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The Influence of Variable Flow Velocity

on Unsteady Airfoil Behavior

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Abstract

In this paper, the effects of an oscillating freestream on the unsteady aerodynamic lift of an airfoil is examined. First, existing theories are investigated and their simplifications and limitations are identified. Then, the theories are compared to show the differences between them for different conditions: first, for constant angle of attack, second, for in-phase pitch motion, and third, for 90° out-of-phase pitch with respect to velocity oscillations. In addition, the exact theory (exising only for constant angle of attack and pitch about midchord) is extended to pitch (including higher harmonics) about an arbitrary axis and plunge motion (also including higher harmonics). The results are also compared to a finite difference scheme of the arbitrary motion theory. It was found, that the arbitrary motion theory is best suited for calculating the unsteady aerodynamic lift even in an oscillating freestream. The results of the exact theory are finally validated using an Euler code for very low Mach numbers.

Nomenclature

a	pitch axis location wrt. midchord, pos. aft	$\alpha, \bar{\alpha}$	angle of attack, nondimensional angle of attack
c	airfoil chord	λ	nondimensional amplitude of velocity oscillation
$C(k_V) = F(k_V) + iG(k_V)$	Theodorsen function	ρ	air density
$\bar{h}_{1S}, \bar{h}_{1C}$	nondimensional amplitude of plunge displacement	$\Phi(s)$	Wagner function
i	imaginary unit	$\psi = \omega_V t = k_V s$	nondimensional time
$J_n(n\lambda)$	Bessel function	ω_V	frequency of velocity oscillation
$k_V = \omega_V t/(2V_0)$	reduced frequency of velocity oscillations	**Indices:**	
L, C_L	lift, lift coefficient	V	velocity
Re	Reynolds number	0	mean or reference value
s	nondimensional distance travelled by the airfoil	c, nc	circulatory, non circulatory
t	time	S, C	sine-, cosine component
V	velocity	qs	quasisteady
w	normal velocity	3/4	at 3/4 chord

1 Introduction

A helicopter rotor blade in forward flight encounters a highly unsteady flowfield. To predict the aeroelastic behavior of the rotor, it is necessary to accurately calculate the aerodynamic loads acting on the blades. These consist of both steady as well as unsteady components. One source of aerodynamic loads is the varying oncoming flow velocity at each blade station. This leads to a dynamic pressure variation containing steady, $1/rev$ and $2/rev$ components. Additional degrees of freedom result from the blade motion in flap, lag and torsion, and the nonuniform inflow. Therefore, a fully unsteady aerodynamic theory must be used to predict the aerodynamic loads. This has been discussed by various authors, for example by Johnson and Kaza [1, 2]. Both state that the lift deficiency function must be generalized to account for the unsteady freestream effects. This generalisation was given by Johnson [3], but in most analysis the Theodorsen lift deficiency function for constant freestream flow [4] is often used instead. However, the direct application of Theodorsen's theory to rotorcraft in forward flight is questionable. A theory including the effect of periodically stretching and compressing the shed wake vorticity distribution behind the pitching, plunging and fore-aft moving airfoil should be used in order to include the effects of varying freestream on the unsteady aerodynamic forces and moments. In this paper, a review on modelling the varying freestream effects will be given, and an exact theory for an airfoil with pitching, plunging and fore-aft motion will be presented. The limitations and assumptions of existing theories will be clearly shown. The objective is first to find an answer to whether or not it is necessary to model the effects of unsteady freestream fluctuations in a rotor loads or aeroelastic analysis in forward flight. The second objective is to show whether or not it is possible to simulate the attached flow behavior using an arbitrary motion theory, comprizing of Duhamel's integral and indicial function approximation (Wagner function) for step changes in angle of attack, pitch rate and plunge velocity.

It is necessary to differentiate between two kinds of velocity changes that a rotor blade encounters in forward flight. First there will be a fore-aft (lead-lag) motion of the rotor blade, and second, an oscillating freestream velocity (gust problem) resulting from the superposition of the rotational velocity and the forward speed of the helicopter, see Fig. 1. The first case (lead-lag) leads to a uniform velocity distribution across the airfoil chord, while the second case (gust) produces a velocity gradient across the chord. For small reduced frequencies both cases may be handled the same way since the gradients in the second case are small. However, this is only an *approximation* and is not valid for large large reduced frequencies. This is because a lead-lag motion will result in very large noncirculatory forces, while in an oscillating freestream the noncirculatory lift will reduce to zero again since several modes along the chord cancel each other (as is the case in a vertical gust field). For the form of the wake behind the airfoil, however, there is no difference between either case because the positioning and velocity of vorticity in the shed wake relative to the airfoil remains the same. A radial station of a helicopter blade, in reality, encounters both phenomena and the velocity changes due to forward flight are physically a gust problem and should be treated as such.

Analytical approaches to the problem of an oscillating airfoil in a varying freestream velocity have been performed by several authors in the past. Fundamental closed form solutions for an oscillating airfoil in a steady freestream were given by Theodorsen

in 1935 [4], and in 1940 in operational form by Sears [5]. Probably the first attempt to derive a closed form solution for the case of unsteady freestream velocity variations was given by Isaacs in 1945, but then only for the case of *constant angle of attack* [6]. In 1946, Isaacs included a periodic change in angle of attack in order to fulfill the needs of helicopter aerodynamicists [7]. His solution, however, was confined to a pitch axis at half chord, and therefore *it was not very* appropriate for helicopter calculations since nearly all helicopter blades have a feathering axis at the quarter chord.

In 1946 Greenberg [8] published *his extension of Theodorsen's theory.* Even today, his results are thought to be the most reliable for application to rotorcraft aeroelastic problems. However, Greenberg made a high frequency assumption about the shed wake behind the airfoil to obtain a solution in terms of the Theodorsen function only. The effect and nature of this assumption was never clarified.

In 1977 a new theory directly related to rotorcraft was developed by Kottapalli [9]. In 1979 [10] and again in 1985 [11] additional results were published. This theory was developed by applying only small lead-lag oscillation amplitudes with respect to the mean velocity. Consequently, Kottapalli limits the validity of his approach to the case of blade flutter in hover. Therefore, the results seem to be of limited help for helicopter applications in forward flight, since the assumption of *small flow oscillation* amplitudes holds only for very small advance ratios.

Johnson published some discussion regarding the problem of a varying velocity in [3]. *Using the same assumptions made by* Isaacs [6, 7], Johnson basically followed Isaacs' theory and gave expressions for lift and moment of an airfoil having plunge as well as pitch motion about an arbitrary pitch axis. The final result is given in form of integrals without giving the appropriate solution of these in terms of Bessel functions. The effect of varying velocity is described by Johnson as: *"On the advancing side, the increased velocity lowers the reduced frequency and hence the lift deficiency function is nearer unity. On the retreating side there is the greatest accumulation of shed vorticity in the wake near the trailing edge, and thus the greatest reduction in lift. In summary ... all these effects basically produce 1/rev variations of the loads."* Johnson's conclusion is that the approximation using the Theodorsen function with the local reduced frequency will work for flow oscillation amplitudes of up to 70% of the mean velocity. For small flow oscillation amplitudes, the Theodorsen function calculated using the mean velocity will be accurate enough, which effectively means neglecting the unsteady freestream fluctuations. However, this statement seems to be based only on one presented result, and it is doubtful whether it holds for other mean reduced freuencies and higher harmonics of the blade response.

