper is mentioned in Section V.

II. Fluid dynamic equations

The three nondimensional potential flow equations for the pressure and velocity fields, known as momentum and continuity, are:

$$(1) \quad \frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial \xi} = -\bar{\nabla} P$$

$$(2) \quad \nabla \cdot \bar{v} = 0$$

where $\bar{v}$ is the local induced velocity vector, t is nondimensional reduced time, ξ is the stream-wise direction (positive downstream), and P is the pressure field. It is assumed that the velocities are expressed as the gradient of a potential function, thus ensuring incompressibility and continuity.

The solution for simple harmonic excitation is found from a complex harmonic balance on the momentum equation. We begin with a derivation for the case of frequency response, and then use Fourier Transform arguments to extend the solution to the general time domain.

We first write the velocity as the real part of a complex quantity:

$$(3) \quad \vec{v}(x, y, z, t) = [\vec{u}(x, y, z) + i\vec{w}(x, y, z)]e^{i\omega t}$$

Next, the complex pressure field P is expressed as the summation of terms that include both discontinuities in pressure and mass sources on the disk:

$$(4) \quad P(x, y, z, t) = -\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} [\tau_n^m \Phi_n^m(\nu, \eta, \psi)] e^{i\omega t}$$

One can then rewrite Eq. (1) and obtain the following equation by substituting Eqs. (3) and (4) into Eq. (1)

$$(5) \quad i\omega [\vec{u}(x, y, z) + i\vec{w}(x, y, z)] + \left[\frac{\partial \vec{u}(x, y, z)}{\partial \xi} + i \frac{\partial \vec{w}(x, y, z)}{\partial \xi} \right] = -\vec{\nabla} P$$

where $\vec{u}(x, y, z)$ and $i\vec{w}(x, y, z)$ are the real and imaginary parts of the complex velocity; where it is implicitly assumed that one takes only the real part of Eq. (3); and where Φ_n^m are a complete set of potential functions with discontinuities across the disk. Terms with $m+n$ odd represent pressure discontinuities across the disk and terms with $m+n$ even represent mass sources at the disk.

In order to obtain a closed-form frequency response, one can collect the real and imaginary parts of the equation, which yields

$$(6) \quad -\omega \vec{w} + \frac{\partial \vec{u}}{\partial \xi} = -\vec{\nabla} P$$

$$(7) \quad \omega \vec{u} + \frac{\partial \vec{w}}{\partial \xi} = 0$$

Taking $\vec{u}$ from Eq. (7) and substituting it back into the derivative of Eq. (6); and then taking $\vec{w}$ from Eq. (6) and substituting it back into Eq. (7), we have

$$(8) \quad \frac{\partial^2 \vec{w}}{\partial \xi^2} + \omega^2 \vec{w} = \omega \vec{\nabla} P$$

$$(9) \quad \frac{\partial^2 \vec{u}}{\partial \xi^2} + \omega^2 \vec{u} = -\frac{\partial}{\partial \xi} \vec{\nabla} P$$

Taking a Laplace transform in ξ of Eqs. (8) and (9) gives

$$(10) \quad \vec{U}(s) = -\frac{s}{s^2 + \omega^2} \vec{\nabla} P(s)$$

$$(11) \quad \vec{W}(s) = \frac{\omega}{s^2 + \omega^2} \vec{\nabla} P(s)$$

where $\vec{W}(s)$ is the Laplace transform of $\vec{w}$, $\vec{U}(s)$ is the Laplace transform of $\vec{u}$, and $\vec{\nabla} P(s)$ is the Laplace transform of $\vec{\nabla} P$.

Based on the convolution inverse of a Laplace transformation, we have

$$(12) \quad \begin{aligned} \vec{u}(\omega, x_0, y_0, \xi_0) &= -\int_{-\infty}^{\xi_0} \cos[\omega(\xi_0 - \xi)] \vec{\nabla} P(\xi) d\xi \\ \vec{w}(\omega, x_0, y_0, \xi_0) &= \int_{-\infty}^{\xi_0} \sin[\omega(\xi_0 - \xi)] \vec{\nabla} P(\xi) d\xi \end{aligned}$$

Because the only boundary condition on the flow is that the flow must go to zero far upstream ($\xi = -\infty$), the lower limits are set at that point. The physical meaning of Eq. (12) is that, in order to obtain the exact solution, one must integrate the gradient of the pressure field along a streamline from far upstream up to the point within the flow field.

For the real part of the flow, the integral includes the kernel $\cos[\omega(\xi_0 - \xi)]$, and for the imaginary part it includes the kernel $\sin[\omega(\xi_0 - \xi)]$. As a special case, it should be noted that $\xi = z$ for axial flow.

The above derivation can be used as a way to determine an exact solution against which to compare the results derived by other methods, (however it is not tractable for practical rotor calculations). Because Eq. (12) is valid both above and below the rotor disk, it provides the starting point from which to derive the velocity below the disk in terms of the velocity above.

III. Formulation for Skewed Flow

In this section, the derivation is aimed to show that, if one knows the velocity field in the upper hemisphere $\xi < 0$, then one can find the velocity field in the lower hemisphere $\xi > 0$ simultaneously. Suppose that one knows the solution of the frequency response outlined above by some method (such as the finite-state method) but that the solution is only valid

in the upper hemisphere. Our goal is to use that solution to find the solution for the complex velocity in the lower hemisphere.

