Linear Time Invariant Models for Integrated Flight and Rotor Control

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Abstract

Formulation of linear time invariant (LTI) models of a nonlinear system about a periodic equilibrium using the harmonic domain representation of LTI model states is well established in the literature. This paper presents a computationally efficient scheme for implementation of a previously developed method for extraction of linear time invariant (LTI) models from a helicopter nonlinear model in forward flight. The fidelity of the extracted LTI models is evaluated using response comparisons between the extracted LTI models and the nonlinear model in both time and frequency domains. For time domain evaluations, individual blade control (IBC) inputs that have been tried in the literature for vibration and noise control studies are used. For frequency domain evaluations, frequency sweep inputs are used to obtain frequency responses of fixed system hub loads to a single blade IBC input. The evaluation results demonstrate the fidelity of the extracted LTI models, and thus, establish the validity of the LTI model extraction process for its use in integrated flight and rotor control studies.

Nomenclature

A	LTI model system matrix
B	LTI model input matrix
C	LTI model output matrix
D	LTP model damping matrix
E	LTI model input-to-output matrix
F_x,F_y,F_z	fixed system hub forces
G	LTP model input matrix
K	LTP model stiffness matrix
L	number of harmonic components of y
M_x,M_y,M_z	fixed system hub moments
M	number of harmonic components of control
N	number of harmonic components of x
P	LTP model output matrix associated with x
Q	LTP model output matrix associated with $\dot{x}$
U	control vector
X	vector of displacement variables
Y	output vector
n_t	number of response points used in model fidelity evaluation
n_o	number of outputs used in model fidelity evaluation
t	time
u	changes in U from trim
x	changes in X from trim
y	changes in Y from trim
Ω	rotor rotational speed
Ξ	equation imbalance
ψ	azimuth angle (= Ωt)
$\dot{x}, \dot{X}$	first time derivatives of x and X
$\ddot{x}, \ddot{X}$	second time derivatives of x and X
$\overline{X}, \overline{U}$	periodic steady state values of X and U

Subscripts

$(\ )_o$	zero-th harmonic component/average component of ( )
$(\ )_{nc}$	n^{th} harmonic cosine component of ( )
$(\ )_{ns}$	n^{th} harmonic sine component of ( )
$(\ )_{aug}$	augmented vector made up of average and harmonic components

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Introduction

Current flight controller designs for helicopters represent a difficult trade-off between controller bandwidth and its impact on rotor stability, rotor vibratory loads, etc. Traditional swashplate controls in terms of collective, longitudinal cyclic and lateral cyclic limit the number of controls available to the control designer in addressing flight and rotor control issues. For example, a four-bladed rotor using IBC has four independent controls available. However, the use of swashplate for control inputs restricts the number of independent controls to only 3, thus limiting the control design space available to the control designer.

Individual Blade Control (IBC) and On-Blade Control (OBC) concepts offer tremendous potential for expanding the available control design space in tailoring control input signals for desired blade responses to address flight and rotor control issues in a unified framework. IBC and OBC concepts offer the potential to develop innovative controllers for mitigation of compressibility effects on advancing blades and reverse flow effects on retreating blades for improved rotor performance, control of an individual blade that may be off-track, mitigation of transient effects associated with rotor speed variations, mitigation of undesirable coupling between body and rotor in large size helicopters due to increased rotor blade flexibility, reduction of maneuver blade and rotor loads, reduction of vibratory hub loads, reduction of blade-vortex interaction noise, etc., while ensuring good flying qualities as specified in the Aeronautical Design Standard (ADS-33). While higher flight control bandwidth can be achieved through innovative integrated flight and rotor control designs, the implications of such higher bandwidth control arising out of IBC and OBC concepts on handling qualities and vehicle-pilot-biodynamic coupling, etc., need to be carefully assessed before they can be fully realized.

Due to the periodic nature of helicopter rotors, the linearized models including rotor states extracted from nonlinear models of a rotorcraft will have periodic coefficients. Though stability analysis of the extracted linear time periodic (LTP) models can be performed using the Floquet stability theory, it does not provide a convenient framework for controller synthesis and design as the available control design tools for LTP systems are few in number [1]. Further, the handling qualities specifications for small amplitude maneuvers as prescribed in the Aeronautical Design Standard (ADS 33) are based on a linear time invariant (LTI) model, and thus cannot be directly accounted for in the controller design process using LTP models. If linearized models in time invariant form are made available, it will open up the choice of available design and analyses tools to a rotorcraft control designer [1].

Methods available in the literature for transformation of LTP models to time-invariant form suffer from certain disadvantages. For instance, in the Lyapunov-Floquet transformation method [2], the system matrix of the LTP model is transformed into a time-invariant form using the time varying Lyapunov-Floquet transformation matrix. However, the control matrix of the transformed model will still be time-periodic. To overcome this difficulty, an auxiliary system is constructed with pseudo control variables which bear no resemblance to the control vector of the LTP model. Controller design is carried out on the auxiliary system and control laws for the time periodic system are constructed from feedback signals of the auxiliary system. However, this method suffers from the disadvantage of needing to compute the state transition matrices of the LTP model over one rotor period in order to construct the Lyapunov-Floquet transformation matrix. Analytical approximations of the coefficient matrices using the shifted Chebyshev polynomials of the first kind provide an efficient means in the state transition matrix computations. While the computational effort is significantly improved by the use of such closed form approximations, the accuracy of the results is significantly influenced by the number of terms used in the analytical approximations, and such an approach becomes numerically very sensitive. The transformation using discrete-time methods, such as time lifting and frequency lifting methods [3], also suffer from the same disadvantage of a need for state transition matrices.

The use of harmonic analyzers as part of the linearization step to extract a time invariant linear model for the specific application of flight control and higher harmonic control is explored in [1]. The extracted LTI model consists of the body states, time averaged rotor states, harmonic analyzer states, pilot controls and higher harmonic controls. Using such a model, Cheng, et al [1] show that it becomes feasible to consider the important coupling between the body states and the higher frequency rotor response in a combined flight and vibration controller design. Using numerical perturbations to individual harmonic components of periodic states of a system, LTI models are formulated with average and harmonic components of rotor response as pseudo states in [4].

It is well known that the use of numerical perturbation techniques for extraction of linear models from a nonlinear model is sensitive to the size of state and control perturbations. Also, in cases where reduced order models are sought from the linearization, the effect of state and control perturbations on the neglected states have to be properly taken into account, thus further increasing the computational complexity of the linearization process. More importantly, the number of harmonic components of rotor states required in the LTI model for retaining the important coupling between body states and rotor states is generally not known *a priori* in a specific application. As a result, one may be forced to consider different LTI model approximations in order to arrive at the appropriate model, thus further increasing the computational complexity of the linearization process.

In [5], a two-step approach for extracting LTI models from a nonlinear model is proposed. First, a linear time periodic (LTP) model with sufficient fidelity is extracted from a nonlinear model about a specified equilibrium condition using a numerical perturbation scheme. In the second step, harmonic decomposition of the LTP model matrices is used to arrive at a LTI model of selected order. This is shown to offer computational flexibility in the LTI model extraction process as one can use closed form expressions relating LTI model matrices and the harmonic components of LTP model matrices.

The present study considers a computationally efficient scheme for combining the two-step method of [5] into a single step by carrying out the numerical perturbation part and the harmonic decomposition part simultaneously. The scheme is integrated within a comprehensive flight dynamic model of a helicopter. The fidelity of the extracted LTI models is evaluated using response comparisons in both time and frequency domains.

