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Abstract

In this paper, an approach to the determination of rotor blade loading from measured strains is presented. The basic idea is to use the orthogonality of the blade mode shapes (displacement and/or moments) for solving this inverse problem. Both loading and response (bending moments and displacement) are regarded as series in the orthogonal mode shapes. The finite element method is used for structure dynamic analysis.

The disadvantage of the least-squares procedure is discussed, and the superiority of the "Orthogonal Analysis method" is demonstrated.

The primary numerical study shows that the approach is feasible and worthy of further exploitation in highly complex loading and structure.

1. Introduction

Predicting rotor blade loading continues to be of primary importance for providing and evaluating improved rotor designs. A long-term effort has been made to advance the technology associated with the aerodynamic and aeroelastic behaviour of helicopter rotors [1] [2] [3]. In view of the achievements which have been made and the problems which still exisit, it seems possible and worthwhile to combine the 'inverse' problem with the 'direct' problem in order to seek an engineering approach to determine rotor blade loadings, including rotor loads, blade stresses, and hub forces and moments. For example, based on aerodynamic theory and experiment, we can develop an aerodynamic model and derive relevant formulae from it to predict rotor blade airloads. The model should include all significant effects and the formulae should ideally be simple. There will inevitably be some coefficients and quantities which are determined by identification technology from the test blade loadings, and the loadings need therefore to be deduced from measured strains on the tested blade -- that is the 'inverse' problem. After determing airloads, the blade stresses and the hub vibratory forces of a new rotor can be calculated, this is the 'direct' problem. Perphaps these formulae, especially, the coefficients and quantities may be limited in some conditions, but they will be easily understood, trusted and can be conveniently used by designers within known limitations. It is this goal that the authors attempt to achieve.

It is obvious that the task is not an easy one, and much research work needs to be done in both the aerodynamics and the structural dynamics areas. This paper presents just part of the effort in the structural dynamics area: that is an approach to the determination of rotor blade loading from measured strains. The determination of airloads from strain guage measurements could of course also be used to gain insight into the nature of airloading distributions for the development of aeroelastic tailoring concepts and vibration reduction.

There are few publications on how to determine a distributed loading on a structure from the response. Tadghighi [4] has proposed a method to determine the aerodynamic loadings from the blade strains, but the author stated: "The accuracy of the results is unknown and the method has not been tested." The model was highly simplified, and could not obtain the response coefficients corresponding to blade modes. Hillary and Ewins [5,6] reviewed the inverse problem and presented a technique which was shown to be successful under some conditions. Unfortunately some problems arose, especially, in the determination of distributed forces over a simple cantilever beam. In their study a harmonic distributed load was represented by discrete point forces and the least-squares method was used, but the results were not totally satisfactory.

A new approach is presented here in which we express both the loading and response as series in orthogonal mode shapes for displacement and/or moments. The "orthogonal analysis method", in this paper is shown to overcome the deficiency of the least-squares method.

The main aim of this paper is only to present the idea mentioned in the first paragraph and an approach and to show its feasiblity; so the analysis is limited to the out-of-plane response and a numerical study is conducted using some simulated experimented data. All of the numerical results are listed in the form of tables to avoid the possibilty that curves conceal the slight differences among corresponding data.

2. Approach to blade loading Determination

The procedure for blade loading Determination from measured strains now follows. It should be stated that we are only dealing with the steady-state operating conditions, and here the finite element is used for the structural analysis.

i) Determine the bending moments $M(r,t)$ and their harmonic coefficients $M\ (r)$, $M_{Nc}(r)$ and $M_{Ns}(r)$ from the blade strain gauge outputs:

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$$M(r,t) = M_o(r) + \sum_N [M_{Nc}(r) \text{ Cos } (N\Omega t) + M_{Ns}(r) \text{ Sin } (N\Omega t)]$$

where r -- the radial coordinate of the measured station

t -- time

Ω -- rotor rotational speed

This Fourier analysis is standard strategy and we will assume that this has been achieved using test measurements for moments at a sufficient number of radial stations.

ii) Calcualte generalized coordinates from the measured moments.