In order to simplify the derivation, we define the following quantities for $-\infty < \xi_0 < 0$.

$$\begin{aligned}
 \bar{C}(\omega, x_0, y_0, \xi_0) & \\
 & \equiv -\int_{-\infty}^{\xi_0} \cos(\omega\xi) \bar{\nabla} P(\xi) d\xi \\
 \bar{S}(\omega, x_0, y_0, \xi_0) & \\
 & \equiv -\int_{-\infty}^{\xi_0} \sin(\omega\xi) \bar{\nabla} P(\xi) d\xi
 \end{aligned}
 \quad (13)$$

For the special case $\tau_j^r = 1$, and all other $m \neq r, n \neq j$, $\tau_n^m = 0$, we can write without loss of generality.

$$\begin{aligned}
 \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) &= \bar{C}(\omega, x_0, y_0, \xi_0) \\
 & \equiv \int_{-\infty}^{\xi_0} \cos(\omega\xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
 \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0) &= \bar{S}(\omega, x_0, y_0, \xi_0) \\
 & \equiv \int_{-\infty}^{\xi_0} \sin(\omega\xi) \bar{\nabla} \Phi_j^r(\xi) d\xi
 \end{aligned}
 \quad (14)$$

where $x_0 = -r_0 \cos \bar{\psi}_0$ and $y_0 = r_0 \sin \bar{\psi}_0$. Then we can rewrite $\bar{u}(\omega, x_0, y_0, \xi_0)$ and $\bar{w}(\omega, x_0, y_0, \xi_0)$ as

$$\begin{aligned}
 \bar{u}(\omega, r_0, \bar{\psi}_0, \xi_0) & \\
 & = \cos(\omega\xi_0) \bar{C}(\omega, x_0, y_0, \xi_0) \\
 & \quad + \sin(\omega\xi_0) \bar{S}(\omega, x_0, y_0, \xi_0) \\
 \bar{w}(\omega, r_0, \bar{\psi}_0, \xi_0) & \\
 & = -\sin(\omega\xi_0) \bar{C}(\omega, x_0, y_0, \xi_0) \\
 & \quad + \cos(\omega\xi_0) \bar{S}(\omega, x_0, y_0, \xi_0)
 \end{aligned}
 \quad (15)$$

The induced velocity above the disk would consequently be given by

$$\begin{aligned}
 \bar{v}(r_0, \bar{\psi}_0, \xi_0, t) &= \bar{u}(\omega, r_0, \bar{\psi}_0, \xi_0) \cos(\omega t) \\
 & \quad - \bar{w}(\omega, r_0, \bar{\psi}_0, \xi_0) \sin(\omega t) \\
 & = \cos(\omega\xi_0) \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) \cos(\omega t) \\
 & \quad + \sin(\omega\xi_0) \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0) \cos(\omega t) \\
 & \quad + \sin(\omega\xi_0) \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) \sin(\omega t) \\
 & \quad - \cos(\omega\xi_0) \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0) \sin(\omega t)
 \end{aligned}
 \quad (16)$$

which is equivalent to:

$$\begin{aligned}
 \bar{v}(r_0, \bar{\psi}_0, \xi_0, t) & \\
 & = \cos[\omega(t - \xi_0)] \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) \\
 & \quad - \sin[\omega(t - \xi_0)] \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0)
 \end{aligned}
 \quad (17)$$

In preparation for the solution of the flow below the disk, we now need to compute $\bar{C}$ and $\bar{S}$ from the finite-state solution above the disk. By breaking the complex Morrillo states into real and imaginary parts $\bar{a}_n^m$ and $\bar{b}_n^m$, we have

$$\begin{aligned}
 \bar{u}(\omega, r_0, \bar{\psi}_0, \xi_0) &= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \bar{a}_n^m \bar{\nabla} \Psi_n^m(\nu, \eta, \bar{\psi}) \\
 & = \cos(\omega\xi_0) \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) \\
 & \quad + \sin(\omega\xi_0) \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0) \\
 \bar{w}(\omega, r_0, \bar{\psi}_0, \xi_0) &= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \bar{b}_n^m \bar{\nabla} \Psi_n^m(\nu, \eta, \bar{\psi}) \\
 & = -\sin(\omega\xi_0) \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) \\
 & \quad + \cos(\omega\xi_0) \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0)
 \end{aligned}
 \quad (18)$$

where $\nu, \eta, \bar{\psi}$ is the ellipsoidal coordinate location of $r_0, \bar{\psi}_0, \xi_0$; and Ψ_n^m is the velocity potential.

Therefore, we can solve for the $\bar{C}$ and $\bar{S}$ integrals from (18) in terms of the known finite-state result.

$$\begin{aligned}
 \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) & \\
 & = \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \bar{a}_n^m \cos(\omega\xi_0) \bar{\nabla} \Psi_n^m(\nu, \eta, \bar{\psi}) \\
 & \quad - \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \bar{b}_n^m \sin(\omega\xi_0) \bar{\nabla} \Psi_n^m(\nu, \eta, \bar{\psi}) \\
 \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0) & \\
 & = \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \bar{a}_n^m \sin(\omega\xi_0) \bar{\nabla} \Psi_n^m(\nu, \eta, \bar{\psi}) \\
 & \quad + \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \bar{b}_n^m \cos(\omega\xi_0) \bar{\nabla} \Psi_n^m(\nu, \eta, \bar{\psi})
 \end{aligned}
 \quad (19)$$

The above equations are valid above the disk plane which is actually for $-\infty < \xi_0 < 0$. However, in the lower hemisphere, where $\xi_0 > 0$, because of the discontinuities on the disk, we must integrate Eq.(15) to the disk of the rotor plane, stop, and then continue below the plane (as in a Cauchy integral). For simplicity, we will assume the single term $P = -\Phi_j^r$, where $\Phi_j^r = \bar{P}_j^r(\nu) \bar{Q}_j^r(i\eta) \cos(r\bar{\psi})$. Any pressure can be represented as a summation of these terms by superposition principle, so that this is a general approach.