Linear Time Invariant (LTI) Models of a Nonlinear System

Consider a nonlinear system of the form

$$f(X, \dot{X}, \ddot{X}, U) = 0 \tag{1}$$

where X , $\dot{X}$ and $\ddot{X}$ are respectively the position, velocity and acceleration vectors, and U is the control vector. Let $((\overline{X}(\psi), \overline{U}(\psi)))$ represents a periodic equilibrium of the system of Eq. (1) such that

$$\overline{X}(\psi + 2\pi) = \overline{X}(\psi), \quad \overline{U}(\psi + 2\pi) = U(\psi) \tag{2}$$

A linearization of Eq. (1) can be obtained by considering changes from equilibrium as

$$x(\psi) = X(\psi) - \overline{X}(\psi), \;\; u(\psi) = U(\psi) - \overline{U}(\psi) \tag{3}$$

and expanding Eq. (1) about the periodic equilibrium in Taylor series to first order as

$$\begin{aligned} & f(\overline{X}, \dot{\overline{X}}, \ddot{\overline{X}}, \overline{U}) + \\ & \left[\frac{\partial f}{\partial \ddot{X}}\right]\ddot{x} + \left[\frac{\partial f}{\partial \dot{X}}\right]\dot{x} + \left[\frac{\partial f}{\partial X}\right]x + \left[\frac{\partial f}{\partial U}\right]u = 0 \end{aligned} \tag{4}$$

The partial derivatives in Eq. (4) are obtained at the selected periodic equilibrium. Since the periodic equilibrium also must satisfy Eq. (1), the above equation reduces to

$$\left[\frac{\partial f}{\partial \ddot{X}}\right]\ddot{x} + \left[\frac{\partial f}{\partial \dot{X}}\right]\dot{x} + \left[\frac{\partial f}{\partial X}\right]x + \left[\frac{\partial f}{\partial U}\right]u = 0 \tag{5}$$

which can be rearranged into the form

$$\ddot{x} = -K(\psi)x - D(\psi)\dot{x} + G(\psi)u \tag{6}$$

where

$$\begin{aligned} K(\psi) &= \left[\frac{\partial f}{\partial \ddot{X}}\right]^{-1}\left[\frac{\partial f}{\partial X}\right] \\ D(\psi) &= \left[\frac{\partial f}{\partial \ddot{X}}\right]^{-1}\left[\frac{\partial f}{\partial \dot{X}}\right] \\ G(\psi) &= -\left[\frac{\partial f}{\partial \ddot{X}}\right]^{-1}\left[\frac{\partial f}{\partial U}\right] \end{aligned} \tag{7}$$

Likewise, defining the output equation of the nonlinear system of Eq. (1) as

$$Y = g(X, \dot{X}, \ddot{X}, U) \tag{8}$$

where Y is a vector of outputs. At a periodic equilibrium, the value of the out

$$\overline{Y} = g(\overline{X}, \dot{\overline{X}}, \ddot{\overline{X}}, \overline{U}) \tag{9}$$

A linearized form of the output equation is obtained by expanding Eq. (8) about the periodic equilibrium in Taylor series to first order as

$$\begin{aligned} Y &= g(\overline{X}, \dot{\overline{X}}, \ddot{\overline{X}}, \overline{U}) \\ &+ \left[\frac{\partial g}{\partial \ddot{X}}\right]\ddot{x} + \left[\frac{\partial g}{\partial \dot{X}}\right]\dot{x} + \left[\frac{\partial g}{\partial X}\right]x + \left[\frac{\partial g}{\partial U}\right]u \end{aligned} \tag{10}$$

Substituting Eqs. (6) and (9) into Eq. (10) results in

$$y = P(\psi)x + Q(\psi)\dot{x} + R(\psi)u \tag{11}$$

where y represents change in the output Y from its equilibrium value $\overline{Y}$, and P, Q and R matrices can be obtained using

$$\begin{aligned} P(\psi) &= \left[\frac{\partial g}{\partial X}\right] - \left[\frac{\partial g}{\partial \ddot{X}}\right]K(\psi) \\ Q(\psi) &= \left[\frac{\partial g}{\partial \dot{X}}\right] - \left[\frac{\partial g}{\partial \ddot{X}}\right]D(\psi) \\ R(\psi) &= \left[\frac{\partial g}{\partial U}\right] + \left[\frac{\partial g}{\partial \ddot{X}}\right]G(\psi) \end{aligned} \tag{12}$$

The linear time periodic (LTP) model of Eqs. (6) and (11) is converted into a linear time invariant (LTI) form using the following approximation to x:

$$x = x_o + \sum_{n=1}^{N} x_{nc} \cos n\psi + x_{ns} \sin n\psi \tag{13}$$

where x_o is the average component and x_{nc} and x_{ns} are respectively the n/rev cosine and sine harmonic components of x. Likewise, control (u) and output (y) are expanded in terms of harmonic components as

$$u = u_o + \sum_{m=1}^{M} u_{mc} \cos m\psi + u_{ms} \sin m\psi \tag{14}$$

$$y = y_o + \sum_{l=1}^{L} y_{lc} \cos l\psi + y_{ls} \sin l\psi \tag{15}$$

Now defining an augmented state, control and output vectors in terms of their respective average and harmonic components as

$$x_{aug} = \left[x_o{}^T .. x_{ic}{}^T \;\; x_{is}{}^T .. x_{jc}{}^T \;\; x_{js}{}^T .. \dot{x}_o{}^T .. \dot{x}_{ic}{}^T \;\; \dot{x}_{is}{}^T .. \dot{x}_{jc}{}^T \;\; \dot{x}_{js}{}^T ..\right]^T$$

$$u_{aug} = \left[u_o{}^T .. \;\; u_{mc}{}^T \;\; u_{ms}{}^T \; ....\right]^T$$

$$y_{aug} = \left[y_o{}^T .. \;\; y_{lc}{}^T \;\; y_{ls}{}^T \; ...\right]^T$$

where x_o is the average component, x_{ic} and x_{is} are respectively the i^{th} harmonic cosine and sine components of x, u_o is the average component and u_{mc} and u_{ms} are respectively the m^{th} harmonic cosine and sine components of u, and y_o is the average component and y_{lc} and y_{ls} are respectively the l^{th} harmonic cosine and sine components of y. Using the augmented state, control and output vectors, the LTP model of Eqs. (6) and (11) can be approximated into a LTI form as

$$\dot{x}_{aug} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_{aug} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{aug} \tag{16}$$

$$y_{aug} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x_{aug} + [E] u_{aug} \tag{17}$$

Closed form expressions for the elements of the LTI model matrices of Eqs. (16) and (17) have been derived in [5]. The final expressions taken from [5] are included here.

$$A_{11} = \begin{bmatrix} 0......0 \;\; 0.......0 \;\; 0........ \\ .... \\ 0......0 \;\; 0.......0 \;\; 0........ \\ 0......0 \;\; 0.......0 \;\; 0........ \\ .... \\ 0......0 \;\; 0.......0 \;\; 0........ \\ 0......0 \;\; 0.......0 \;\; 0........ \\ .... \end{bmatrix} \tag{18}$$

$$A_{12} = \begin{bmatrix} I........0 \;\; 0.........0 \;\; 0......... \\ .... \\ 0........I \;\; 0.........0 \;\; 0......... \\ 0........0 \;\; I.........0 \;\; 0......... \\ .... \\ 0........0 \;\; 0.........I \;\; 0......... \\ 0........0 \;\; 0.........0 \;\; I......... \\ .... \end{bmatrix} \tag{19}$$

$$A_{21} = \begin{bmatrix} -H_{oK}...(-H_{oK^{ic}} + i\Omega H_{oD^{is}}) \;\; (-H_{oK^{is}} - i\Omega H_{oD^{ic}})...(-H_{oK^{jc}} + j\Omega H_{oD^{js}}) \;\; (-H_{oK^{js}} - j\Omega H_{oD^{jc}})..................... \\ .... \\ -H_{icK}...(i^2\Omega^2 I - H_{icK^{ic}} + i\Omega H_{icD^{is}}) \;\; (-H_{icK^{is}} - iH_{icD^{ic}})...(-H_{icK^{jc}} + j\Omega H_{icD^{js}}) \;\; (-H_{icK^{js}} - j\Omega H_{icD^{ic}}).......... \\ -H_{isK}...(-H_{isK^{ic}} + i\Omega H_{isD^{is}}) \; (i^2\Omega^2 I - H_{isK^{is}} - i\Omega H_{isD^{ic}}).... (-H_{isK^{jc}} + j\Omega H_{isD^{js}} \;\; (-H_{isK^{js}} - j\Omega H_{isD^{jc}}).............. \\ .... \\ -H_{jcK}...(-H_{jcK^{ic}} + i\Omega H_{jcD^{is}}) \;\; (-H_{jcK^{is}} - i\Omega H_{jcD^{ic}})...(j^2\Omega^2 I - H_{jcK^{jc}} + j\Omega H_{jcD^{js}}) \;\; (-H_{jcK^{js}} - j\Omega H_{jcD^{jc}})... \\ -H_{jsK}...(-H_{jsK^{ic}} + i\Omega H_{jsD^{is}}) \; (-H_{jsK^{is}} - i\Omega H_{jsD^{ic}})... (-H_{jsK^{jc}} + j\Omega H_{jsD^{js}}) \; (j^2\Omega^2 I - H_{jsK^{js}} - j\Omega H_{jsD^{jc}}).... \\ .... \end{bmatrix} \tag{20}$$