There are two methods in this step. One is the least-squares procedure and the other is the "orthogonal analysis method" using the orthogonality of the blade mode shapes (displacement and moments).

a) The least-squares procedure.

The measured bending moments at any particular radial station r, $M_o(r)$, $M_{Nc}(r)$ and $M_{Ns}(r)$, may be expressed as

$$\left.\begin{aligned} M_o(r) &= \sum_j q_o^{(j)} M_R^{(j)}(r) \\ M_{Nc}(r) &= \sum_j q_{Nc}^{(j)} M_R^{(j)}(r) \\ M_{Ns}(r) &= \sum_j q_{Ns}^{(j)} M_R^{(j)}(r) \end{aligned}\right\} \qquad (1)$$

where $M_R^{(j)}(r)$ is the jth modal moments of the rotating blade (at rotational speed Ω), and is readily determined from the blade natural frequency and mode calculation; $q^{(j)}$ is the jth generalized coordinate, where superscript j represents the jth mode, and subscript R represents "Rotating".

Let

$$[q] = \begin{bmatrix} q_o^{(1)} & q_{1c}^{(1)} & q_{1s}^{(1)} & \cdots\cdots & q_{Nc}^{(1)} & q_{Ns}^{(1)} \\ q_o^{(2)} & q_{1c}^{(2)} & q_{1s}^{(2)} & \cdots\cdots & q_{Nc}^{(2)} & q_{Ns}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ q_o^{(m)} & q_{1c}^{(m)} & q_{1s}^{(m)} & \cdots\cdots & q_{Nc}^{(m)} & q_{Ns}^{(m)} \end{bmatrix} \qquad (2)$$

$$[M] = \begin{bmatrix} M_o(r_1) & M_{1c}(r_1) & M_{1s}(r_1) & \cdots\cdots & M_{Nc}(r_1) & M_{Ns}(r_1) \\ M_o(r_2) & M_{1c}(r_2) & M_{1s}(r_2) & \cdots\cdots & M_{Nc}(r_2) & M_{Ns}(r_2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ M_o(r_n) & M_{1c}(r_n) & M_{1s}(r_n) & \cdots\cdots & M_{Nc}(r_n) & M_{Ns}(r_n) \end{bmatrix} \qquad (3)$$

$$[M_R] = \begin{bmatrix} M_R^{(1)}(r_1) & M_R^{(2)}(r_1) & \cdots\cdots & M_R^{(m)}(r_1) \\ M_R^{(1)}(r_2) & M_R^{(2)}(r_2) & \cdots\cdots & M_R^{(m)}(r_2) \\ \vdots & \vdots & & \vdots \\ M_R^{(1)}(r_n) & M_R^{(2)}(r_n) & \cdots\cdots & M_R^{(m)}(r_n) \end{bmatrix} \qquad (4)$$

then the least-squares solution to [q] is given by:

$$[q] = ([M_R]^T [M_R])^{-1} [M_R]^T [M] \qquad (5)$$

where n is the number of measured stations, m is the number of used modes and also that of the unknown $q^{(j)}$ per harmonic component, and $n > m$.

The least-squares procedure is widely used as a "best fit" and is less susceptable to numerical oscillation than collocation where n=m. However it is still found that when the measured data [M] contain significant components of higher order than those used in the least-squares procedure then the results deteriorate and even those corresponding to the lower order modes are corrupted. In order to solve this problem, a new approach is presented using the orthogonality of the blade mode shapes (displacement and moments). In short, this is called "orthogonal analysis method".

(b) Orthogonal analysis method.