Other authors refer to different problems with time varying velocities, especially accelerated motions, but not to harmonically varying freestreams. Some of them are to be found in [12, 13, 14, 15].

Most of the experimental work done in this area of research is the measurement of the aerodynamic coefficients in a wind tunnel. A number of experiments with airfoils oscillating in a constant freestream velocity have been conducted, for example [18, 19]. Only few experiments have been done in an oscillating freestream velocity environment, which is of interest here.

Probably the first experiments on this problem were done by Fejer, Saxena and Morkovin in 1976 [20, 21]. The parameters achieved were $\lambda = 0.18$, $k = 0.18$ and 0.9, $Re = 2.5 \times 10^5$ and a trip was mounted to force the boundary layer to be turbulent. Here and in later tests [22, 23] it has been found that in periodically changing flows, dynamic stall of airfoils can assume a variety of forms depending on the frequency and amplitude of the oscillations. The airfoil coefficients do not behave in a quasisteady manner, and it was concluded that for the case of helicopter dynamic stall the freestream flow fluctuations must be taken into account and cannot be neglected.

Parallel to the analytical work of Kottapalli at Georgia Institute of Technology, some experiments were also conducted by Pierce, Kunz and Malone [24] in 1976. $\lambda = 0.177$ and $Re = 2.02 \times 10^5$ could be achieved. The pitch frequency was set to 6 times of the flow oscillation frequency in order to have one airfoil oscillation during the more or less linear regime of accelerating flow, and one in the appropriate regime of decelerating flow. Steady tests showed thin airfoil stall characteristics on the airfoil. Dynamic tests showed a large effect of flow oscillations on the dynamic stall behavior.

At about the same time, the French team of Maresca, Favier and Rebont started *a series of experiments with an airfoil* undergoing fore-aft motions, plunge motions and pitch motions in a steady stream [25, 26, 27]. They achieved high values of λ, but the mean velocity of the flow was very small $Re = 2.5 \times 10^5$. In 1982 the same authors presented some additional measurements of combined motion for oscillations below the static stall angle, as well as for those going beyond stall, and compared the results for lift, drag and moment with the appropriate plunge oscillations in a constant freestream flow [28]. The hysteresis loops were found to be entirely different. Moreover, at $Re = 1.44 \times 10^5$ one must be careful to assume the flow below the static stall angle as attached since the airfoil is very likely to experience thin airfoil stall. Additional measurements were conducted and presented in 1988 [29]. It was shown that the phase of the flow velocity and the angle of attack oscillations is an important parameter and changes the lift hysteresis behavior in a significant manner. The data presented in 1992 [30] also refer to rather low Reynolds numbers.

As a result of the foregoing, it can be stated that there is only limited airfoil data for freestream fluctuations available to compare with theory, and the data already published are mostly confined to the dynamic stall phenomenon, not to the case of attached flow. In case of the tests having angles of attack smaller than the static stall angle, the flow will also not be attached because of the small Reynolds numbers, leading to thin airfoil stall characteristics with separation regimes beginning at very small angles of attack. Therefore it will be very difficult, if not impossible, to compare the theories with exisitng experimental data.

Until now, there is no other theory available for this problem. Also, comparisons between the various theories are very scarse. This gap has been closed in recent research by the author [16, 17] *showing in detail the results and differences of the* various theories for different conditions. Also, the assumptions and simplifications made by the various authors are clarified. In addition, the results are compared with those of a finite difference scheme of the arbitrary motion theory. In this paper the main results from [16] are presented.

2   Theories for Unsteady Freestream

The values of normalised velocity amplitudes, $\lambda = \Delta V/(\Omega R)$, as well as the range of reduced frequencies at which helicopter blade sections are operating are of significant interest. In forward flight, λ can take any value from zero to unity or even more in regions of reversed flow. The reduced frequency here is defined by the mean section normal velocity in hover, Ωr. Taking a typical value of $R/c = 20$, the distribution of reduced frequencies depends on the geometry only: $k_V = (\omega_V c)/(2V) = 0.025/(r/R)$. So the reduced frequencies at a typical rotor blade section range from 0.025 at the tip, to 0.125 at the root. The reduced frequencies are not very high, since only the 1/rev motion was taken into account, but high enough to justify the need of an unsteady aerodynamic theory in rotor calculations. When considering lead-lag motion of the higher modes the rotor blade, the reduced frequencies are considerably higher but the amplitudes will be much smaller. Thus, an analytic theory cannot be simplified for small values of λ or small k_V. In this study, the following types of motion have been investigated:

$$
\begin{aligned}
V(t) &= V_0(1 + \lambda \sin \omega_V t) \\
\alpha(t) &= \alpha_0 \left(\bar{\alpha}_0 + \bar{\alpha}_{1S} \sin \omega_V t + \bar{\alpha}_{1C} \cos \omega_V t \right) = \alpha_0 \bar{\alpha}_0 + \alpha_{dyn} \\
h(t) &= \frac{c}{2}\alpha_0 \left(\bar{h}_{1S} \sin \omega_V t + \bar{h}_{1C} \cos \omega_V t \right)
\end{aligned}
\tag{1}
$$

In the following sections, Theodorsen's theory is combined with an unsteady freestream, and Greenberg's, Kottapalli's and Isaacs' theory are given in terms of Fourier series for easy application and comparison. For convenience, all results will be written in nondimensional form by dividing by the lift at the reference angle of attack α_0 and the mean velocity V_0, i. e. by L_0.

$$
L_0 = \frac{\rho}{2} V_0^2 c C_{L0} \qquad\qquad C_{L0} = 2\pi\alpha_0
\tag{2}
$$

It must be kept in mind, that all the theoretical approaches above were formulated with certain assumptions. In summary, these are:

1. Two-dimensional flow (i.e., no spanwise effects or curved wake forms included)
2. Incompressible flow (i.e., infinite speed of sound)
3. Small disturbances (i.e., thin airfoil, small angles, small frequencies)
4. No friction forces (i.e., infinite Reynolds number = nonviscous flow)
5. Planar, infinite wake (i.e., no distortion, no diffusion)
6. Constant freestream velocity across the chord (i. e., only lead-lag motion considered)

Therefore, the results can be valid only in the incompressible attached flow regime. Especially the last item of the list is interesting since all authors claimed to handle the unsteady freestream effect yet in reality they provided a solution for the lead-lag problem with some additional simplifications (except Isaacs [6, 7] with an exact solution for lead-lag effects).