$$A_{22} = \begin{bmatrix} -H_{oD} \cdots -H_{oD^{ic}} \quad -H_{oD^{is}} \cdots -H_{oD^{jc}} \quad -H_{oD^{js}} \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -H_{icD} \cdots -H_{icD^{ic}} \quad (-2i\Omega I - H_{icD^{is}}) \cdots -H_{icD^{jc}} \quad -H_{icD^{js}} \cdots \\ -H_{isD} \cdots (2i\Omega I - H_{isD_{ic}}) \quad -H_{isD_{is}} \cdots -H_{isD_{jc}} \quad -H_{isD_{js}} \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -H_{jcD} \cdots -H_{jcD^{ic}} \quad -H_{jcD^{is}} \cdots -H_{jcD^{jc}} \quad (-2j\Omega I - H_{jcD^{js}}) \cdots \\ -H_{jsD} \cdots -H_{jsD^{ic}} \quad -H_{jsD^{is}} \cdots (-2j\Omega I - H_{jsD^{jc}}) \quad -H_{jsD^{js}} \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix} \tag{21}$$

$$B_1 = \begin{bmatrix} 0 \\ . \\ 0 \\ 0 \\ . \\ 0 \\ 0 \\ . \end{bmatrix} \tag{22}$$

$$B_2 = \begin{bmatrix} H_{oG} \cdots H_{oG^{mc}} \quad H_{oG^{ms}} \cdots \\ \cdots\cdots\cdots\cdots\cdots \\ H_{icG} \cdots H_{icG^{mc}} \quad H_{icG^{ms}} \cdots \\ H_{isG} \cdots H_{isG^{mc}} \quad H_{isG^{ms}} \cdots \\ \cdots\cdots\cdots\cdots\cdots \\ H_{jcG} \cdots H_{jcG^{mc}} \quad H_{jcG^{ms}} \cdots \\ H_{jsG} \cdots H_{jsG^{mc}} \quad H_{jsG^{ms}} \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix} \tag{23}$$

$$C_1 = \begin{bmatrix} H_{oP} \cdots (H_{oP^{ic}} - i\Omega H_{oQ^{is}}) \quad (H_{oP^{is}} + i\Omega H_{oQ^{ic}}) \cdots (H_{oP^{jc}} - j\Omega H_{oQ^{js}}) \quad (H_{oP^{js}} + j\Omega H_{oQ^{ic}}) \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ H_{lcP} \cdots (H_{lcP^{ic}} - i\Omega H_{lcQ^{is}}) \quad (H_{lcP^{is}} + iH_{lcQ^{ic}}) \cdots (H_{lcP^{jc}} - j\Omega H_{lcQ^{js}}) \quad (H_{lcP^{js}} + jH_{lcQ^{jc}}) \cdots \\ H_{lsP} \cdots (H_{lsP^{ic}} - i\Omega H_{lsQ^{is}}) \quad (H_{lsP^{is}} + i\Omega H_{lsQ^{ic}}) \cdots (H_{lsP^{jc}} - j\Omega H_{lsQ^{js}} \quad (H_{lsP^{js}} + j\Omega H_{lsQ^{jc}}) \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix} \tag{24}$$

$$C_2 = \begin{bmatrix} .H_{oQ} \cdots H_{oQ^{ic}} \quad H_{oQ^{is}} \cdots H_{oQ^{jc}} \quad H_{oQ^{js}} \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ ..H_{lcQ} \cdots H_{lcQ^{ic}} \quad H_{lcQ^{is}} \cdots H_{lcQ^{jc}} \quad H_{lcQ^{js}} \cdots \\ H_{lsQ} \cdots H_{lsQ^{ic}} \quad H_{lsQ^{is}} \cdots H_{lsQ^{jc}} \quad H_{lsQ^{js}} \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{bmatrix} \tag{25}$$

$$E = \begin{bmatrix} H_{oR} \cdots H_{oR^{mc}} \quad H_{oR^{ms}} \cdots \\ \cdots\cdots\cdots\cdots\cdots \\ H_{lcR} \cdots H_{lcR^{mc}} \quad H_{lcR^{ms}} \cdots \\ H_{lsR} \cdots H_{lsR^{mc}} \quad H_{lsR^{ms}} \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{bmatrix} \tag{26}$$

Closed form expressions for the various elements of A_{12}, A_{22}, B_2, C_1, C_2 and E can be obtained in terms of harmonic components of the LTP model matrices *D, K, G, P, Q* and *R* [5]. For example, if the matrix $D(\psi)$ is expanded in terms of its harmonic components as

$$D(\psi) = D_o + \sum_{k=1}^{\infty} (D_{kc} \cos k\psi + D_{ks} \sin k\psi) \tag{27}$$

where D_o is the average component and D_{kc} and D_{ks} are respectively the k^{th} harmonic cosine and sine components of the matrix $D(\psi)$ such that

$$D_o = \frac{1}{2\pi}\int_0^{2\pi} D(\psi)d\psi$$
$$D_{kc} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\cos k\psi\, d\psi \quad (28)$$
$$D_{ks} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\sin k\psi\, d\psi$$
$$k = 1, 2, 3, ...$$

then

$$H_{oD} = \frac{1}{2\pi}\int_0^{2\pi} D(\psi)d\psi = D_o$$
$$H_{oD^{ic}} = \frac{1}{2\pi}\int_0^{2\pi} D(\psi)\cos i\psi\, d\psi = \frac{D_{ic}}{2} \quad (29)$$
$$H_{oD^{is}} = \frac{1}{2\pi}\int_0^{2\pi} D(\psi)\sin i\psi\, d\psi = \frac{D_{is}}{2}$$
$$i = 1, 2, 3, ...$$

$$H_{icD} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\cos i\psi\, d\psi = D_{ic}$$
$$H_{isD} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\sin i\psi\, d\psi = D_{is} \quad (30)$$
$$i = 1, 2, 3, ...$$

$$H_{icD^{jc}} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\cos j\psi \cos i\psi\, d\psi$$
$$= D_o + \frac{D_{kc}}{2} \quad \text{for } i = j \text{ where } k = i + j$$
$$= \frac{D_{kc} + D_{lc}}{2} \quad \text{for } i \neq j \text{ and } i > j \quad (31)$$
$$\text{where } k = i + j,\ l = i - j$$
$$= \frac{D_{kc} + D_{mc}}{2} \quad \text{for } i \neq j \text{ and } j > i$$
$$\text{where } k = i + j,\ m = j - i$$
$$i = 1, 2, ..\ \text{and}\ j = 1, 2, ..$$

$$H_{icD^{js}} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\sin j\psi \cos i\psi\, d\psi$$
$$= \frac{D_{ks}}{2} \quad \text{for } i = j \text{ where } k = i + j$$
$$= \frac{D_{ks} - D_{ls}}{2} \quad \text{for } i \neq j \text{ and } i > j \quad (32)$$
$$\text{where } k = i + j,\ l = i - j$$
$$= \frac{D_{ks} + D_{ms}}{2} \quad \text{for } i \neq j \text{ and } j > i$$
$$\text{where } k = i + j,\ m = j - i$$
$$i = 1, 2, ..\text{and}\ j = 1, 2, ...$$

$$H_{isD^{jc}} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\cos j\psi \sin i\psi\, d\psi$$
$$= \frac{D_{ks}}{2} \quad \text{for } i = j \text{ where } k = i + j$$
$$= \frac{D_{ks} + D_{ls}}{2} \quad \text{for } i \neq j \text{ and } i > j \quad (33)$$
$$\text{where } k = i + j,\ l = i - j$$
$$= \frac{D_{ks} - D_{ms}}{2} \quad \text{for } i \neq j \text{ and } j > i$$
$$\text{where } k = i + j,\ m = j - i$$
$$i = 1, 2, ...\ \text{and}\ j = 1, 2, ...$$

$$H_{isD^{js}} = \frac{1}{\pi}\int_0^{2\pi} D(\psi)\sin j\psi \sin i\psi\, d\psi$$
$$= D_o - \frac{D_{kc}}{2} \quad \text{for } i = j \text{ where } k = i + j$$
$$= \frac{D_{lc} - D_{kc}}{2} \quad \text{for } i \neq j \text{ and } i > j \quad (34)$$
$$\text{where } k = i + j,\ l = i - j$$
$$= \frac{D_{mc} - D_{kc}}{2} \quad \text{for } i \neq j \text{ and } j > i$$
$$\text{where } k = i + j,\ m = j - i$$
$$i = 1, 2, ..\ \text{and}\ j = 1, 2, ..$$

Likewise, expressions similar to the above can be obtained for the elements of the LTI model matrices involving *K, G, P, Q* and *R*.