In this method the measured bending momnets $M_o(r)$, $M_{Nc}(r)$ and $M_{Ns}(r)$ are not initially regarded as series in the rotating blade model moments $M_R^{(j)}$ but as series in the non-rotating blade modal moments $M_{NR}^{(j)}$, i.e.

$$\left.\begin{aligned} M_o(r) &= \sum_j g_o^{(j)} M_{NR}^{(j)}(r) \\ M_{Nc}(r) &= \sum_j g_{Nc}^{(j)} M_{NR}^{(j)}(r) \\ M_{Ns}(r) &= \sum_j g_{Ns}^{(j)} M_{NR}^{(j)}(r) \end{aligned}\right\} \tag{6}$$

Applying conventional orthogonal analysis to the moments, we obtain ;

$$g^{(j)} = \int_o^L \frac{1}{EI(r)} M(r)\, M_{NR}^{(j)}(r)\, dr \Big/ P^2_{jNR}\, I_{NR}^{(j)} \tag{7}$$

where $g^{(j)}$ may be $g_o^{(j)}$, $g_{Nc}^{(j)}$ and $g_{Ns}^{(j)}$ corresponding to $M_o(r)$. $M_{Nc}(r)$ and $M_{Ns}(r)$; $M(r)$ may be $M_o(r)$, $M_{Nc}(r)$ and $M_{Ns}(r)$;

and

P_{jR} ----- the jth natural frequency of non-rotating blade;

$I_{NR}^{(j)}$ ----- generalized mass in the jth mode of non-rotating blade;

or

$P^2_{jNR}\ I_{NR}^{(j)}$ ----- generalized stiffness in the jth mode of non-rotating blade;

$EI(r)$ ----- bending stiffness.

Having $g^{(j)}$, the displacement $W(r)$ corresponding to the measured bending mements $M(r)$ can be calculated from :

$$[W] = [W_{NR}]\ [g] \tag{8}$$

where

$$[g] = \begin{bmatrix} g_o^{(1)} & g_{1c}^{(1)} & g_{1s}^{(1)} & \cdots & g_{Nc}^{(1)} & g_{Ns}^{(1)} \\ g_o^{(2)} & g_{1c}^{(2)} & g_{1s}^{(2)} & \cdots & g_{Nc}^{(2)} & g_{Ns}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_o^{(m)} & g_{1c}^{(m)} & g_{1s}^{(m)} & \cdots & g_{Nc}^{(m)} & g_{Ns}^{(m)} \end{bmatrix} \tag{9}$$

$$[W] = \begin{bmatrix} W_o(r_1) & W_{1c}(r_1) & W_{1s}(r_1) & \cdots\cdots & W_{Nc}(r_1) & W_{Ns}(r_1) \\ W_o'(r_1) & W_{1c}'(r_1) & W_{1s}'(r_1) & \cdots\cdots & W_{Nc}'(r_1) & W_{Ns}'(r_1) \\ W_o(r_2) & W_{1c}(r_2) & W_{1s}(r_2) & \cdots\cdots & W_{Nc}(r_2) & W_{Ns}(r_2) \\ W_o'(r_2) & W_{1c}'(r_2) & W_{1s}'(r_2) & \cdots\cdots & W_{Nc}'(r_2) & W_{Ns}'(r_2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ W_o(r_n) & W_{1c}(r_n) & W_{1s}(r_n) & \cdots\cdots & W_{Nc}(r_n) & W_{Ns}(r_n) \\ W_o'(r_n) & W_{1c}'(r_n) & W_{1s}'(r_n) & \cdots\cdots & W_{Nc}'(r_n) & W_{Ns}'(r_n) \end{bmatrix} \tag{10}$$