2.1   Theodorsen's Theory and Unsteady Freestream

To apply Theodorsen's result to unsteady freestream, it is necessary to include the freestream variations into the noncirculatory and circulatory parts. This may be referred to as the direct effect of velocity changes on the lift development; the additional phase lags and amplifications due to the wake are not included. The Theodorsen function is defined by $C(k) = F(k) + iG(k)$. This leads to the following result for the lift in the form of a Fourier series

$$
\begin{aligned}
\frac{L_{nc}}{L_0} &= \frac{k_V}{2} \left\{ \left[\lambda\bar{\alpha}_0 + \bar{\alpha}_{1S} + k_V(a\bar{\alpha}_{1C} - \bar{h}_{1C}) \right] \cos\omega_V t + \lambda\bar{\alpha}_{1C}\cos 2\omega_V t + \left[-\bar{\alpha}_{1C} + k_V(a\bar{\alpha}_{1S} - \bar{h}_{1S}) \right] \sin\omega_V t + \lambda\bar{\alpha}_{1S}\sin 2\omega_V t \right\} \\[4pt]
\frac{L_c}{L_0} &= \bar{\alpha}_0\left(1 + \frac{\lambda^2}{2}\right) + \frac{\lambda}{2}\left[f_{1S} + F(k_V)\bar{\alpha}_{1S} - G(k_V)\bar{\alpha}_{1C}\right] + \left\{ f_{1C} + \frac{\lambda^2}{4}\left[F(k_V)\bar{\alpha}_{1C} + G(k_V)\bar{\alpha}_{1S}\right] \right\}\cos\omega_V t \\[4pt]
&\quad + \left\{ 2\lambda\bar{\alpha}_0 + f_{1S} + \frac{3\lambda^2}{4}\left[F(k_V)\bar{\alpha}_{1S} - G(k_V)\bar{\alpha}_{1C}\right] \right\}\sin\omega_V t - \frac{\lambda}{2}\left[\lambda\bar{\alpha}_0 + f_{1S} + F(k_V)\bar{\alpha}_{1S} - G(k_V)\bar{\alpha}_{1C}\right]\cos 2\omega_V t \\[4pt]
&\quad + \frac{\lambda}{2}\left[f_{1C} + F(k_V)\bar{\alpha}_{1C} + G(k_V)\bar{\alpha}_{1S}\right]\sin 2\omega_V t - \frac{\lambda^2}{4}\left\{\left[F(k_V)\bar{\alpha}_{1C} + G(k_V)\bar{\alpha}_{1S}\right]\cos 3\omega_V t + \left[F(k_V)\bar{\alpha}_{1S} - G(k_V)\bar{\alpha}_{1C}\right]\sin 3\omega_V t\right\}
\end{aligned}
\tag{3}
$$

with the coefficients

$$
\begin{aligned}
f_{1S} &= F(k_V)\left[\bar{\alpha}_{1S} - k_V\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right] - G(k_V)\left[\bar{\alpha}_{1C} + k_V\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1S} + \bar{h}_{1S}\right)\right] \\
f_{1C} &= F(k_V)\left[\bar{\alpha}_{1C} + k_V\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1S} + \bar{h}_{1S}\right)\right] + G(k_V)\left[\bar{\alpha}_{1S} - k_V\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right]
\end{aligned}
\tag{4}
$$

The appropriate lift coefficients are evaluated simply by the following formula

$$
\frac{C_L(t)}{C_{L0}} = \frac{L(t)}{L_0}\frac{1}{(1 + \lambda\sin\omega_V t)^2}
\tag{5}
$$

From these equations, the quasisteady theory result follows as a special case. This assumes very small frequencies, and therefore the noncirculatory part becomes zero while the Theodorsen function takes the values $F(kv) = 1$ and $G(kv) = 0$. Therefore

$$
\begin{aligned}
\frac{L_{c,qs}}{L_0} =\ & \bar{\alpha}_0 \left(1 + \frac{\lambda^2}{2}\right) + \lambda \left[\bar{\alpha}_{1S} - \frac{kv}{2}\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right] + \left[\bar{\alpha}_{1C}\left(1 + \frac{\lambda^2}{4}\right) + kv\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1S} + \bar{h}_{1S}\right)\right]\cos\omega v t \\
& + \left[2\lambda\bar{\alpha}_0 + \bar{\alpha}_{1S}\left(1 + \frac{3\lambda^2}{4}\right) - kv\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right]\sin\omega v t \\
& - \frac{\lambda}{2}\left[\lambda\bar{\alpha}_0 + 2\bar{\alpha}_{1S} - kv\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right]\cos 2\omega v t - \frac{\lambda^2}{4}\bar{\alpha}_{1C}\cos 3\omega v t \\
& + \frac{\lambda}{2}\left[\qquad 2\bar{\alpha}_{1C} + kv\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1S} + \bar{h}_{1S}\right)\right]\sin 2\omega v t - \frac{\lambda^2}{4}\bar{\alpha}_{1S}\sin 3\omega v t
\end{aligned}
$$

$$(6)$$

Even from this simple result, it can be seen that the lift response includes a $3/rev$ component because of the multiplication of the trigonometric functions. When the compression and stretching of the shed wake is taken into account, then the vorticity in the shed wake does not have a sinusoidal form but more of a kind of Fourier series of harmonics. The conclusion is that there will also be a series of harmonics in the lift and moment response that is not predicted by quasisteady assumptions. Additionally, if the airfoil is set at a constant angle of attack and has no pitch or plunge motion, both Theodorsen's theory and quasisteady theory lead to the same circulatory lift since no lift deficiency function is in effect. Thus, the use of quasisteady theory or Theodorsen's theory in an unsteady freestream velocity is questionable, in general.

Despite this, the quasisteady theory is a reasonable simplification for small reduced frequencies, but it is unclear whether this statement holds also for large flow oscillation amplitudes λ, even when the reduced frequency is small. This will be clarified using results from more complete theories.

2.2 Isaacs' Theory

This theory assumes a $1/rev$ variation in angle of attack *about midchord* with the same frequency as in the freestream variations. Again, the result can again be expressed in the form of a Fourier series.

$$
\frac{L_{nc}}{L_0} = \frac{kv}{2}\left[(\lambda\bar{\alpha}_0 + \bar{\alpha}_{1S})\cos\omega v t - \bar{\alpha}_{1C}\sin\omega v t + \lambda(\bar{\alpha}_{1C}\cos 2\omega v t + \bar{\alpha}_{1S}\sin 2\omega v t)\right]
$$

$$
\frac{L_c}{L_0} = \left[\bar{\alpha}_0\left(1 + \frac{\lambda^2}{2}\right) + \lambda\left(\bar{\alpha}_{1S} - \frac{kv}{4}\bar{\alpha}_{1C}\right)\right](1 + \lambda\sin\omega v t) + \sum_{m=1}^{\infty}(l_m\cos m\omega v t + l'_m\sin m\omega v t)
$$

$$(7)$$

with the coefficients

$$
l_m + il'_m = -2\frac{m}{i^m}\sum_{n=1}^{\infty}\left\{F_n[J_{n+m}(n\lambda) - J_{n-m}(n\lambda)] + iG_n[J_{n+m}(n\lambda) + J_{n-m}(n\lambda)]\right\}
$$

$$(8)$$

Here

$$
F_n + iG_n = [F(nkv) + iG(nkv)]\frac{H_n + iH'_n}{n^2}
$$

$$(9)$$

with

$$
\begin{aligned}
H_n &= \frac{J_{n+1}(n\lambda) - J_{n-1}(n\lambda)}{2}\left(\lambda\bar{\alpha}_0 - \bar{\alpha}_{1S} - \frac{kv}{2}\bar{\alpha}_{1C}\right) - \frac{2J_n(n\lambda)}{n\lambda}\bar{\alpha}_{1S} \\
H'_n &= \frac{J_{n+1}(n\lambda) - J_{n-1}(n\lambda)}{n}\bar{\alpha}_{1C} + \frac{J_n(n\lambda)}{\lambda}\left[\bar{\alpha}_{1C}(1 - \lambda^2) - \frac{kv}{2}\bar{\alpha}_{1S}\right]
\end{aligned}
$$

$$(10)$$

Setting $\bar{\alpha}_{1S} = \bar{\alpha}_{1C} = 0$ and $\bar{\alpha}_0 = 1$ one obtains the expression for constant angle of attack. A closer examination of Isaacs' result (Eq. 8) indicates certain limitations in its application since there are two nested summations involved.