LTI Models from FLIGHTLAB

The LTI model extraction method described in the previous section is implemented within FLIGHTLAB™ [6] using the generalized force formulation written as

$$\Xi = f(U, X, \dot{X}, \ddot{X}) \quad (35)$$

where Ξ is the equation imbalance of Eq. (1). The control at each time step is iteratively solved to drive Ξ to zero for trim. The nonlinear model is first trimmed at a specified flight condition. Then a reference blade is set to the zero azimuth position and the periodic trim condition is recorded over one rotor revolution. With the reference blade at selected azimuthal steps, the values of Ξ due to perturbations in state/control are computed at each azimuthal step, and system matrices are obtained by computing the partial derivatives of Ξ with respect to individual state/control through central finite differencing. For example,

$$\frac{\partial f}{\partial X} \approx \frac{f(\overline{X}+\Delta X, \dot{\overline{X}}, \ddot{\overline{X}}, \overline{U}) - f(\overline{X}-\Delta X, \dot{\overline{X}}, \ddot{\overline{X}}, \overline{U})}{2\Delta X} \quad (36)$$

where ΔX is the selected value of the numerical perturbation. Likewise, partial derivatives of the output with respect to state/control perturbations are obtained. The numerically computed partial derivatives are used to assemble the LTP model matrices $D(\psi)$, $K(\psi)$, etc., using Eqs. (7) and (12) at the current azimuthal step. Also, additional components, such as $D(\psi)cos(k\psi)$, $D(\psi)sin(k\psi)$, $K(\psi)cos(k\psi)$, $K(\psi)sin(k\psi)$, *etc.*, $k=0, 1, 2,\ldots$ required for harmonic decomposition (see Eq. 28) are also generated at each azimuthal step of the linearization. Therefore, once the linearization process is completed over one rotor revolution, it only takes a few algebraic operations (using Eqs. 18 through 26 and 29 through 34) to obtain a LTI model of selected order. The linearization can be configured to generate either a full linearized model or a reduced order model as desired. For reduced order models, a quasi-static model reduction technique is applied by selecting the dynamically retained states while residualizing the neglected dynamics.

LTI Model Fidelity Evaluations

A generic helicopter model available in FLIGHTLAB™ is used for LTI model fidelity evaluations in this study. The vehicle weight is 15000 lb and it has a four bladed articulated rotor, conventional tail rotor, horizontal stabilizer and a vertical fin. The analog and digital SAS portions of the control system are disabled in this study. The nonlinear model includes one rigid plus one elastic mode for flap as well as lead-lag motions of each blade and a 15-state dynamic inflow model. The blade feathering is assumed to be rigid

The number of average states in the LTI model is 55 which includes 8 body states (vehicle mass center velocity components, angular velocity components, body pitch and roll attitudes), 15 inflow states (4 harmonic distributions combining with 4th power radial representation [6]), and 32 Multi Blade Coordinate (MBC) rotor states (16 for the rigid flap and lead-lag motions and 16 for the elastic flap and lag motions). The number of harmonic components of rotor MBC states is 256 for the case when one includes 1/rev to 4/rev harmonic sine and cosine components (64 for rigid mode flap, 64 for elastic mode flap, 64 for rigid mode lead-lag and 64 for elastic mode lead-lag) resulting in a LTI model order of 311. When one includes 1/rev to 8/rev sine and cosine harmonics of rotor MBC states, the resulting LTI model order is 567, which includes 55 average states and 512 harmonic sine and cosine components of rotor MBC states. In case of an isolated rotor representation, the body states are absent, thus reducing the LTI model order by 8.

Individual blade control (IBC) inputs excite higher frequencies. A careful study is required to make an assessment on the number of harmonic states needed for good fidelity. As the number of harmonic rotor states required increases, the computational effort involved in the extraction of LTI model also increases. An assessment is made of the number of floating point operations (FLOPS) needed for a LTI model extraction. It is seen that with the current approach, the number of FLOPS increases linearly with an increase in the number of harmonic states of the LTI model. This is in contrast to roughly a quadratic increase in FLOPS with the number of harmonic states using the numerical scheme proposed in [4] which involves individual harmonic component perturbations.

Metrics for Evaluation of Model Fidelity

Tischler and Remple [7] suggest the use of the following metrics for checking the fidelity of flight mechanics models identified from test data in time and frequency domains:

Time-domain metric:

$$J^{(1)} = \sqrt{\frac{1}{n_t . n_o} \sum_{i=1}^{n_t} \left[(\Delta y_{data} - \Delta y)_i^T W (\Delta y_{data} - \Delta y)_i \right]} \quad (37)$$

Frequency-domain metric:

$$\begin{aligned} J^{(2)} &= \frac{20}{n_\omega} \sum_{\omega_1}^{\omega_{n_\omega}} W_\gamma \left[W_g \left(|T_c| - |T| \right)^2 + W_p (\angle T_c - \angle T)^2 \right] \\ &\quad where \;\; W_\gamma(\omega) = [1.58 \cdot (1 - e^{-\gamma_{xy}^2})]^2, \\ &\quad W_g = 1.0 \\ &\quad W_p = 0.01745 \end{aligned} \quad (38)$$

The above metrics are adapted in this study by treating y_{data} as the response from the nonlinear model and y as the response from the LTI model. Δy in Eq. (37) is the perturbation time history of response from trim, n_t is the number of response points and n_o is the number of outputs. In Eq. (38), T_c is the transfer function from the nonlinear model, T is the transfer function from the LTI model, γ_{xy} is the coherence function, and n_ω is the number of discrete frequency points used.

The normalized fixed system hub forces and moments are used as outputs for model evaluations in this study.

$$y = \begin{bmatrix} \dfrac{F_x}{m} & \dfrac{F_y}{m} & \dfrac{F_z}{m} & \dfrac{M_x}{I_{xx}} & \dfrac{M_y}{I_{yy}} & \dfrac{M_z}{I_{zz}} \end{bmatrix}^T \quad (39)$$

It is suggested in [7] that 1 deg/s error is equivalent to 1 ft/s or 1 ft/s^2. Here, this equivalence is extended to 1 deg/s^2 as well. Hence, the fixed system hub forces and moments are normalized by the vehicle mass and the corresponding mass moments of inertia, respectively, as shown in Eq. (39). The elements of the weighting matrix W in J of Eq. (37) are appropriately selected to achieve this equivalency. It is suggested in [7] that for good model fidelity, the value of the time domain error index ($J^{(1)}$) needs to be less than 1 ~ 2 and the value of the frequency domain error index ($J^{(2)}$) needs to be less than 100.

Results

LTI model fidelity evaluations are carried out for a forward flight case of 0.15 advance ratio. The types of IBC inputs used are taken to be similar to those used in the literature for vibration and noise control applications (for example, see [8-10]). In time domain evaluations, the simulation time is set at 5 seconds for all cases. In each case, simulation begins at trim, and the selected input is applied at 1 sec into the simulation. The input is turned off at 3 sec into the run, and the simulation is continued till 5 sec. All fidelity evaluations in this study are carried out using the generic helicopter model with an elastic blade representation and a 15-state dynamic inflow model.

Time Domain Evaluations

Higher harmonic inputs (2/rev, 3/rev, 4/rev, etc.) are used in the literature for reductions in vibration, noise and rotor power [8 – 10]. For reducing rotor power, a 2/rev individual blade control (IBC) input is suggested in [8]. In order to evaluate the fidelity of the extracted LTI models for their use in active rotor power reduction studies, a 2/rev IBC input of 2^o magnitude (similar in magnitude to what has been tried in [8]) and (an arbitrarily selected) 125^o phase is used in the LTI model fidelity evaluations. The resulting fixed hub load variations with time as predicted from FLIGHTLAB and from the extracted LTI model are compared in Fig. 1a. The LTI model includes up to 4/rev harmonic components of rotor MBC states. Figure 1b is a zoom-in of results from Fig. 1a. The time-domain error index computed using Eq. (37) is obtained as 0.316 indicating good fidelity of the extracted LTI model. It is interesting to see that 2/rev IBC inputs impact the steady state components of rotor thrust and torque as evident from the response predictions of F_z and M_z in Fig. 1a.