and

$$[W_{NR}] = \begin{bmatrix} W_{NR}^{(1)}(r_1) & W_{NR}^{(2)}(r_1) & \cdots\cdots & W_{NR}^{(m)}(r_1) \\ W_{NR}^{'(1)}(r_1) & W_{NR}^{'(2)}(r_1) & \cdots\cdots & W_{NR}^{'(m)}(r_1) \\ W_{NR}^{(1)}(r_2) & W_{NR}^{(2)}(r_2) & \cdots\cdots & W_{NR}^{(m)}(r_2) \\ \vdots & \vdots & & \vdots \\ W_{NR}^{(1)}(r_n) & W_{NR}^{(2)}(r_n) & \cdots\cdots & W_{NR}^{(m)}(r_n) \\ W_{NR}^{'(1)}(r_n) & W_{NR}^{'(2)}(r_n) & \cdots\cdots & W_{NR}^{'(m)}(r_n) \end{bmatrix} \qquad (11)$$

is the mode shapes of the non-rotating blade.

For a given blade, the same bending moments must correspond to the same displacement whether rotating or non-rotating. So the next analysis can be transformed to the rotating condititon.

Applying the orthogonal analysis to displacement, the unknown $q_o^{(j)}$, $q_{Nc}^{(j)}$ and $q_{Ns}^{(j)}$ can be obtained from

$$[q] = [m_e]^{-1} [W_R]^T [m] [W] \qquad (12)$$

where

$[q]$ ----- the same as matrix (2)
$[m]$ ----- mass matrix of the blade
$[m_e]$ ----- generalized mass matrix of the rotating blade
$[W_R]$ ----- the mode shapes of the rotating blade, similar to matrix (11)
$[W]$ ----- displacement obtained from Equation (8)

It is important to note that the accuracy of the orthongonal analysis method in evaluating $q^{(j)}$ should be independent of the number of modes which contained in the measured data M or used in M_{NR}, $\dot{W}_{NR}$, W_R

(iii) Calculate the loading coefficient $[F_j]$ from the generalized coordinate $[q]$.

The distributed blade loading, may now be expanded as a series in the mode shapes. In the commonly used finite element model, the external forces $[F]$ arise naturally as

$$[F] = [m]\,[W_R]\,[F_j] \qquad (14)$$

where

$$[F_j] = \begin{bmatrix} F_o^{(1)} & F_{1c}^{(1)} & F_{1s}^{(1)} & \cdots & F_{Nc}^{(1)} & F_{Ns}^{(1)} \\ F_o^{(2)} & F_{1c}^{(2)} & F_{1s}^{(2)} & \cdots & F_{Nc}^{(2)} & F_{Ns}^{(2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_o^{(m)} & F_{1c}^{(m)} & F_{1s}^{(m)} & \cdots & F_{Nc}^{(m)} & F_{Ns}^{(m)} \end{bmatrix} \qquad (15)$$

The relationships between the reponses, generalized coordinate $[q]$, and model force coefficient $[F_j]$ can be found from the forced vibration analysis of the blade as follows :

Let

$$\left\{F_o^{(j)}\right\} = \begin{Bmatrix} F_o^{(1)} \\ F_o^{(2)} \\ \vdots \\ F_o^{(m)} \end{Bmatrix} \qquad \left[\left\{F_{ic}^{(j)}\right\} \;\; \left\{F_{is}^{(j)}\right\}\right] = \begin{bmatrix} F_{ic}^{(1)} & F_{is}^{(1)} \\ F_{ic}^{(2)} & F_{is}^{(2)} \\ \vdots & \vdots \\ F_{ic}^{(m)} & F_{is}^{(m)} \end{bmatrix} \qquad (16)$$

$(i = 1,2,3 \ldots\ldots N)$

then

$$\left.\begin{aligned} \left\{F_o^{(j)}\right\} &= \left[P_{jR}^2\right]\left\{q_o^{(j)}\right\} \\ \left[\left\{F_{ic}^{(j)}\right\}\left\{F_{is}^{(j)}\right\}\right] &= \left(\left[P_j^2\right] - \left[(N\Omega)^2\right]\right)\left[\left\{q_{ic}^{(j)}\right\}\left\{q_{is}^{(j)}\right\}\right] \end{aligned}\right\} \qquad (17)$$

where $\left[P_{jR}^2\right]$ and $(N\Omega)^2$ are diagonal matrices whose elements are the square of the jth rotating blade natural frequency P^2_{jR} and $N^2\Omega^2$ repectively.