1. The first sum (over m) represents the harmonic content of the lift response. If the interest is mainly in the rotor performance, one can neglect the higher harmonics and will obtain sufficiently accurate results with the first few harmonics alone.

2. The second sum (over n) has to be calculated for every item in the first sum. Since here Bessel functions of the first kind and n-th integer order are involved, as well as the computation of the Theodorsen function, this part requires considerable computational time when it is necessary to calculate higher harmonics. One must keep in mind that the Theodorsen function also consists of Bessel functions of the first and second kind. This series, therefore, has to be terminated after computing a sufficient number of elements in order to minimize computational time.

For the special case (thought to be typical for helicopters in 1945) of constant angle of attack, a reduced frequency $kv = 0.0424$ and a freestream oscillation amplitude of $\lambda = 0.4$, Isaacs gave a numerical example for the total lift ratio L/L_0 and compared it to the quasisteady theory leading to the result: "...so that for this case the effects herein considered [1] are not large."

This issue often comes to mind when it comes to justifying the flow oscillation effect. Since it is based only on this special case of moderate flow amplitude (nowadays helicopters encounter much greater values of λ, even larger than unity) it is not to be taken as the general case. Only a systematic study with a variety of parametric variations including all reduced frequencies of

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[1] Unsteady freestream effects are meant here

interest, as well as all flow oscillation amplitudes, will be able to justify the necessity of including these effects. The quasisteady formulation yields for $a = 0$ (rotation about midchord)

$$
\begin{aligned}
\frac{L_{c,qs}}{L_0} =\ & \bar{\alpha}_0 \left(1 + \frac{\lambda^2}{2}\right) + \lambda \left(\bar{\alpha}_{1S} - \frac{kv}{4}\bar{\alpha}_{1C}\right) + \left[\bar{\alpha}_{1C}\left(1 + \frac{\lambda^2}{4}\right) + \frac{kv}{2}\bar{\alpha}_{1S}\right]\cos\omega_V t \\
& + \left[2\lambda\bar{\alpha}_0 + \bar{\alpha}_{1S}\left(1 + \frac{3\lambda^2}{4}\right) - \frac{kv}{2}\bar{\alpha}_{1C}\right]\sin\omega_V t - \frac{\lambda}{2}\left(\lambda\bar{\alpha}_0 + 2\bar{\alpha}_{1S} - \frac{kv}{2}\bar{\alpha}_{1C}\right)\cos 2\omega_V t \\
& + \frac{\lambda}{2}\left(2\bar{\alpha}_{1C} + \frac{kv}{2}\bar{\alpha}_{1S}\right)\sin 2\omega_V t - \frac{\lambda^2}{4}(\bar{\alpha}_{1C}\cos 3\omega_V t + \bar{\alpha}_{1S}\sin 3\omega_V t)
\end{aligned}
\tag{11}
$$

Comparing the two expressions (the quasisteady result Eq. 11 and the unsteady result Eq. 7), one can see that the mean values are the same in both cases. The dynamic part, however, is different since it includes the lift deficiency function for dynamic pitch in oscillating flow. This consists of the Theodorsen function for the pitch oscillation as well as of Bessel functions for the unsteady velocity effect.

It is interesting whether or not the well known result from Theodorsen for pure angle of attack oscillations about the midchord axis in a steady freestream can be extracted by setting $\lambda = 0$. From the behavior of the Bessel functions, the sum over all m reduces to only the first element, and the same is in effect for the sum over n. Then it can easily be seen that it is identical to Theodorsen's result, as required. Therefore, Isaacs' theory of combined periodic flow and angle of attack oscillations with arbitrary phase angle *between both of these motions* can be considered as the best available theory for attached flow. However, when it comes to practical application, the amount of computational effort involved with the repeated evaluation of Bessel functions places limitations on this theory.

2.3 Generalisation of Isaacs' Theory

Since Isaacs' derivation [7] was made for a fixed pitch axis at midchord, the results are not very useful because in helicopter applications the pitch axis is usually at the quarter chord. Thus, a more general formulation is required where the position of the pitch axis, a, is a varable parameter. Additionally, Isaacs' theory does not include the effect of plunge motion, $h(t)$, although this degree of freedom is very important in helicopter aerodynamics. The recent research by the author [16] includes all degrees of freedom in two dimensions: pitch motion (including higher harmonics) about an arbitrary location of pitch axis *on the chord*, fore-aft motion (1/rev) with velocity amplitudes smaller than the velocity of the freestream itself, plunge motion (including higher harmonics). This extension of Isaacs' theory has not been presented previously, and therefore it is given here for the first time. The complete derivation is very lengthy and is not shown here, but is included in [16].

For the special case of harmonically varying fore-aft motion, angle of attack and plunge motion like

$$
\begin{aligned}
V(t) &= V_0(1 + \lambda\sin\omega_V t) \qquad\qquad |\lambda| < 1 \\
\alpha(t) &= \alpha_0\left(\bar{\alpha}_0 + \sum_{n=1}^{\infty}\bar{\alpha}_{nS}\sin n\omega_V t + \bar{\alpha}_{nC}\cos n\omega_V t\right) \\
h(t) &= \alpha_0\frac{c}{2}\sum_{n=1}^{\infty}(\bar{h}_{nS}\sin n\omega_V t + \bar{h}_{nC}\cos n\omega_V t)
\end{aligned}
\tag{12}
$$

the integral equation can be solved and one gets the following result for the lift

$$
\begin{aligned}
\frac{L_{nc}}{L_0} =\ & \frac{kv}{2}\Biggl\{\left[\lambda\bar{\alpha}_0 + \bar{\alpha}_{1S} + kv(a\bar{\alpha}_{1C} - \bar{h}_{1C}) - \frac{\lambda}{2}\bar{\alpha}_{2C}\right]\cos\psi - \left[\bar{\alpha}_{1C} - kv(a\bar{\alpha}_{1S} - \bar{h}_{1S}) + \frac{\lambda}{2}\bar{\alpha}_{2S}\right]\sin\psi \\
& + \sum_{n=2}^{\infty}n\left[\bar{\alpha}_{nS} + nkv(a\bar{\alpha}_{nC} - \bar{h}_{nC}) + \frac{\lambda}{2}\left(\bar{\alpha}_{(n-1)C} - \bar{\alpha}_{(n+1)C}\right)\right]\cos n\psi \\
& + \sum_{n=2}^{\infty}n\left[-\bar{\alpha}_{nC} + nkv(a\bar{\alpha}_{nS} - \bar{h}_{nS}) + \frac{\lambda}{2}\left(\bar{\alpha}_{(n-1)S} - \bar{\alpha}_{(n+1)S}\right)\right]\sin n\psi\Biggr\}
\end{aligned}
$$

$$
\frac{L_c}{L_0} = \left\{\left(1 + \frac{\lambda^2}{2}\right)\bar{\alpha}_0 + \lambda\left[\bar{\alpha}_{1S} - \frac{kv}{2}\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right) - \frac{\lambda}{4}\bar{\alpha}_{2C}\right]\right\}(1 + \lambda\sin\psi) + \sum_{m=1}^{\infty}(l_m\cos m\psi + l_m'\sin m\psi)
\tag{13}
$$

with $\psi = \omega_V t$. The coefficients l_m, l_m' are built up in the same way as in Eq. 8 and Eq. 9, but the values of H_n and H_n' include the position of axis of rotation a, as well as the amplitude of plunge motion $\bar{h}_{nC}$ and $\bar{h}_{nS}$, and those of pitch in $\bar{\alpha}_{nC}, \bar{\alpha}_{nS}$. In the case of pure $1/rev$ and steady components, the coefficients H_n and H_n' can be written in a form very similar to Isaacs.