The real and imaginary parts of the velocity below-plane will be:

$$\begin{aligned}
\bar{u}(\omega, r_0, \bar{\psi}_0, \xi_0) &= \int_{-\infty}^0 \cos[\omega(\xi_0 - \xi)] \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&\quad + \int_0^{\xi_0} \cos[\omega(\xi_0 - \xi)] \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&= \cos(\omega \xi_0) \bar{C}(\omega, r_0, \bar{\psi}_0, 0) \\
&\quad + \sin(\omega \xi_0) \bar{S}(\omega, r_0, \bar{\psi}_0, 0) \\
&\quad + \cos(\omega \xi_0) \int_0^{\xi_0} \cos(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&\quad + \sin(\omega \xi_0) \int_0^{\xi_0} \sin(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
\bar{w}(\omega, r_0, \bar{\psi}_0, \xi_0) &= -\int_{-\infty}^0 \sin[\omega(\xi_0 - \xi)] \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&\quad - \int_0^{\xi_0} \sin[\omega(\xi_0 - \xi)] \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&= -\sin(\omega \xi_0) \bar{C}(\omega, r_0, \bar{\psi}_0, 0) \\
&\quad + \cos(\omega \xi_0) \bar{S}(\omega, r_0, \bar{\psi}_0, 0) \\
&\quad - \sin(\omega \xi_0) \int_0^{\xi_0} \cos(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&\quad + \cos(\omega \xi_0) \int_0^{\xi_0} \sin(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi
\end{aligned} \tag{20}$$

where the integrals are broken into two parts: one from downstream infinity to the disk, and one from the disk to the desired point below the disk. By noticing the symmetry character in the system, we can rewrite the integral segments below the disk in terms of integral segments above the rotor disk, which gives:

$$\begin{aligned}
&\int_0^{\xi_0} \cos(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&= (-1)^{j+1} \int_{-\xi_0}^0 \cos(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&= (-1)^{j+1} [\bar{C}(\omega, r_0, \tilde{\psi}_0, 0) - \bar{C}(\omega, r_0, \tilde{\psi}_0, -\xi_0)] \\
&\int_0^{\xi_0} \sin(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&= (-1)^j \int_{-\xi_0}^0 \sin(\omega \xi) \bar{\nabla} \Phi_j^r(\xi) d\xi \\
&= (-1)^j [\bar{S}(\omega, r_0, \tilde{\psi}_0, 0) - \bar{S}(\omega, r_0, \tilde{\psi}_0, -\xi_0)]
\end{aligned} \tag{21}$$

From Fig. 1, it can be seen that, without considering the signs, the downstream streamline emanating from $r_0, \bar{\psi}_0$ should be identical in functionality to the upstream streamline emanating from $r_0, \tilde{\psi}_0$ (where $\tilde{\psi}_0 = \bar{\psi}_0 + \pi$). The sign terms in Eq. (21) are determined as follows.

a) For $r + j$ odd, the pressure potential is of opposite sign above and below the disk, whereas the gradient of pressure is the same sign above and below the disk. For $r + j$ even, the opposite is true. Since the integrals involve gradient of pressure, there is a factor of $(-1)^{r+j+1}$

b) The fact that $\tilde{\psi}_0$ is on the opposite side of disk as $\bar{\psi}_0$ implies that every $\bar{\psi}$ along the upstream streamline will differ by π from $\bar{\psi}$ along the downstream streamline. Therefore, the $\sin(r\bar{\psi})$ and $\cos(r\bar{\psi})$ terms in Φ_j^r will yield a factor of $(-1)^r$.

Consequently, a total sign change for the cosine integral will be:

$$(22) \quad \text{sign} = (-1)^{r+j+1} (-1)^r = (-1)^{2r+j+1} = (-1)^{j+1}$$

In the sine integral, an extra (-1) appears because $\sin(\omega \xi) = -\sin(-\omega \xi)$. Thus, the coefficients $(-1)^{j+1}$ and $(-1)^j$ appear in the last lines of Eq. (21). We then have $\bar{u}$ and $\bar{w}$ for $\xi_0 > 0$ from Eqs. (20) and (21):

$$\begin{aligned}
\bar{u}(\omega, r_0, \bar{\psi}_0, \xi_0) &= \cos(\omega \xi_0) \left\{ \begin{aligned} &\bar{C}(\omega, r_0, \bar{\psi}_0, 0) \\ &+ (-1)^{j+1} \bar{C}(\omega, r_0, \tilde{\psi}_0, 0) \\ &- (-1)^{j+1} \bar{C}(\omega, r_0, \tilde{\psi}_0, -\xi_0) \end{aligned} \right\} \\
&\quad + \sin(\omega \xi_0) \left\{ \begin{aligned} &\bar{S}(\omega, r_0, \bar{\psi}_0, 0) \\ &+ (-1)^j \bar{S}(\omega, r_0, \tilde{\psi}_0, 0) \\ &- (-1)^j \bar{S}(\omega, r_0, \tilde{\psi}_0, -\xi_0) \end{aligned} \right\} \\
\bar{w}(\omega, r_0, \bar{\psi}_0, \xi_0) &= -\sin(\omega \xi_0) \left\{ \begin{aligned} &\bar{C}(\omega, r_0, \bar{\psi}_0, 0) \\ &+ (-1)^{j+1} \bar{C}(\omega, r_0, \tilde{\psi}_0, 0) \\ &- (-1)^{j+1} \bar{C}(\omega, r_0, \tilde{\psi}_0, -\xi_0) \end{aligned} \right\} \\
&\quad + \cos(\omega \xi_0) \left\{ \begin{aligned} &\bar{S}(\omega, r_0, \bar{\psi}_0, 0) \\ &+ (-1)^j \bar{S}(\omega, r_0, \tilde{\psi}_0, 0) \\ &- (-1)^j \bar{S}(\omega, r_0, \tilde{\psi}_0, -\xi_0) \end{aligned} \right\}
\end{aligned} \tag{23}$$