Equation (17) is for an undamped system, but for damped systems, the relationship can be extended without any difficulty.

iv) Determine the loading [F] from mode force coefficient $[F_j]$.

After calculating $[F_j]$, the loading [F] can be obtained easily from Equation (14);

$$[F] = [m]\,[W_R]\,[F_j]$$

3. Numerical Study and discussion

As an example, a hingless rotor blade with constant section is considered. Firstly, the natural frequencies, the mode shapes for the displacement and the bending moments (in both non-rotating and rotating cases) are calculated using the finite element method. The other related quantities, such as generalized mass $[m_e]$ and stiffness $[K_e]$ are obtained simultaneously. 25 spanwise stations are used in all calculations.

1) The blade loadings [F] have been calculated in many test cases by using assumed measured bending momnets which are obtained from the given coefficients $[q_o]$ multiplying the modal moments of the rotating blade $[M_R]$, however numerical results are presented here for only two cases. One is that the given bending moments only consist of the first four modal moments (case I). The other has the tenth order mode in addition to the first four (case II). In Tables 1,2,3 and 4, the given coefficients $[q_o]$, the determined [q], $[F_j]$ and [F] are tabled for the above two cases respectively. For these results, the orthogonal analysis method is used in the second step of the procedure. When the least-squares method is used, and the number of selected modes$\geqslant$10, the results are nearly the same.

In order to check accuracy, the dynamic reponse analysis is conducted by using the derived blade loading[F] for the same structure system. The results(mode force coefficient $[F_{j1}]$ and response $[q_1]$ are tabled in Tables 5 and 6 respectively.

By comparing Table 1 with Table 2,3 with 5,and 2 with 6, we clearly see that the approach is feasible and the methods are accurate.

2) As we mentioned, the least-squares procedure, using given data incorporating large higher order modes, can produce poor results, even for the lower order modes. In table 7, the determined coefficients [q] are tabled corresponding to three successive approximations using two, five and eight modal moments respectively. The given data $[q_o]$ are the same as Table 1 - case II. The results are clearly not good. On the other hand, the results from the orthogonal analysis method for the same cases, given in the Table 8, are much better and completely overcome the deficiency of the least-squares method.

3) It is obvious that the accuracy of this approach, especially when using the orthogonal analysis method in the second step, depends on the accuracy of the orthongonality analysis for the mode shapes (displacement and non-rotating modal moments). Therefore, it should be emphasised that the finite element method should be used for generating the orthogonal series. But it is not very easy to ensure the accuracy of the modal moments and the necessary orthogonality intergration. We have tried several ways and in this paper we have used the "dynamic stiffness method"[7] for calculating modal moments and the concept of "the continuous mass finite element" [8] for the orthogonal analysis. The results of the orthogonality test for the mode shapes of both rotating and non-rotating displacements and non-rotating moments are given in the Table 9. They are $[W_{NR}]^T[K_N]\,[W_{NR}]$. ($[K_N]$ is the stiffness matrix of non-rotating blade), $[W_R]^T[m]\,[W_R]$, and $\int_0^L \frac{1}{EI} M^{(i)}_{NR}(r)\, M^{(j)}_{NR}(r)\, dr$. The results are obviously very good and it is this which ensures the accuracy of the approach for the loading determination.

4. Summary and conclusion

At the begining of this paper, an idea was presented for seeking an engineering approach to determine rotor blade loadings by combining the "inverse" problem with the "direct" problem. A part of this inverse problem is to determine blade loading from measured strains, which is the main contents of this paper.

The approach presented in this paper is based on the mode superposition and the orthogonality analysis. The accuracy of the approach depends on the accuracy of measured strains and the number of measured stations, the number of used modes, the accuracy of the mode shape and bending moment calculation, and the accuracy of the orthogonality analysis.