$$
\begin{aligned}
H_n &= \frac{J_{n+1}(n\lambda) - J_{n-1}(n\lambda)}{2}\left[\lambda\bar{\alpha}_0 - \bar{\alpha}_{1S} - kv\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right] - \frac{2J_n(n\lambda)}{n\lambda}\bar{\alpha}_{1S} \\
H_n' &= \frac{J_{n+1}(n\lambda) - J_{n-1}(n\lambda)}{n}\bar{\alpha}_{1C} + \frac{J_n(n\lambda)}{\lambda}\left[\bar{\alpha}_{1C}(1 - \lambda^2) - kv\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1S} + \bar{h}_{1S}\right)\right]
\end{aligned}
\tag{14}
$$

This may be used to show the effect of another pitch axis location or plunge motion on the lift development. The full formulation may be used for calculating the unsteady airloads of higher harmonic motion in a $1/rev$ varying freestream.

2.4 Greenberg's Theory

Greenberg extended Theodorsen's theory of harmonic airfoil motion in a constant freestream flow to the case of an additional periodically varying freestream flow conditions [8]. However, he also defines the freestream *velocity to be constant over the chord*, and this implies an unsteady fore-aft motion of the airfoil and not a varying freestream, see [8]. Additionally, Greenberg applies a high frequency assumption to the wake integrals in order to obtain pure periodic wake forms and thus simplifying the derivation. This assumption was never clarified; in the next section it will be shown to be a small λ approximation for parts of the derivation. With the coefficients f_{1S} and f_{1C} as defined before, and

$$f_{2S} = F(2kv)\bar{\alpha}_{1S} - G(2kv)\bar{\alpha}_{1C} \qquad\qquad f_{2C} = F(2kv)\bar{\alpha}_{1C} + G(2kv)\bar{\alpha}_{1S} \tag{15}$$

the result of Greenberg, written in terms of a Fourier series, is

$$\frac{L_{nc}}{L_0} = \frac{kv}{2}\left\{\left[\lambda\bar{\alpha}_0 + \bar{\alpha}_{1S} + kv(a\bar{\alpha}_{1C} - \bar{h}_{1C})\right]\cos\omega v t + \lambda\bar{\alpha}_{1C}\cos 2\omega v t + \left[-\bar{\alpha}_{1C} + kv(a\bar{\alpha}_{1S} - \bar{h}_{1S})\right]\sin\omega v t + \lambda\bar{\alpha}_{1S}\sin 2\omega v t\right\}$$

$$\begin{aligned}
\frac{L_c}{L_0} = {}& \bar{\alpha}_0\left[1 + \frac{\lambda^2}{2}F(kv)\right] + \frac{\lambda}{2}\left[f_{1S} + \bar{\alpha}_{1S}\right] + \left[\lambda\bar{\alpha}_0 G(kv) + f_{1C} + \frac{\lambda^2}{4}f_{2C}\right]\cos\omega v t \\
& + \left[\lambda\bar{\alpha}_0[1 + F(kv)] + f_{1S} + \frac{\lambda^2}{4}f_{2S} + \frac{\lambda^2}{2}\bar{\alpha}_{1S}\right]\sin\omega v t - \frac{\lambda}{2}\left[\lambda\bar{\alpha}_0 F(kv) + f_{1S} + f_{2S}\right]\cos 2\omega v t \\
& + \frac{\lambda}{2}\left[\lambda\bar{\alpha}_0 G(kv) + f_{1C} + f_{2C}\right]\sin 2\omega v t - \frac{\lambda^2}{4}\left(f_{2C}\cos 3\omega v t + f_{2S}\sin 3\omega v t\right)
\end{aligned} \tag{16}$$

2.5 Kottapalli's Theory

Kottapalli [9] also assumed the instantaneous velocity distribution along the chord as constant. The additional restriction of small oscillation amplitudes of this lead-lag motion omits all terms of higher order in λ and limits the applicability of this theory to the case of a hovering rotor, or one at low advance ratios in forward flight. Since the noncirculatory part is the same as in Isaacs' or Greenberg's result, it is not considered here. The result in form of a Fourier series is

$$\begin{aligned}
\frac{L_c}{L_0} = {}& \bar{\alpha}_0 + \lambda\left[\bar{\alpha}_{1S} - \frac{kv}{2}\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right)\right] + \left[\lambda\bar{\alpha}_0 G(kv) + f_{1C}\right]\cos\omega t + \left[\lambda\bar{\alpha}_0(1 + F(kv)) + f_{1S}\right]\sin\omega t \\
& - \lambda\left[\frac{kv}{2}f_{3C} + f_{1S}\right]\cos 2\omega t - \lambda\left[\frac{kv}{2}f_{3S} - f_{1C}\right]\sin 2\omega t
\end{aligned} \tag{17}$$

with the coefficients f_{1S}, f_{1C} like defined before and

$$\begin{aligned}
f_{3S} &= F(2kv)\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1S} + \bar{h}_{1S}\right) - G(2kv)\left(\frac{1-2a}{2}\bar{\alpha}_{1C} + \bar{h}_{1C}\right) \\
f_{3C} &= F(2kv)\left(\left(\frac{1-2a}{2}\right)\bar{\alpha}_{1C} + \bar{h}_{1C}\right) + G(2kv)\left(\frac{1-2a}{2}\bar{\alpha}_{1S} + \bar{h}_{1S}\right)
\end{aligned} \tag{18}$$

Immediately one can see that Kottapalli's derivation includes only two harmonics in contrast to three harmonics even in quasisteady theory. Here, the assumption of small flow oscillation amplitudes is responsible since all terms of higher order in λ are missing and the $3/rev$ was multiplied by λ^2 in the quasisteady, Theodorsen's and Greenberg's theories.

2.6 Arbitrary Motion Theory in an Unsteady Freestream (AMT)

After investigating the various thin airfoil theories that are all set up for harmonic motion of the airfoil or the freestream, it is of utmost interest, whether or not the theory of arbitrary motion will lead to the same results as the exact theory in the case of an unsteady freestream. This method is based on the superposition principle and the use of Duhamel's integral in combination with the indicial response of lift (or moment) due to a sudden change in any of the degrees of freedom. This method has been described several times, for example in [31, 32, 33].