Note that, in the above, terms from the integrals below the disk are rewritten in terms of integrals above the disk (i.e., the terms with $(-1)^j$ or $(-1)^{j+1}$ and which are functions of $\tilde{\psi}_0$

$$\begin{aligned}
(24) \quad & \bar{v}(r_0, \bar{\psi}_0, \xi_0, t) \\
&= \cos[\omega(t - \xi_0)] \bar{C}(\omega, r_0, \bar{\psi}_0, \xi_0) \\
&\quad - \sin[\omega(t - \xi_0)] \bar{S}(\omega, r_0, \bar{\psi}_0, \xi_0) \\
&\quad + (-1)^{j+1} \left\{ \cos[\omega(t - \xi_0)] \bar{C}(\omega, r_0, \bar{\psi}_0, 0) \right. \\
&\quad \left. + \sin[\omega(t - \xi_0)] \bar{S}(\omega, r_0, \bar{\psi}_0, 0) \right\} \\
&\quad - (-1)^{j+1} \left\{ \cos[\omega(t - \xi_0)] \bar{C}(\omega, r_0, \bar{\psi}_0, -\xi_0) \right. \\
&\quad \left. + \sin[\omega(t - \xi_0)] \bar{S}(\omega, r_0, \bar{\psi}_0, -\xi_0) \right\}
\end{aligned}$$
$$(25) \quad \begin{aligned} \bar{v}(r_0, \bar{\psi}_0, \xi_0, t) &= \bar{v}(r_0, \bar{\psi}_0, 0, t - \xi_0) \\ &+ \bar{v}^*(r_0, \tilde{\psi}_0, 0, t - \xi_0) - \bar{v}^*(r_0, \tilde{\psi}_0, -\xi_0, t) \end{aligned}$$
$$(26) \quad \vec{v}^* = (-1)^{j+1} [\vec{u} \cos(\omega t) + \vec{w} \sin(\omega t)]$$

The mathematical basis for generalizing the frequency domain in Eq. (24) to the time domain in Eq. (25) is that any general time-domain solution can be expressed as a Fourier Transform in terms of frequency components. As Eq. (24) is true for any frequency in the transform, then it is true in general for the inverse transform when written in terms of $\bar{v}^*$ as in Eq. (25).

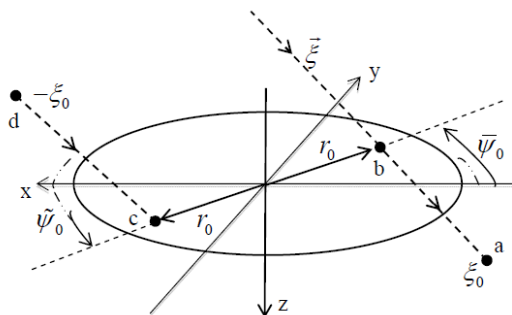


Figure 1 shows the 3-D perspective of the co-state method. If one needs to calculate the

In the general time domain, $\bar{v}^*$ is defined as the solution to the adjoint equations—i.e., the solution of the differential equations in which the time derivative terms are multiplied by (-1) and each forcing function in those time-domain equations—i.e., each $\tau_n^m(t)$ in Eq. (3)—is multiplied by $(-1)^{n+1}$.

IV. Results

Here, we concentrate on the z components (perpendicular to the disk plane in the direction of the free-stream) of skewed flow only. Two skewed angles, $\chi = 30^\circ$ and $\chi = 75^\circ$, and two frequencies, $\omega = 0$ and $\omega = 4$, are considered. Velocity is plotted on a plane one rotor radius below the rotor disk $z = 1$ ($\xi = 1$) where the rotor wake is the region $-1 < x_0 < 1$.

Figures 2 through 5 show the frequency response to the first collective pressure input, Φ_1^0 . In Figures 2 and 4 ($\omega=0$), there are no imaginary parts in the solution. In Figures 2 and 3 ($\chi=30^\circ$), the co-state method gives excellent convergence both on-disk and off disk; however, in Figures 4 and 5 ($\chi=75^\circ$),

the convergence of the co-state method is slow and with oscillations. This is typical also of the convergence on the disk, and will be discussed shortly.

Figures 6 through 9 are for oscillations of a cyclic pressure distribution, Φ_2^1 . Figures 6 and 8 give the static pressure ($\omega = 0$) responses, and Figures 7 and 9 give the velocity distributions for dynamic pressure ($\omega = 4$). Again, we can see that the results for small skew angle $\chi = 30^\circ$ is essentially exact, but for large skew angle $\chi = 75^\circ$ it is still good but not completely converged. The convergence is better for this cyclic response than it was for collective. In general, the convergence of the lowest collective mode is always the slowest.