When the measured moments contains some significant components of higher order than those used in the analysis, (for example a discontinuous bending moment ditribution), then the least-squares procedure can produce poor results, even for the lower order modes. The "orthogonal analysis method" presented in this paper can overcome the drawback, and ensure the accuracy of the generalized coordinate.

The accuracy of this approach has been demonstrated on a simple unifrom hingless rotor blade, and it now remains to show that this inverse solution works accurately and conveniently for complex realistically loaded and instrumented rotor blades.

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TABLE 1 Given Coefficients $[q_o]$

Case I

j	$q_o^{(j)}$	$q_{1c}^{(j)}$	$q_{1s}^{(j)}$
1	1.000000	2.022000	-.160300
2	.500000	.568300	-.655800
3	.000000	-.019550	-.074940
4	.000000	-.022980	.017980
5	.000000	.000000	.000000
6	.000000	.000000	.000000
7	.000000	.000000	.000000
8	.000000	.000000	.000000
9	.000000	.000000	.000000
10	.000000	.000000	.000000

Case II

j	$q_o^{(j)}$	$q_{1c}^{(j)}$	$q_{1s}^{(j)}$
1	1.000000	2.022000	-.160300
2	.500000	.568300	-.655800
3	.000000	-.019550	-.074940
4	.000000	-.022980	.017980
5	.000000	.000000	.000000
6	.000000	.000000	.000000
7	.000000	.000000	.000000
8	.000000	.000000	.000000
9	.000000	.000000	.000000
10	.010000	.010000	.010000

TABLE 2 Determined Generalized Coordinate [q]

Case I

j	$q_o^{(j)}$	$q_{1c}^{(j)}$	$q_{1s}^{(j)}$
1	1.004257	2.031637	-.159987
2	.503466	.574280	-.657557
3	.000966	-.018255	-.076084
4	.000297	-.022606	.017619
5	.000147	.000174	-.000154
6	.000084	.000104	-.000090
7	.000053	.000067	-.000057
8	.000035	.000043	-.000038
9	.000024	.000029	-.000027
10	.000017	.000011	-.000024

Case II

j	$q_o^{(j)}$	$q_{1c}^{(j)}$	$q_{1s}^{(j)}$
1	1.007170	2.034600	-.157074
2	.503127	.573940	-.657896
3	.001153	-.018067	-.075896
4	.000167	-.022736	.017489
5	.000218	.000245	-.000083
6	-.000010	.000010	-.000184
7	.000112	.000127	.000003
8	.000045	.000053	-.000028
9	.000042	.000047	-.000009
10	.010226	.010221	.010185

TABLE 3 Determined Force Coefficient $[F_j]$

Case I

j	$F_o^{(j)}$	$F_{1c}^{(j)}$	$F_{1s}^{(j)}$
1	1340.623951	329.809419	-25.971118
2	4789.135697	4789.326521	-5483.838374
3	34.806432	-636.692132	-2653.617115
4	30.143552	-2269.917058	1769.135405
5	34.417257	40.649817	-36.075957
6	39.773578	49.001607	-42.720867
7	45.882735	56.164751	-49.014899
8	51.192706	63.169051	-55.991059
9	56.756285	68.564561	-63.972953
10	59.938477	41.217103	-56.823704

Case II

j	$F_o^{(j)}$	$F_{1c}^{(j)}$	$F_{1s}^{(j)}$
1	1344.517833	330.282317	-25.498223
2	4785.903757	4736.497375	-5436.667520
3	41.575857	-530.142943	-2647.067881
4	16.947887	-2232.965346	1756.387116
5	51.140551	57.289611	-19.436162
6	-4.804641	4.528615	-87.193860
7	97.595344	109.807513	2.627863
8	65.328380	77.793087	-41.367022
9	100.310723	112.097378	-20.440141
10	36908.399772	36377.706903	35749.666095