In incompressible flow the circulatory lift is determined from the normal velocity at 3/4 chord of the airfoil, while the noncirculatory lift is the result of the instantaneous local accelerations. Thus, the total lift is

$$L = \pi\rho\frac{c^2}{4}\left[\ddot{h}(t) + V(t)\dot{\alpha}(t) + \dot{V}(t)\alpha(t) - a\frac{c}{2}\ddot{\alpha}(t)\right] + 2\pi\rho V(t)\frac{c}{2}\left[w_{3/4}(0)\phi(s) + \int_0^s \frac{dw_{3/4}(\sigma)}{d\sigma}\phi(s-\sigma)d\sigma\right] \tag{19}$$

where $\phi(s)$ is Wagner's deficiency function for the lift [34], s the distance travelled by the airfoil (in half chords) and $w_{3/4}(t)$ the instantaneous value of normal velocities at the three quarter chord point. The normal velocity depends on the angle of attack $\alpha(t)$, the flap or plunge motion $h(t)$, the position of the pitch axis $ac/2$, and the time-dependent velocity $V(t)$. This velocity may originate from freestream variations or lead-lag motion of the airfoil or a combination of both. However, it is assumed here to depend on time only, so the velocity distribution along the chord is the same everywhere. This is done in order to compare results of arbitrary motion theory with those of the other theories discussed so far. Thus, the normal velocity at the three quarter chord is

$$w_{3/4}(t) = V(t)\alpha(t) + \dot{h}(t) + \frac{c}{2}\left(\frac{1-2a}{2}\right)\dot{\alpha}(t) \tag{20}$$

There are two approaches that can be taken. First, for a given forcing function one can analytically integrate to obtain a closed form solution; second, one can let the type of motion be prescribed and apply a finite difference method. Here only the second

approach is used; results obtained with the first approach are given in [16]. Duhamel's integral yields for the circulatory part of the lift

$$L_c = 2\pi\frac{\rho}{2}V(t)c\left[w_{3/4}(0)\phi(s) + \int_0^s \frac{dw_{3/4}(\sigma)}{d\sigma}\phi(s-\sigma)d\sigma\right] = 2\pi\frac{\rho}{2}V(t)c\,w_{3/4,eff} \tag{21}$$

and the normal velocity at 3/4 chord is written as

$$w_{3/4}(t) = V(t)\alpha(t) + \dot{h}(t) + \frac{c}{2}\left(\frac{1-2a}{2}\right)\dot{\alpha}(t) \tag{22}$$

Now the derivative $dw_{3/4}(\sigma)/d\sigma$ is

$$\frac{dw_{3/4}(\sigma)}{d\sigma} = \frac{dV(\sigma)}{d\sigma}\alpha(\sigma) + V(\sigma)\frac{d\alpha(\sigma)}{d\sigma} + \frac{d\dot{h}(\sigma)}{d\sigma} + \frac{c}{2}\left(\frac{1-2a}{2}\right)\frac{d\dot{\alpha}(\sigma)}{d\sigma} \qquad \tag{23}$$

The method of finite differences introduces the calculation at different time steps with a stepwidth being rather small relative to the highest frequency encountered. Therefore, normally about 45 to 60 steps are made within one cycle. However, this implies the use of some mechanism to describe the state between the time steps, and this is usually done by a zero order hold. By this a finite difference approximation can be made for the integrals, when using one of the common exponential series approximations for the Wagner function.

$$\phi(s) = 1 + \sum_{k=1}^{N} A_k e^{b_k s} \tag{24}$$

Then, for the sample with index n being the current sample, the expression in the brackets in Eq. 21 for the effective normal velocity at 3/4 chord becomes $w_{3/4,eff} = w_{3/4,n}$.

$$w_{3/4,n} = \sum_{i=0}^{n}\left[V_i\Delta\alpha_i + \alpha_i\Delta V_i + \frac{c}{2}\left(\frac{1-2a}{2}\right)\Delta\dot{\alpha}_i + \Delta\dot{h}_i\right] - \sum_{j=1}^{4}\sum_{k=2}^{N} X_{n,k}^{(j)} \tag{25}$$

Herein, the X are called deficiency functions and contain the information of the time history of the different degrees of freedom. They are [33]

$$X_{n,k}^{(j)} = X_{n-1,k}^{(j)}e^{b_k\Delta s} + A_k\Delta^{(j)}e^{b_k\Delta s/2} \tag{26}$$

and can be combined in order to reduce the computational effort. The values A_k and b_k are those of the usual approximation to the Wagner function; for example Jones approximation [35]. If a higher order approximation is used, such as that of [36, 37], than additional deficiency functions are added, as indicated by the upper limit N. This is not usually desirable, since more terms lead to additional computational effort without leading to any significant gains in the accuracy of the results. One has to note that $4N$ deficiency functions have to be computed (or N, if all $\Delta^{(j)}$ are put together), and therefore for practical applications one must keep N as small as possible. The values denoted by $\Delta^{(j)}$ are the differential changes of the four derivatives in the current sample [33], i.e.,

$$\Delta^{(1)} = V_n\Delta\alpha_n \qquad \Delta^{(2)} = \Delta V_n\alpha_n \qquad \Delta^{(3)} = \frac{c}{2}\left(\frac{1-2a}{2}\right)\Delta\dot{\alpha}_n \qquad \Delta^{(4)} = \Delta\dot{h}_n \tag{27}$$

and the increment in the distance travelled by the airfoil Δs is

$$\Delta s = \frac{2}{c}\int_t^{t+\Delta t} V(t)\,dt = \left(\frac{V_n + V_{n-1}}{c}\right)\Delta t \tag{28}$$

The total response of lift due to arbitrary motion of the airfoil can be calculated by updating the deficiency functions at each sample.

$$\frac{L_{c,n}}{L_0} = \frac{V_n}{V_0}\frac{w_{3/4,n}}{V_0\alpha_0} \tag{29}$$

When this approach is applied to a constant freestream, Theodorsen's result can be reproduced to an accuracy depending on the coefficients of the indicial function ϕ. In this case $\lambda = 0$ and $\Delta s = (2V/c)\Delta t = \Delta\psi/k_V$ with $\psi = \omega t = k_V\bar{s}$ being the rotor azimuth.

This approach now can be applied to any type of airfoil motion, for example harmonic motion. This will now be the subject of later investigation. In all the cases presented, the number of steps in one cycle was set to 64. This is somewhat high, and therefore is on the conservative side. So here space steps are used instead of time steps, and therefore no difficulties occur when it comes to high frequencies where a time spacing leads to fewer steps within one cycle than at lower frequencies. It must be noted, that compressibility effects can also be implemented as was shown by [33, 38].

3 Results and Discussion

3.1 Lift Transfer Function for Constant Angle of Attack

The equations presented previously are not very helpful for a physical understanding of the problem, since there will be a response with a whole range of frequencies to the input of only one frequency in $V(t)$. Since the lift is proportional to the square of the velocity, the input consists of steady, 1/rev and 2/rev parts, and the output will mainly consist of these harmonics, including some phase lag effects. The circulatory lift coefficient, based on the instantaneous dynamic pressure, is far from

uniform, as predicted by quasisteady theory, and this is shown in Fig. 2 for a reduced frequency of $k_V = 0.2$ with $\lambda = 0, ..., 0.8$ in steps of 0.2. The results of Isaacs theory were calculated by including terms up to the 20*th* harmonic, and for each harmonic up to the 25*th* order in the reduced frequency and in the freestream oscillation amplitude λ. It is required to include as many terms as necessary to show the correct solution. The higher order terms become smaller and approach zero because of the factor n^2 in the denominator of Eq. 8, and because of the behavior of the Bessel functions for large arguments. For larger values of λ, even more terms must be used to obtain a converged solution.