Figures 10-13 are responses to an elliptical pressure oscillation. The normal component of velocity is given versus z all at $x_0 = 0.5$ (i.e., on a streamline passing through the disk at $r_0 = 0.5$). Figures 10 and 12 are with $\omega = 0$, so no imaginary parts are displayed. We can see good agreement between the closed form solutions and the co-state method within the streamline cylinder which contains the rotor disk.

Figures 14-17 are responses to a cyclic pressure distribution. Normal velocity is given versus z with $\omega = 4$. Here, velocities are shown both inside and outside of the rotor disk. Again, for small skew angle ($\chi = 30^\circ$), we have almost the identical results as from convolution. For large skew angle ($\chi = 75^\circ$), the result along a streamline within the cylinder ($x_0 = 0.5$) is virtually exact, Figure 16; but for $x_0 = 1.2$ (outside of the rotor disk), a small error occurs, Fig. 17. This follows directly from the corresponding error on the rotor disk in the Peters-Morillo model with $\chi = 75^\circ$ (See Ref. 6).

The error on the disk with the Morillo model has been shown to converge the best when fewer even terms are included as the skew angle increases). In fact, the optimum is such that—when all even terms are eliminated at 90° skew angle—the He result with no even terms is exact. It would therefore seem that the adjoint method could also be improved by an optimization of even potential functions with skew angle. (Presently even terms equal odd.)

Figures 18 through 21, show comparisons of the co-state method with even terms equal to odd terms against results for no even terms for a large skew angle ($\chi = 75^\circ$). The blue solid lines in each figure are obtained with the whole matrix, which contains both even and odd terms. The red dash lines are based on odd terms only.

From these figures, one can tell that the results obtained from the co-state method with only odd terms are more accurate than the ones with both odd and even terms. This verifies the findings of Morillo that the number of even terms should be optimized on the basis of skew angle.

V. Summary and Conclusions

The cost of completing the inflow theory to obtain all components of flow throughout the flow field is adding more states to calculate the additional information. For example in a case in which one is interested in four harmonics of the vertical component of flow on the disk, the Peters-He model, which has only the odd functions, would need 15 inflow states in order to obtain the normal flow on the disk including both cosine and sine terms.

For computing **all** components of the flow above the disk, one must add the even functions, which increases the number of states to 30 for the Peters-Morillo Model. In order to calculate the flow below the disk as well, one must additionally compute the adjoint velocity field from an additional 30 co-states, which means 60 states in all. One should note, however, that the co-states often follow in a trivial manner once the original states are found.

For example, in frequency response, the adjoint is just the complex conjugate of the complex velocity field multiplied by $(-1)^n$. Even for the general case, the solution of the co-states always involves less numerical effort than the computation of the states themselves. In particular, the co-states are only needed for past time, with the range of past time equal to the depth into the wake for which the solution is desired. This can be seen in Eq. (25) in which the flow at a point ξ_0 depends on the co-states at the time at which that flow particle was at the rotor disk, $t - \xi_0$.

If one has saved the forcing functions of the normal states from that duration of past time and then multiplies each τ_n^m by $(-1)^{j+1}$ to obtain the adjoint forcing function, it follows that the computation of the co-states involves only solution of a set of linear equations uncoupled from the rest of the rotorcraft simulation. This implies that the computation is very straightforward.

Moreover, because the linear adjoint equations are unstable, one must time march backwards from the present time to $t - \xi_0$ with zero initial conditions at the present time. It can be shown that Eq. (25) cancels the homogeneous part of the adjoint solution, so

that the initial conditions on this backwards-time computation are irrelevant. Therefore, the initial conditions can be taken to be zero, which makes the second term on the right-hand side of Eq. (25) equal to zero. Thus, the adjoint velocity is needed only on the disk. This implies that the co-state method can also be used with the He model to find the z-component of flow downstream even though the He model only computes the z component of flow on the disk.

In conclusion, a new methodology has been developed for computing the flow field anywhere below the rotor disk for any wake skewed angle with Finite-state. It is derived in a mathematically rigorous way from the potential flow equations. The comparisons with the closed-form solutions demonstrate the effectiveness of the new approach. The cost of the operation is that one needs to compute the co-states for the adjoint velocity as well.

Acknowledgments

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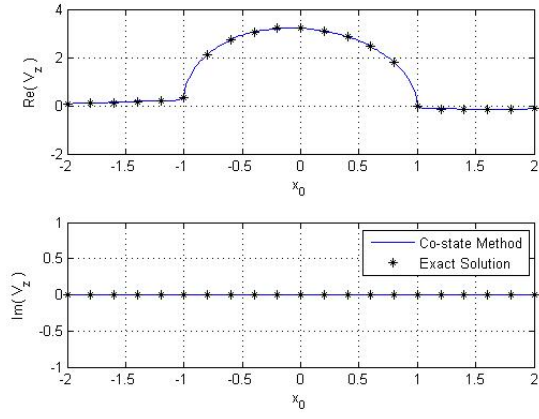