It is well known that N/rev vibration in the fixed system arises from blade force variations in the rotating frame at (N-1)/rev, N/rev and (N+1)/rev vibrations, where N is the number of blades [11]. Hence, as suggested from several studies in the literature (for example, [8]), it is expected that IBC inputs at these frequencies can be used for vibration control. An IBC input consisting of 3/rev, 4/rev and 5/rev components is used as a way to test the fidelity of the extracted LTI models for their use in active vibration control studies. The magnitudes of the harmonic components of IBC are selected to be 1.5^o of 3/rev, 1^o of 4/rev and 0.5^o of 5/rev. These values are similar to the IBC harmonic component magnitudes used in [8]. The phases of the individual harmonic components of IBC are selected arbitrarily. The extracted LTI model includes up to 4/rev harmonic components of rotor states. The fixed system hub load responses to the selected IBC input as predicted from FLIGHTLAB is compared with those predicted using the LTI model in Fig. 2a with a zoom-in of the results shown in Fig. 2b. The time-domain error index computed using Eq. (37) is obtained as 0.612 indicating good model fidelity of the extracted LTI model, suggesting that the proposed LTI model extraction process can be used in active vibration control studies.

It is suggested in [9] that a combination of 6/rev and 7/rev may be used for simultaneous vibration and noise control. In order to verify the LTI model fidelity for its use in active vibration and noise control studies, a test case IBC input with 6/rev and 7/rev components of magnitudes (0.5^o of 6/rev and 0.5^o of 7/rev) similar to those considered in [9] is used. Two different orders of LTI model approximations are used, one that includes up to 4/rev harmonic components of rotor states and the second one that includes up to 8/rev harmonic components of rotor states. The predicted fixed system hub load responses from FLIGHTLAB are compared with those from the LTI model predictions in Fig. 3a with a zoom-in of the results shown in Fig. 3b. It is seen that the inclusion of up to 8/rev harmonic components of rotor states in the LTI model improves the LTI model fidelity significantly (error index of 0.037) when compared to that of the LTI model with only up to 4/rev harmonic components of rotor states (error index 0.71). The higher frequency variations in the fixed hub load responses seen in the FLIGHTLAB results are well captured in the predictions from the LTI model that includes up to 8/rev harmonic components of rotor states (see Fig. 3b).

Next, the LTI model fidelity is evaluated using pulse inputs of IBC when a blade is passing through a selected azimuthal range. This input has been suggested in the literature [10] for avoidance of blade vortex interactions (BVI) using a trailing edge flap actuation. A similar type of input is used with IBC in the present study. The selected IBC input as a function of rotor azimuth angle is shown in Fig. 4. The isolated rotor with the elastic blade and 15-state dynamic inflow model is used in this case. The predicted flapping response is shown in Fig. 5 in terms of vertical deflection from the hub (shown in inches) at three different locations along the radius of a reference blade. A visual comparison of the isolated rotor elastic blade flapping responses from FLIGHTLAB and those from the LTI model indicates that the fidelity of the extracted LTI model is good, suggesting that the proposed LTI model extraction process can be used in active BVI control studies.

Frequency Domain Evaluations

The Comprehensive System Identification from Frequency Responses (CIFER) [12] is used to obtain frequency responses between the fixed system hub loads and a single blade IBC input. Both FLIGHTLAB and LTI models are excited through a single blade IBC frequency sweep input. The frequency sweep magnitude is set at 1 deg and the frequency is linearly varied from 0.3 rad/sec to 135 read/sec (=5Ω) with time. The duration of the frequency sweep is set at 120 seconds and the azimuthal increment (sampling rate) is set at $\Delta\psi=2.5^o$. Five different sizes of moving windows (24 sec, 12sec, 8 sec, 2 sec and 1 sec) are used in the construction of a composite frequency response from the frequency sweep input and output data. The generic helicopter with the elastic blade and 15-state dynamic inflow model is used.

The predicted frequency responses between the fixed system rotor thrust (F_z) and rotor torque (M_z) to single blade IBC input are shown in Figs. 6 and 7, respectively. The frequency domain error index for model fidelity is computed using Eq. (38), which are obtained as 17.6 and 16.8 for the cases of F_z and M_z cases, respectively. These values are well within the bound of 100 suggested in [7], indicating a good fidelity of the extracted LTI model.

Concluding Remarks

Formulation of linear time invariant (LTI) models of a nonlinear system about a periodic equilibrium using the harmonic domain representation of LTI model states is well established in the literature. A computationally efficient scheme for extraction of linear time invariant (LTI) models of a nonlinear helicopter model about a periodic equilibrium is developed in this study. The proposed computational approach makes use of previously developed closed form expressions relating various elements of a LTI model with harmonic components of a corresponding linear time periodic (LTP) model. A numerical perturbation scheme is used to compute various elements of a LTP model at discrete azimuthal steps over one rotor revolution from a helicopter nonlinear model about a periodic equilibrium. Simultaneously, computations needed for decomposition of LTP model matrices into harmonic components are performed. Once the linearization for a LTP model is completed over one rotor revolution, it takes only a few algebraic operations to assemble a LTI model of selected order. The proposed numerical scheme is seen to improve computational speed by an order magnitude when it is compared with the numerical scheme from the literature involving individual harmonic components of state/control perturbations.

The proposed computational scheme is implemented within FLIGHTLAB™ and is used to extract LTI models of a generic helicopter nonlinear model in forward flight. The fidelity of the extracted LTI models is evaluated in both time and frequency domains by using error metrics from the literature. Simulation comparisons are made between the nonlinear model and the extracted linear models using predicted fixed system hub load responses to typical individual blade control (IBC) inputs that have been suggested in the literature for vibration and noise control applications. The evaluation results demonstrate the fidelity of the extracted LTI models, and thus, establish the validity of the LTI model extraction process for its use in integrated flight and rotor control studies.

Further evaluation of the developed LTI model extraction process is needed for its use in active rotor control studies involving on-blade control (OBC) actuation such as trailing edge flaps, active twist, etc.

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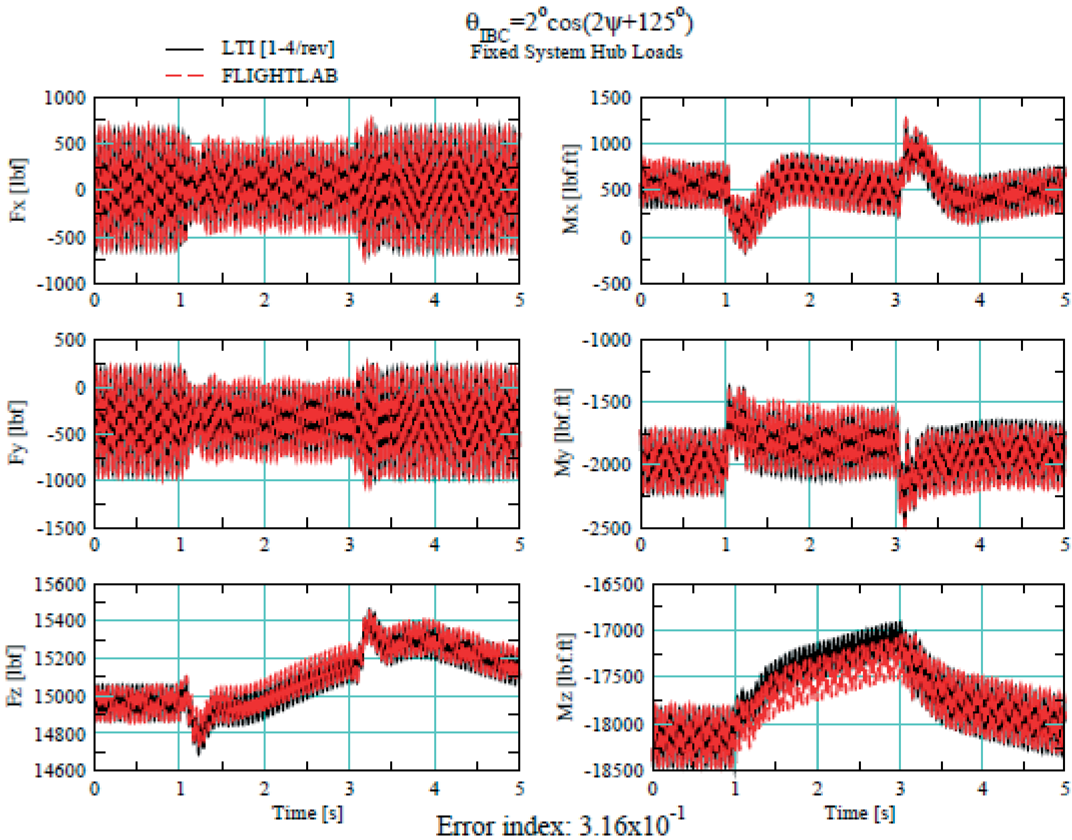

Figure 1a. Predicted Fixed System Hub Load Variations to 2/rev IBC Input.