These results show the typical effects of unsteady aerodynamics already known from constant freestream theory. First, there is a phase lag resulting in a lag in the lift buildup with respect to the change in velocity. Second, there is an effect on the circulatory lift amplitude resulting in a smaller value of maximum lift (where the velocity is at maximum) and more lift in the regime where the velocity is a minimum. Both quasisteady and Theodorsen's theory give the same result for a constant angle of attack and lead to a lift coefficient ratio of 1 independent of λ or k_V. A step in the right direction is given by Greenberg's theory, but here the lift in the area of high velocity is significantly underpredicted (C_{Lc} has to be multiplied with V^2 to compute the lift. V_{max} is at $\Psi = 90^\circ$ so small differences in C_{Lc} lead to large differences in the lift here). In the area of smallest velocity, the lift calculated by Greenberg's theory is smaller than that obtained by Isaacs. This means that the wake effects are not well represented in this theory. The results of Kottapalli's theory, derived for small values of λ, show acceptable agreement only for small λ as expected. Here $\lambda = 0.2$ seems to be a limit for application. Special attention has to be given to the AMT results: they are so close to the exact solution of Isaacs that there are negligible differences. The only difference depends on the quality of approximation to the Wagner function; here the coefficients given by Jones [35] were used. Thus, the AMT is not only a very fast algorithm, but also the most accurate way to predict the unsteady aerodynamic coefficients at constant angle of attack.

3.2   Lift Transfer Function for Sinusoidal Pitch Oscillations

The angle of attack is assumed to consist only of its sinusoidal part, say $\bar{\alpha}_0 = \bar{\alpha}_{1C} = 0$ and $\bar{\alpha}_{1S} = 1$. The lift response is shown in the time domain in Fig. 3. Two interesting observations can be made:

1. At the maximum velocity ($\Psi = 90^\circ$), the unsteady lift for large freestream amplitudes is between the results obtained with quasisteady and with Theodorsen's theory, with a small phase lag. The lift amplitude reduction is not as large as Theodorsen's theory would predict.

2. At the minimum velocity ($\Psi = 270^\circ$), the unsteady lift for high freestream amplitudes is closer to zero as in the quasisteady case or in Theodorsen's theory. This can be seen very clearly in the lift coefficient, for example at $\lambda = 0.8$.

The reason for this surprising behavior is due to the effect of stretching and compressing the shed wake vorticity, respectively. The stretching leads to a smaller effective reduced frequency, while the compression leads to larger effective reduced frequencies with a more significant reduction of circulatory lift. This observation is in agreement with Johnson's results [3].

It is interesting to note that in the region of high velocity the lift is significantly underpredicted by Greenberg's theory. This means that the effective reduced frequency is too high here, leading to a lift deficiency that is also too large. In the region of lowest velocity, the additional loss in lift is not completely predicted by Greenberg's theory, so here the effective reduced frequency is too small, leading to more lift than predicted by the exact theory of Isaacs. Over all, it can be seen that the mean lift will be underpredicted with increasing λ so that the statement made by Greenberg of "good agreement with Isaacs' theory" in [8] is not necessarily correct. While in Isaacs' theory the constant part of the lift is directly proportional to $\lambda\bar{\alpha}_{1S}$, in Greenberg's formulation the constant part of the lift depends on the Theodorsen function and is proportional to $0.5\lambda\bar{\alpha}_{1S}[1 + F(k_V) - 0.5k_V G(k_V)]$, see Eq. 16. Therefore, the final value for high reduced frequencies is only 0.75 of that of Isaacs' theory.

Much better agreement than at constant angle of attack is found between Kottapalli's and Isaacs' theory in the range of flow oscillation amplitudes up to $\lambda = 0.2$. It can be seen that the additional lift loss in the low velocity region is overpredicted by Kottapalli's theory, but the lift in the high velocity region is underpredicted with increasing λ. The mean value, however, is the same as for Isaacs' theory, since it is proportional to $\lambda\bar{\alpha}_{1S}$ and does not depend on the reduced frequency (unlike Greenberg's result). From these results, again, the observation can be made that Kottapalli's theory is useful only for small values of λ.

The AMT represents the unsteady lift behavior in an almost perfect manner. The behavior of the lift coefficient in the region of smallest velocity is correct in the trend, but not completely correct in magnitude. Especially for larger values of λ the mean lift is slightly smaller than that of Isaacs. This is likely due to the Jones' approximation to the Wagner function.

3.3   Lift Transfer Function for Cosine Pitch Oscillations

Now $\bar{\alpha}_0 = \bar{\alpha}_{1S} = 0$ and $\bar{\alpha}_{1C} = 1$ so the pitch variation is 90° out of phase with the freestream variation. From the time domain response, shown in Fig. 4, the following can be observed:

1. As for sinusoidal motion, the unsteady lift response of Isaacs theory is between the quasisteady result than the result obtained with Theodorsen's theory. This is because the stretching of the shed wake vorticity leads to a smaller effective reduced frequency, where the velocity is a maximum.

2. In the region with lowest velocity, a lift overshoot occurs. This is in contrast to the sinusoidal pitch motion where the lift deficiency function shows a reduction in lift.

It is evident, that the combination of Theodorsen's theory with an unsteady freestream cannot be used to predict the lift coefficient. However, since the total velocity is small here, the difference in lift is not very significant.

From Greenberg's result it can be seen that the overall agreement with Isaacs' theory is good for this case, and the lift overshoot in the decelerating flow region is also predicted in the correct trend, but not in magnitude.

The differences between Kottapalli's and Isaacs' theory are small up to values of $\lambda = 0.2$. For higher amplitudes, the lift is increasingly underpredicted in the region of high velocity while it is overpredicted in the smaller velocity region.

No significant differences can be seen in the lift development between the results obtained by AMT and Isaacs. Thus, for all three cases of constant, in-phase and out-of-phase pitch oscillations, AMT is the best available theory to represent the results of the exact theory in an easy manner.

3.4 AMT - Reduced Algorithm

Often, instead of using the full algorithm with all deficiency functions, only a reduced algorithm is used, viewing the changes in freestream velocity as quasisteady and thus neglecting the deficiency terms related to V. It was shown in [16] that this reduced algorithm leads to acceptable results in the lift, but not in the lift coefficient. Additionally, using an analytic derivation of Eq. 19 and replacing the upper limit of the integral, $s = \bar{s} - (\lambda/kv)coskv\bar{s}$, by its mean value, $\bar{s}$, it has been shown to identically reproduce Greenberg's results. Thus, *the high frequency assumption for the wake integrals in Greenberg's theory really means a small λ approximation for parts of the wake.* This is generally not applicable in rotorcraft calculation in forward flight.

3.5 Comparison with Euler Results

A comparison of the results obtained with Isaacs theory and with an Euler code developped at DLR for constant angle of attack at a reduced frequency of $kv = 0.2$ is shown in Fig. 5. Since the Euler code cannot compute the incompressible case, the mean Mach number has been set to 0.1 with variations of up to 80%. Excellent agreement is found and the very small differences between these two results can be neglected. It must be noted, that the computing time of the Euler code is several orders in magnitude larger than that of the analytical expression of Isaacs and again this approach is much more computational intensive than the formulation via AMT. Therefore, AMT is the most reliable *and* the fastest way to calculate the unsteady aerodynamic coefficients in unsteady freestream flow environment.