Figure 2 Velocity for $\xi=1$ with τ_1^0 for $\omega=0$ and $\chi=30^\circ$

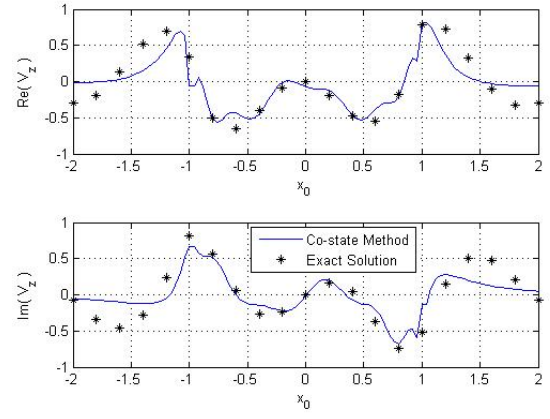


Figure 5 Velocity for $\xi=1$ with τ_1^0 for $\omega=4$ and $\chi=75^\circ$

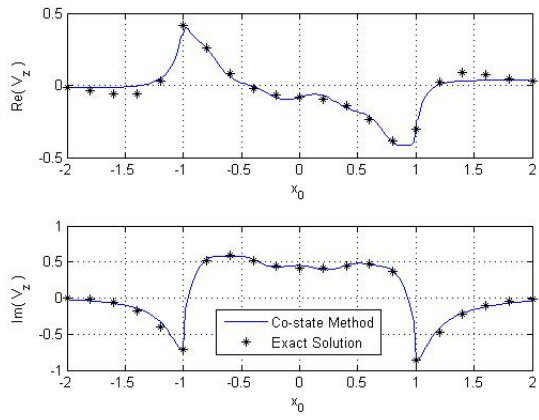


Figure 3 Velocity for $\xi=1$ with τ_1^0 for $\omega=4$ and $\chi=30^\circ$

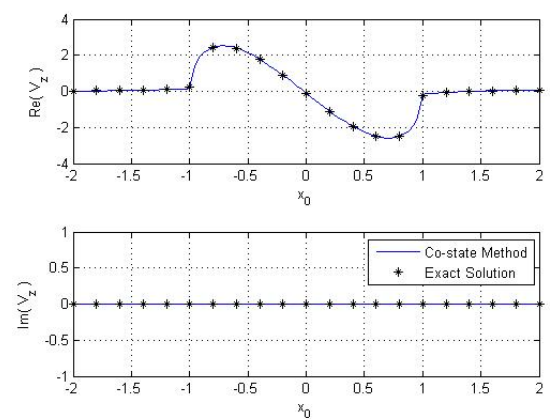


Figure 6 Velocity for $\xi=1$ with τ_2^1 for $\omega=0$ and $\chi=30^\circ$

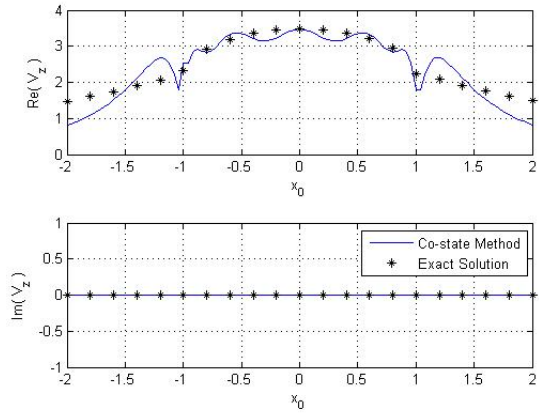


Figure 4 Velocity for $\xi=1$ with τ_1^0 for $\omega=0$ and $\chi=75^\circ$

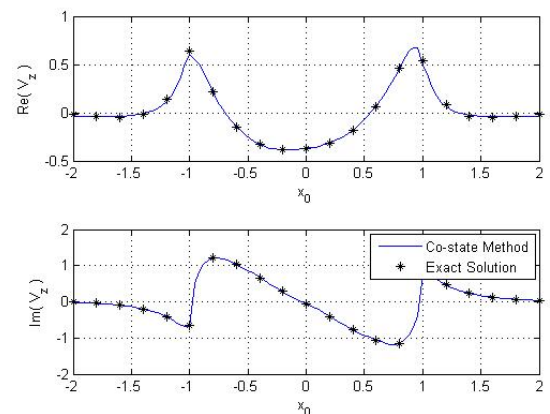


Figure 7 Velocity for $\xi=1$ with τ_2^1 for $\omega=4$ and $\chi=30^\circ$

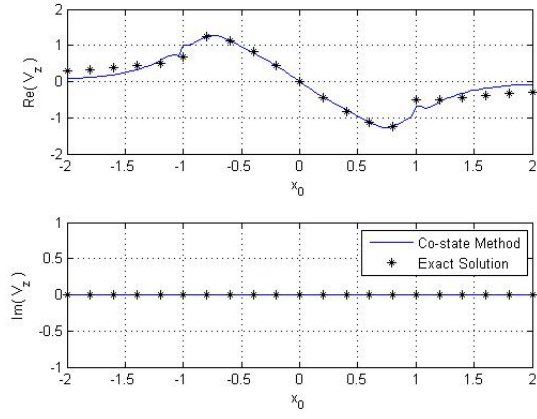


Figure 8 Velocity for $\xi=1$ with τ_2^1 for $\omega=0$ and $\chi=75^\circ$

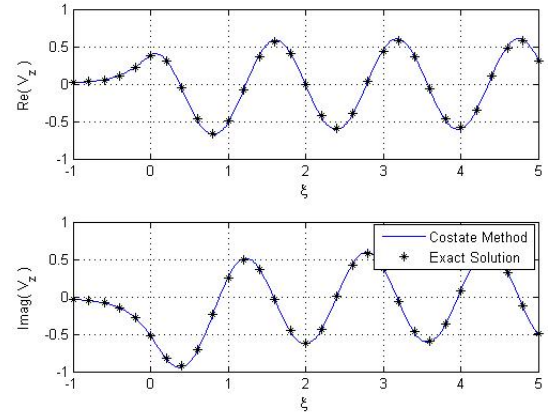