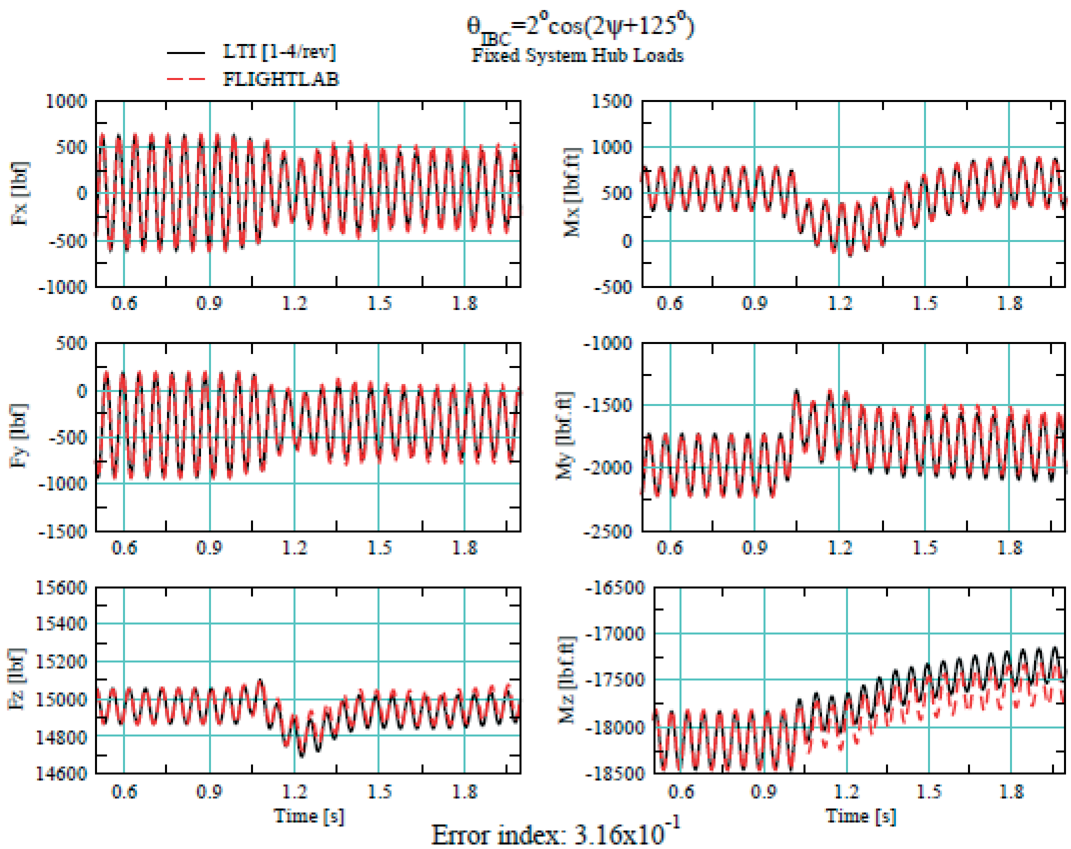

Figure 1b. A Zoom-In of Fig. 1a.

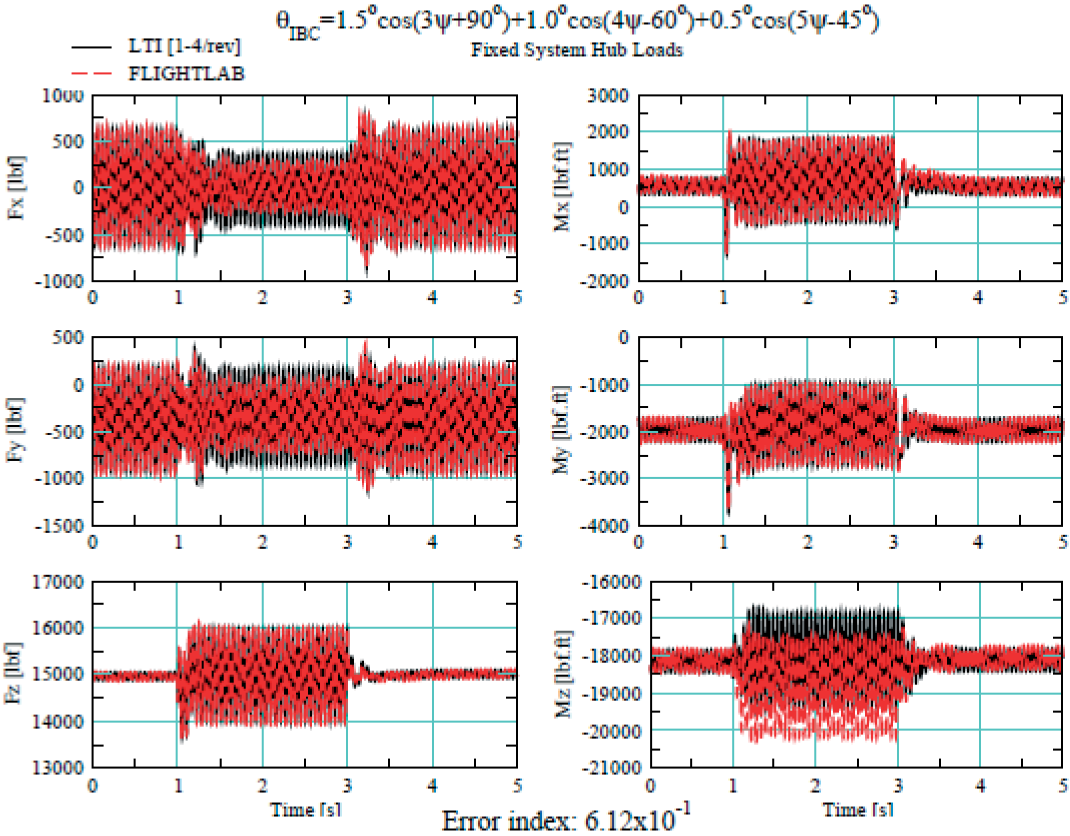

Figure 2a. Predicted Fixed Stem Hub Load Variations to IBC Inputs with 3/rev, 4/rev and 5/rev Components.