TABLE 4 Determined loading [F]

Case I

i	$F_o(r_i)$	$F_{1c}(r_i)$	$F_{1s}(r_i)$
1	37.726547	15.851478	-38.945821
2	54.327920	32.547128	-55.344995
3	49.771091	34.215936	-48.715807
4	37.727949	35.975735	-34.165580
5	27.527363	35.636877	-20.261001
6	17.597157	30.541120	-4.926755
7	7.394124	19.772276	11.983198
8	-2.107452	4.913055	23.270020
9	-9.741764	-11.211558	41.401623
10	-15.444276	-26.233732	50.346538
11	-19.999737	-36.755363	55.285716
12	-23.846093	-47.754717	56.171895
13	-26.584743	-52.169721	52.358627
14	-27.691745	-51.348079	43.753854
15	-27.321114	-45.732596	31.803750
16	-26.131658	-35.847606	13.909819
17	-24.482576	-26.564804	6.995478
18	-22.140115	-16.435336	-3.053207
19	-18.839769	-7.601080	-10.675213
20	-14.867508	-.981093	-14.968824
21	-10.881965	2.805223	-15.187328
22	-7.280241	3.778194	-11.795169
23	-4.054711	2.742302	-6.662937
24	-1.377861	1.027282	-2.147172

Case II

i	$F_o(r_i)$	$F_{1c}(r_i)$	$F_{1s}(r_i)$
1	179.950245	158.009764	103.212466
2	1.121563	-30.676035	-128.568159
3	-215.164022	-230.669021	-313.600764
4	6.373888	4.597222	-65.544093
5	292.011908	300.002101	244.104222
6	216.577709	229.425066	193.957191
7	-132.642809	-120.249374	-128.038452
8	-288.570543	-281.485235	-258.128270
9	-50.177374	-51.660430	.952751
10	245.616007	234.715885	311.296155
11	187.612105	168.765392	262.806470
12	-151.393910	-175.283645	-71.357034
13	-317.206512	-342.718387	-233.190039
14	-89.561996	-113.218003	-13.116070
15	223.304414	204.793542	282.329888
16	201.346035	190.540083	246.297507
17	-124.706442	-126.770531	-93.210249
18	-316.879237	-311.091778	-297.709649
19	-115.764846	-104.507164	-107.581297
20	214.734692	228.585607	214.597976
21	239.523592	253.120079	235.127528
22	-71.984584	-60.912205	-76.485568
23	-312.858827	-305.965694	-315.370933
24	-203.027711	-200.558515	-203.732968

TABLE 5 Force Coefficient $[F_{j1}]$

Case I

j	$F_{\circ 1}^{(j)}$	$F_{1c1}^{(j)}$	$F_{1s1}^{(j)}$
1	1340.625844	329.809558	-25.971187
2	4789.123731	4789.316150	-5483.825304
3	34.805342	-636.697137	-2653.632431
4	30.145046	-2269.917032	1769.136410
5	34.413847	40.647889	-36.371355
6	39.773476	49.000105	-42.717313
7	45.873725	58.163314	-49.010629
8	51.136382	63.166631	-55.986218
9	56.749974	5[illegible].562856	-[illegible]3.967155
10	59.930876	41.214657	-86.817209

Case II

j	$F_{\circ 1}^{(j)}$	$F_{1c1}^{(j)}$	$F_{1s1}^{(j)}$
1	1344.466250	330.234003	-25.546742
2	4785.837515	4786.427748	-5486.713707
3	41.530439	-630.192218	-2647.127512
4	16.893154	-2283.016529	1755.036914
5	51.107133	57.257693	-19.461552
6	-4.852806	4.484067	-87.233352
7	97.574833	109.789583	2.615639
8	65.777129	77.745758	-41.407091
9	100.286657	112.077927	-20.452084
10	36908.773442	35878.085509	36750.053743