Figure 11 Velocity for $x_0=0.5$ with τ_1^0 for $\omega=4$ and $\chi=30^\circ$

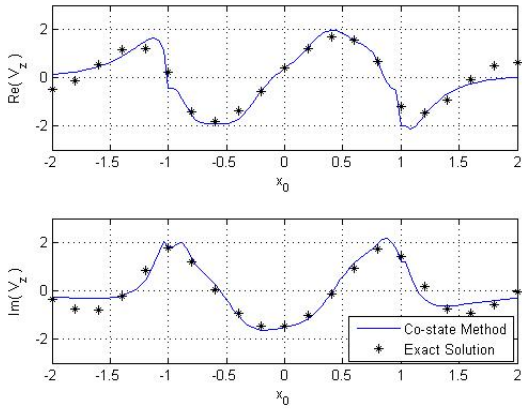


Figure 9 Velocity for $\xi=1$ with τ_2^1 for $\omega=4$ and $\chi=75^\circ$

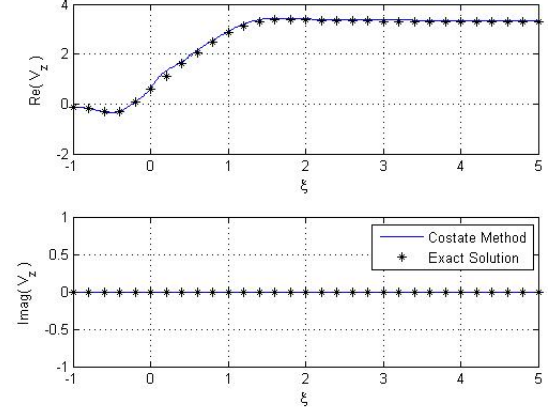


Figure 12 Velocity for $x_0=0.5$ with τ_1^0 for $\omega=0$ and $\chi=75^\circ$

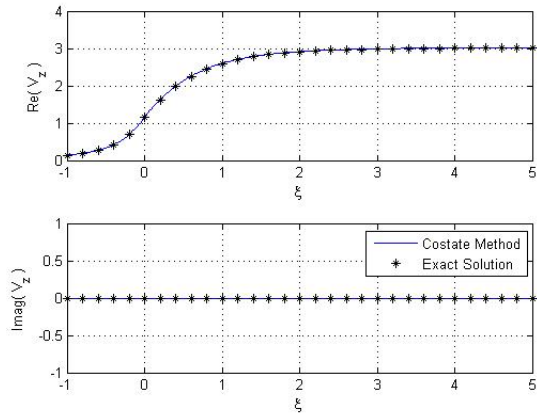


Figure 10 Velocity for $x_0=0.5$ with τ_1^0 for $\omega=0$ and $\chi=30^\circ$

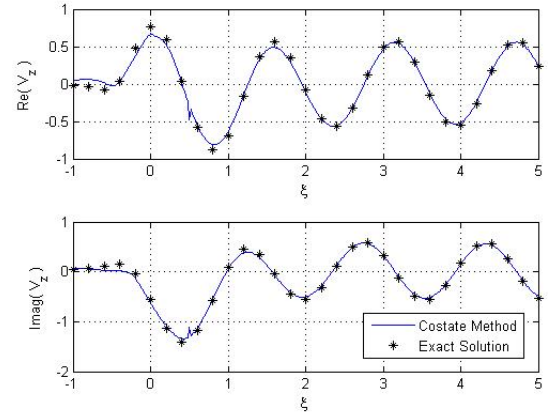


Figure 13 Velocity for $x_0=0.5$ with τ_1^0 for $\omega=4$ and $\chi=75^\circ$

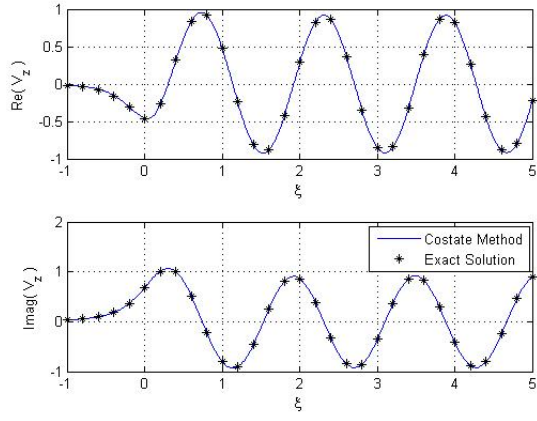


Figure 14 Velocity for $x_0 = 0.5$ with τ_2^1 for $\omega = 4$ and $\chi = 30^\circ$

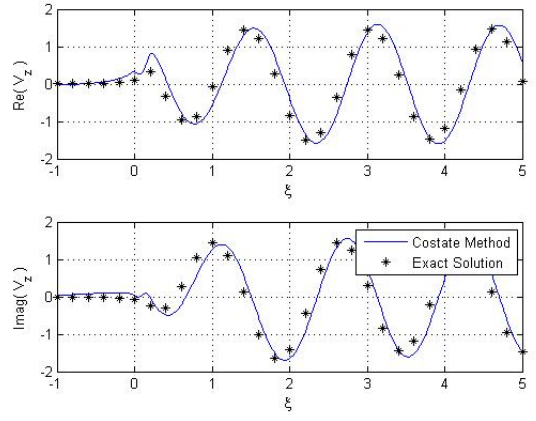


Figure 17 Velocity for $x_0 = 1.2$ with τ_2^1 for $\omega = 4$ and $\chi = 75^\circ$