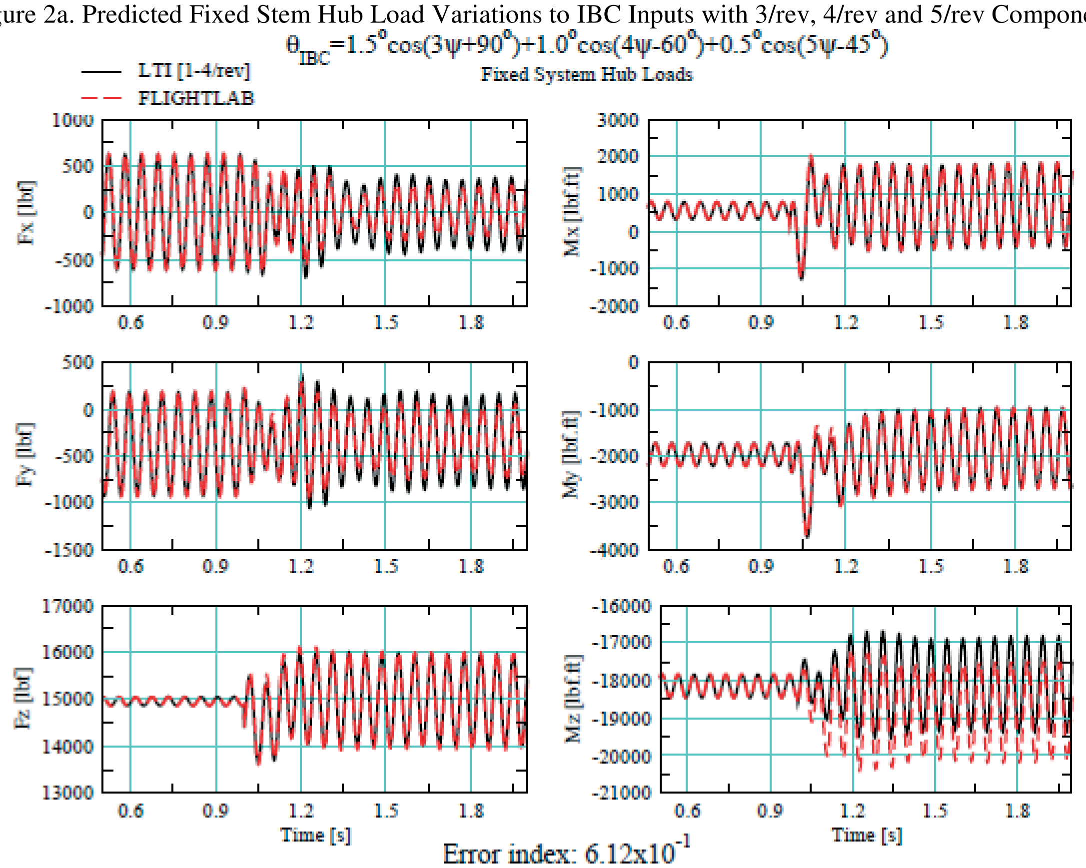

Figure 2b. A zoom-In of Fig. 2a.

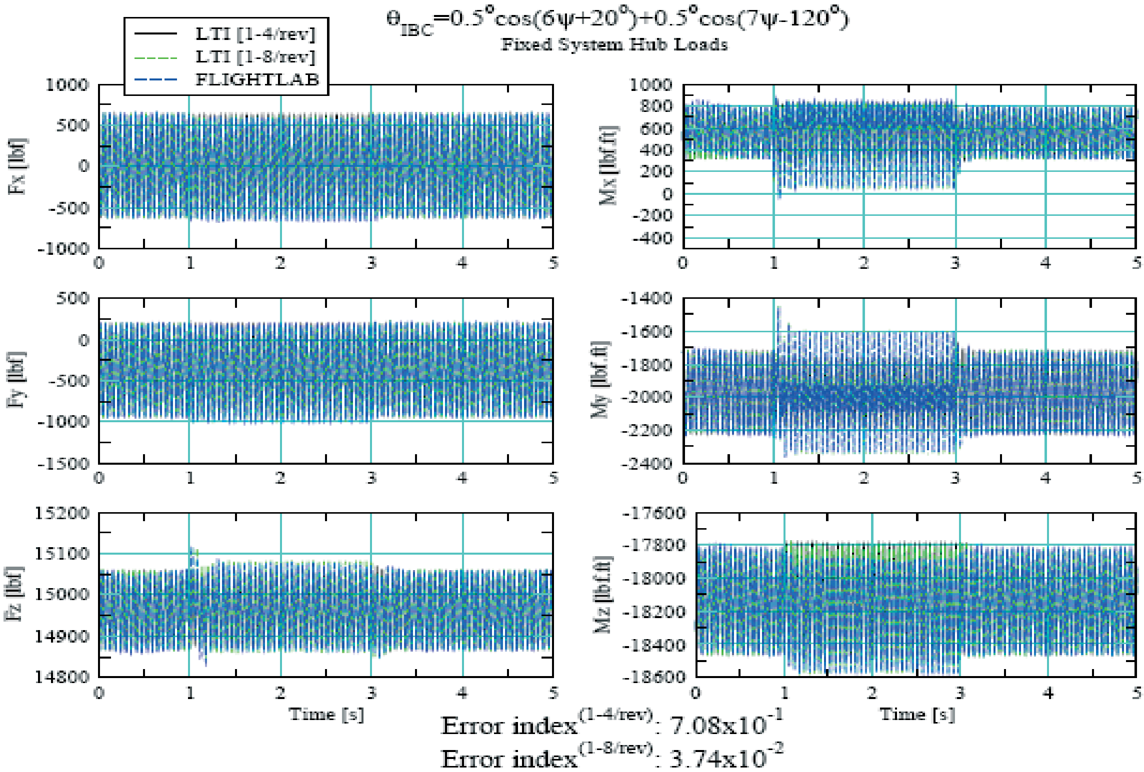

Figure 3a. Predicted Fixed Stem Hub Load Variations to IBC Inputs with 6/rev and 7/rev Components.

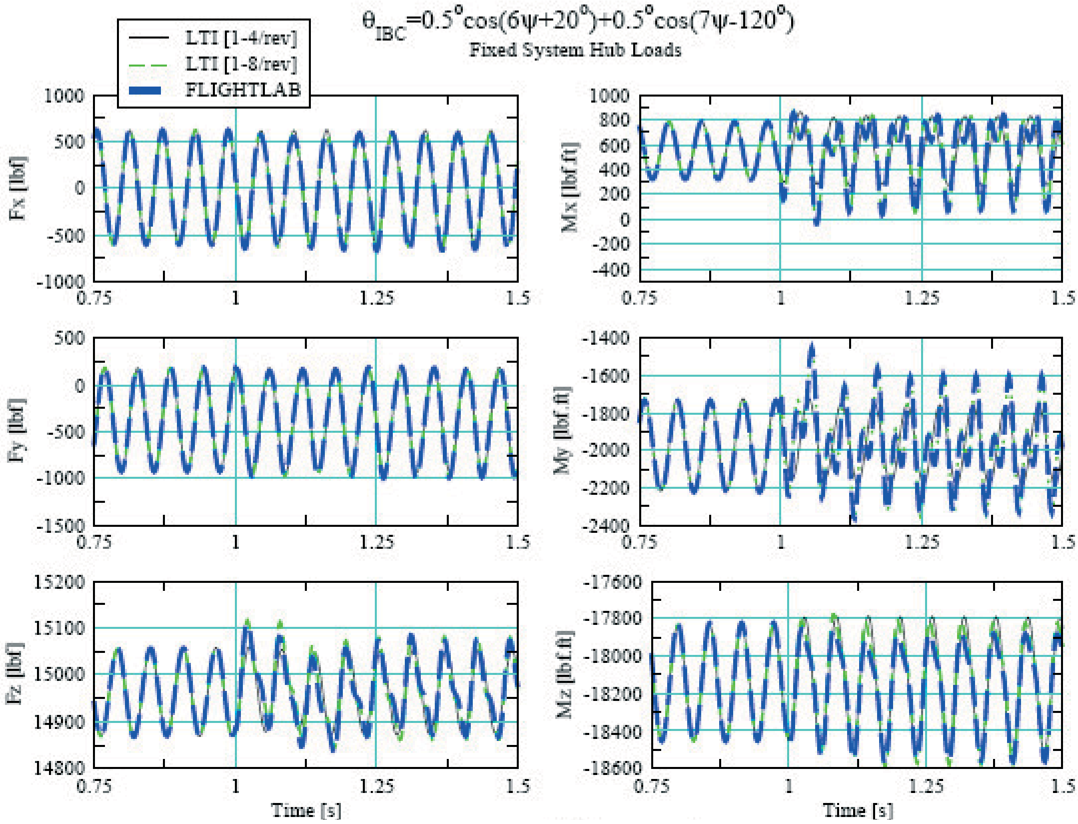

Figure 3b. A Zoom-In of Fig. 3a.

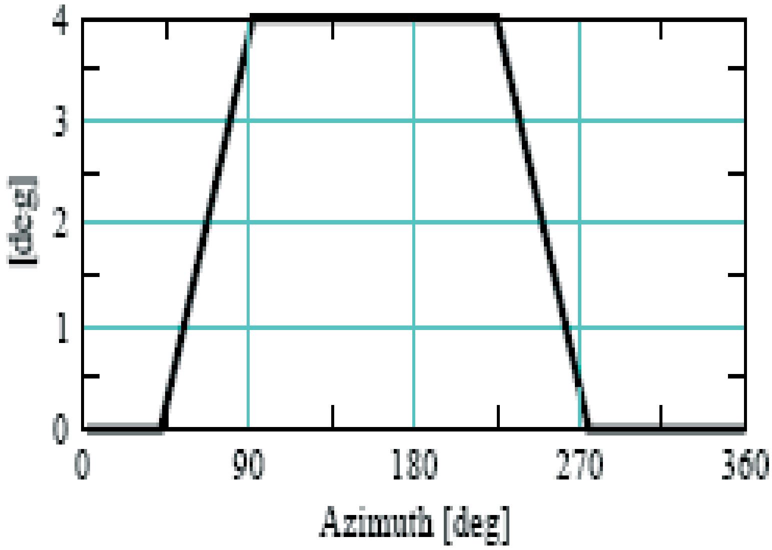

Figure 4. Selected Azimuth Dependent IBC Input.