TABLE 6 Generalized Coordinate $[q_1]$

Case I

j	$q_{\circ 1}^{(j)}$	$q_{1c1}^{(j)}$	$q_{1s1}^{(j)}$
1	1.004257	2.031688	-.159987
2	.503466	.574278	-.657556
3	.003966	-.018255	-.076084
4	.003297	-.022636	.017619
5	.000147	.000174	-.000154
6	.000084	.000104	-.000090
7	.000053	.000067	-.000057
8	.000035	.000043	-.000038
9	.000024	.000029	-.000027
10	.000017	.000011	-.000024

Case II

j	$q_{\circ 1}^{(j)}$	$q_{1c1}^{(j)}$	$q_{1s1}^{(j)}$
1	1.007170	2.034303	-.157373
2	.503127	.573932	-.657902
3	.001153	-.018059	-.075898
4	.000167	-.022736	.017488
5	.000218	.000245	-.000083
6	-.000010	.000009	-.000184
7	.000112	.000127	.000003
8	.000045	.000053	-.000028
9	.000042	.000047	-.000009
10	.010226	.010221	.010185

TABLE 7 Determined Generalized coordinate [q] by least-squares procedure

j	q_o	q_{1c}	q_{1s}
only first two modal moments			
1	.695975	1.727398	-1.041325
2	.513153	.548778	-.692038
only first five modal moments			
1	.787462	1.809462	-.372838
2	.515747	.584047	-.640053
3	-.005734	-.026234	-.081674
4	.004231	-.018749	.022211
5	-.002351	-.002351	-.002351
only first eight modal moments			
1	.832941	1.854941	-.327359
2	.512921	.581221	-.642879
3	-.005595	-.025145	-.080535
4	.003453	-.019527	.021433
5	-.002151	-.002151	-.002151
6	.001652	.001652	.001652
7	-.001083	-.001083	-.001083
8	.001039	.001039	.001039

TABLE 8 Determined Generalized coordinate [q] by Orthogonal analysis method

j	q_o	q_{1c}	q_{1s}
only first two modal moments			
1	1.006396	2.033259	-.155153
2	.502354	.574306	-.656402
only first five modal moments			
1	1.007173	2.034601	-.157078
2	.503119	.573947	-.657886
3	.001167	-.018074	-.075903
4	.000132	-.022684	.017545
5	.000206	.000377	.000069
only first eight modal moments			
1	1.007172	2.034602	-.157072
2	.503121	.573934	-.657902
3	.001165	-.018056	-.075885
4	.000146	-.022756	.017469
5	.000241	.000267	-.000061
6	-.000058	-.000037	-.000231
7	.000137	.000144	.000023
8	-.000105	-.000113	-.000187

TABLE 9 The results of the Orthogonality test

j \ i	1	2	3	4	5
	$[W_{NR}]^T [K_N] [W_{NR}]$				
1	3.785262	.000000	.000000	.000000	.000000
2	.000000	143.661339	.000000	.000000	.000000
3	.000000	.000000	1165.517311	.000000	.000000
4	.000000	.000000	.000000	4475.430491	.000000
5	.000000	.000000	.000000	.000000	12228.524402
	$[W_R]^T [m] [W_R]$				
1	.089296	.000000	.000000	.000000	.000000
2	.000000	.057082	.000000	.000000	.000000
3	.000000	.000000	.066333	.000000	.000000
4	.000000	.000000	.000000	.072614	.000000
5	.000000	.000000	.000000	.000000	.074879
	$\int_o^L \frac{1}{E\,I} M_{NR}^{(i)}(r)\, M_{NR}^{(j)}(r)\, dr$				
1	3.785256	.000121	.000064	.001778	-.007572
2	.000116	143.664795	.001766	-.009046	.069863
3	.000189	.000470	1165.691250	.022154	-.122979
4	.000254	-.000493	-.013550	4477.814930	.562505
5	.000344	.000632	.007905	.296218	12244.985720