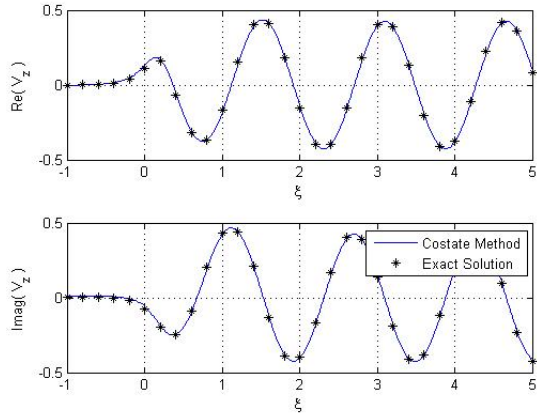


Figure 15 Velocity for $x_0 = 1.2$ with τ_2^1 for $\omega = 4$ and $\chi = 30^\circ$

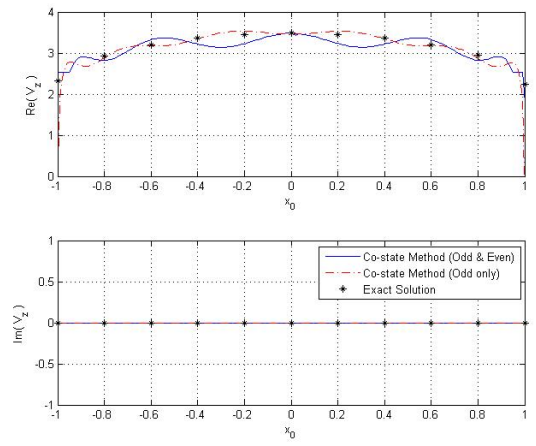


Figure 18 Comparison of the co-state method with different number of even terms for $\xi = 1$ with τ_1^0 for $\omega = 0$ and $\chi = 75^\circ$

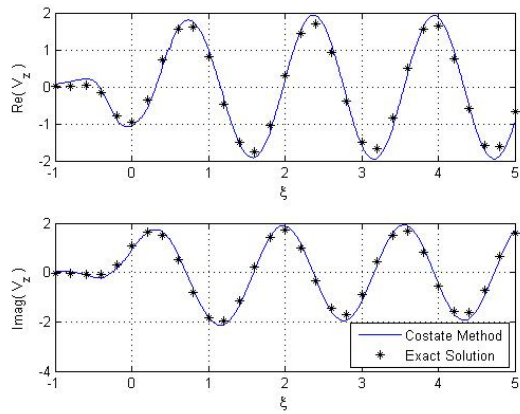


Figure 16 Velocity for $x_0 = 0.5$ with τ_2^1 for $\omega = 4$ and $\chi = 75^\circ$

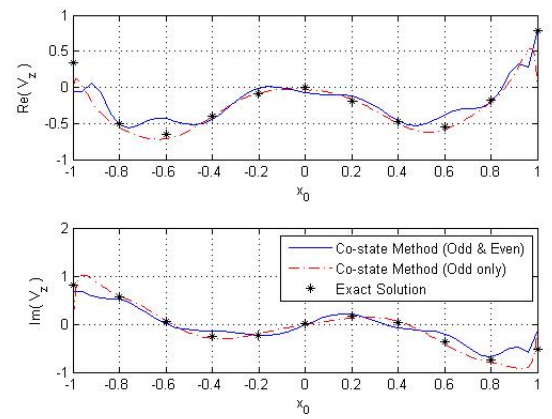


Figure 19 Comparison of the co-state method with different number of even terms for $\xi = 1$ with τ_1^0 for $\omega = 4$ and $\chi = 75^\circ$

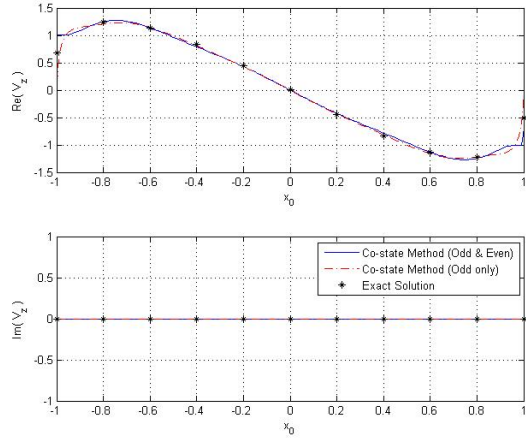


Figure 20 Comparison of the co-state method with different number of even terms for $\xi = 1$ with τ_2^1 for $\omega = 0$ and $\chi = 75^\circ$

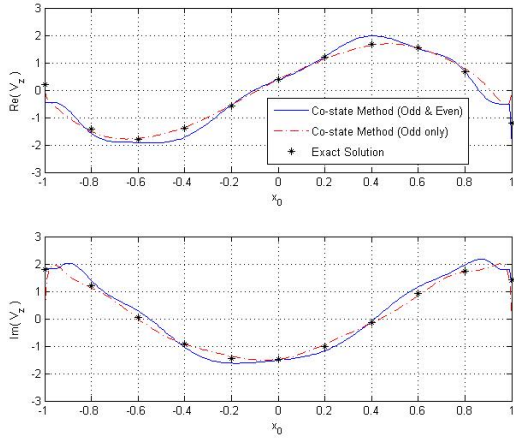


Figure 21 Comparison of the co-state method with different number of even terms for $\xi = 1$ with τ_2^1 for $\omega = 4$ and $\chi = 75^\circ$