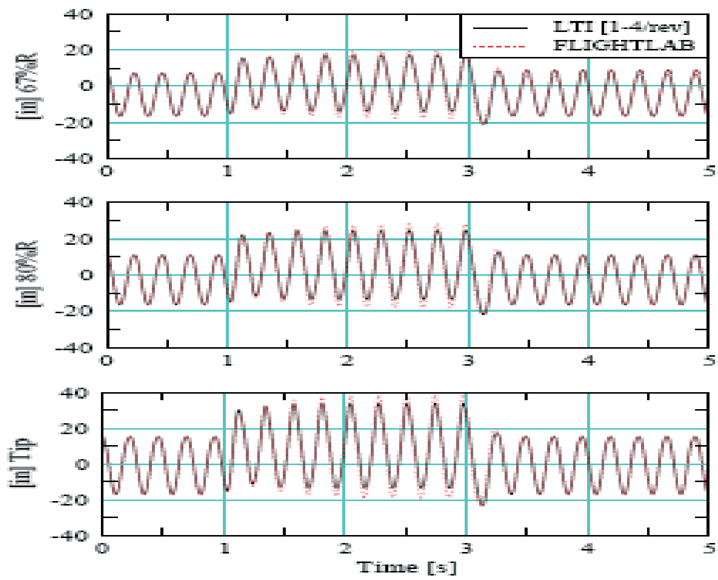

Figure 5. Predicted Elastic Blade Vertical Deflection (inches) in Response to the Selected IBC Pulse Input.

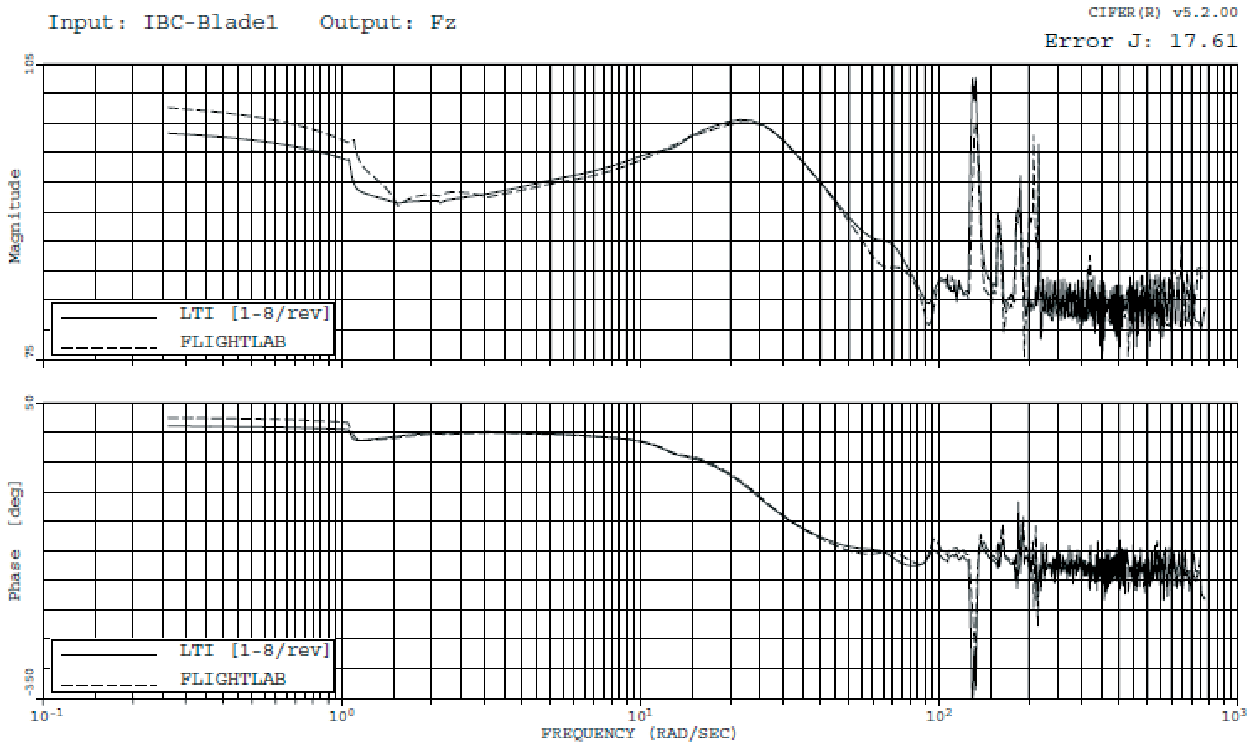

Figure 6. Predicted Frequency Response of Rotor Thrust to a Single Blade IBC Input.

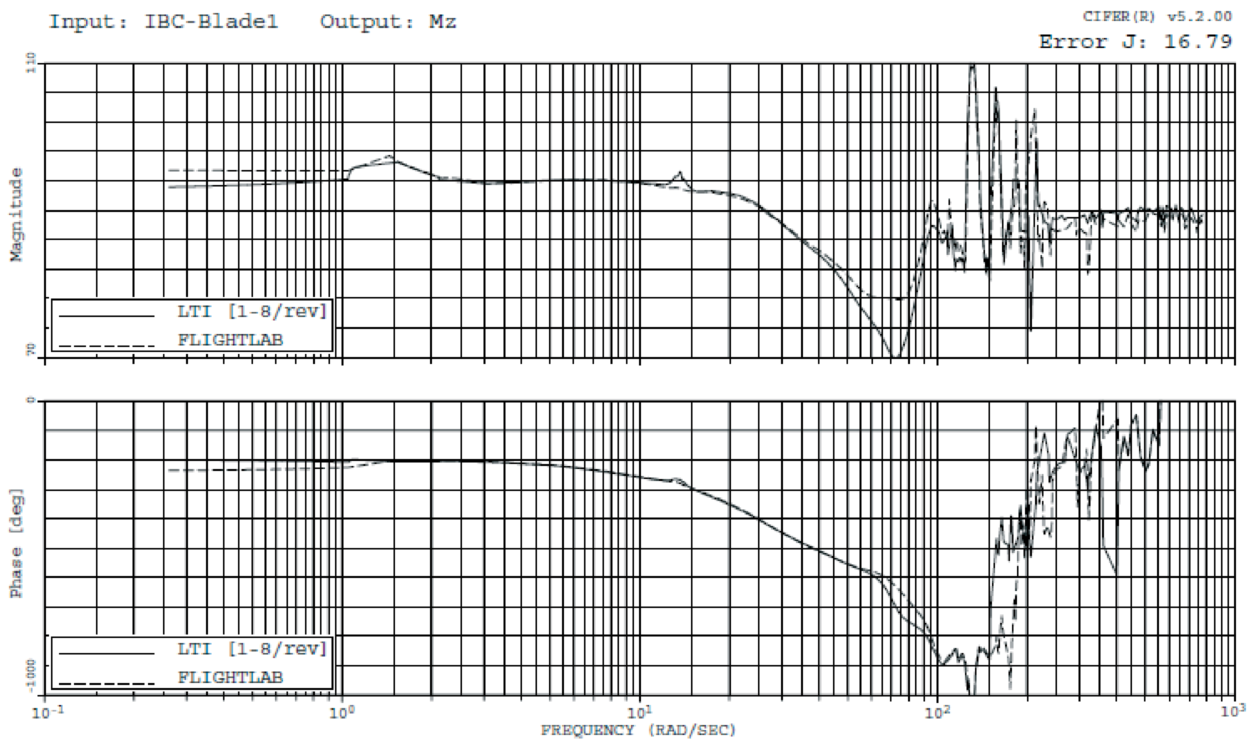

Figure 7. Predicted Frequency Response of Rotor Torque to a Single Blade IBC Input.