4 Summary and Conclusions

In this study five theories handling the effect of unsteady freestream have been analysed. These are: Isaacs' theory, Greenberg's theory, Theodorsen's theory combined with unsteady freestream, Kottapalli's theory and the arbitrary motion theory (AMT). It was found, that *all* of these theories handle the case of a fore-aft moving airfoil instead of an unsteady freestream. This latter case should be more correcly viewed as a system of horizontally propagating gusts. A helicopter rotor blade section in forward flight encounters both unsteady freestream (the superposistion of rotation and forward flight velocity components) and fore-aft motion (through lead-lag). It was found, that in the range of reduced frequencies encountered by a helicopter blade the results will be very similar. Thus, the interpretation of unsteady freestream as an equivalent to fore-aft motion can be viewed as a good approximation in the helicopter case. All of the theories cited above lead to the same noncirculatory expressions, and all of them reduce to Theodorsen's theory when the freestream oscillation amplitude becomes zero. The general effect of an oscillating freestream is a "stretching and compressing" of the shed wake vorticity behind the airfoil. From the analysis and comparisons in this paper the following conclusions can be made:

1) Isaacs' theory is the only theory that gives an analytic solution without additional simplifications, and therefore can be considered as the only "exact theory". The lift for oscillating freestream flow conditions is represented as an infinite Fourier series. The induced phase lags and amplifications depend on the type of motion of the airfoil. Therefore, at constant angle of attack there is a significant lift coefficient overshoot, where the velocity is smallest, but in case of sinusoidally varying angle of attack (in-phase motion) an additional lift deficiency occurs. A cosine motion ($90°$ out-of-phase) also leads to lift coefficient overshoots, but they are not as significant as in the case of constant angle of attack.

2) Greenberg's theory is similar to Theodorsen's theory, but includes the unsteady freestream as additional degree of freedom and the result for the lift contains up to three harmonics. To obtain a simple closed form solution, an additional simplification to the form of the wake was made. That was that an infinite frequency assumption makes the wake vorticity sinusoidal again. It was shown with an analytical derivation via arbitrary motion theory, that this is equivalent to neglecting the flow oscillation amplitude for the induced velocities. Therefore Greenberg's high frequency assumption physically is an assumption of quasisteady convection velocity for the shed wake. This makes Greenberg's theory questionable for high freestream oscillation amplitudes, and it was found that the differences with the exact theory of Isaacs are significant above $\lambda \approx 0.4$. For constant or oscillating angle of attack the basic behavior was correctly represented, but the magnitudes and phase angles were not well represented in the important constant and $1/rev$ parts of lift response.

3) Kottapalli's theory uses an assumption for small freestream amplitudes and thus reduces this theory for the cases of aeroelastic investigations in hover, or very small forward flight conditions. The agreement with Isaacs' theory for that range of freestream oscillations was found to be slightly better than that of Greenberg's results. Because of the assumption made in Kottapalli's theory, only up to the second harmonics describe the lift response.

4) Theodorsen's theory combined with an unsteady freestream essentially can be viewed as quasisteady changes in velocity and the Theodorsen function is only applied to angle of attack and plunge motion. The characteristic lift coefficient overshoots cannot be predicted by this method. It was proved that with an analytical derivation via arbitrary motion theory from the reduced algorithm (omitting the deficiency functions for the changes in velocity), that this is equivalent to neglecting the flow oscillation amplitude for the induced velocities.

5) Arbitrary motion theory (AMT): the finite difference approach using the superposition principle and Duhamel's integral leads nearly exatly to the same results as for Isaacs' theory, when the angle of attack is constant or oscillating $90°$ out-of-phase. For sinusoidal angle of attack motion (in-phase) there are increasing differences with increasing reduced frequencies for the constant and $1/rev$-part of the lift response. In the range of reduced frequencies encountered by a rotor blade, this seems not to be a severe limitation. In all cases the dynamic lift response is represented correctly, depending on the approximation used for the Wagner function. This is proof that the arbitrary motion theory can accurately calculate the lift even in unsteady freestream

conditions. The often used "reduced algorithm", considering the freestream variations as quasisteady, leads to good results for the lift, but the characteristic overshoots in the lift coefficient related to the compression of the shed wake vorticity (at the retreating side of the rotor), are not represented.

The conclusion is, that when the lift coefficient is the subject of investigation, Isaacs' theory or the arbitrary motion theory with all the appropriate deficiency functions are necessary to calculate the correct lift coefficient overshoots or deficiencies. If the lift itself is the subject, then for small freestream amplitudes all theories are useful, for medium amplitudes Isaacs, Greenberg's and AMT are valid, and for high oscillation amplitudes Isaacs' or arbitrary motion theory with all deficiency functions are necessary to accurately calculate the lift response.

As an additional contribution to the analytical side of the problem, Isaacs' theory (that was derived for $1/rev$ oscillations in angle of attack only about midchord) has been generalized to the case of an infinite Fourier series in angle of attack about an arbitrary axis, including also an infinite Fourier series for plunge motion. As a recommendation for future research, this derivation can be used for a general unsteady aerodynamic theory, featuring infinite Fourier series in all types of motion (also fore-aft motion) and with different fundamental frequencies for pitch, plunge and freestream oscillations.

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Legend to Fig. 2-5:

$$V = V_0(1+\lambda\sin \omega_V t)$$
$$\lambda = 0.0, \ 0.2, \ 0.4, \ 0.6, \ 0.8$$
Fig. 2+5: $\alpha = \alpha_0 = $ const.
Fig. 3: $\alpha = \alpha_s \sin \omega_V t$
Fig. 4: $\alpha = \alpha_c \cos \omega_V t$
$k_V = 0.20$

Isaacs ————————

Quasisteady — — — — — — —

Greenberg — — — — — — —

Theodorsen —— —— —— ——

Kottapalli — · — — · — —

AMT (fin. diff.) · — · · · · — · ·

$$V(t) = V_0 \left(1 + \lambda \ \sin \left[\omega_V(t - t_0)\right]\right)$$

$$V(t) = V_0 \left(1 + \lambda \ \sin \left[\omega_V(t - t_0) + k_V(s_0 - x)\right]\right)$$
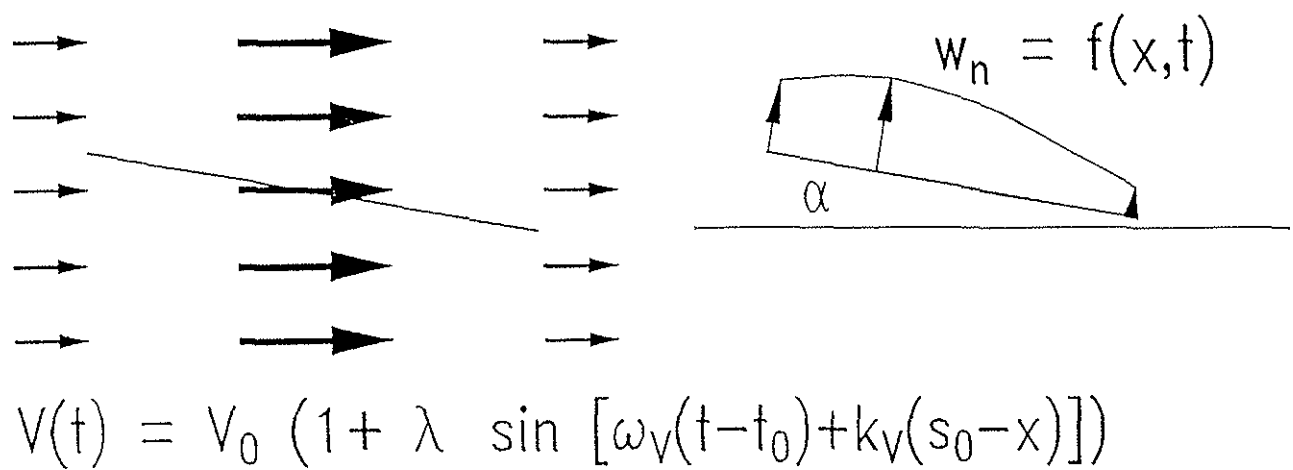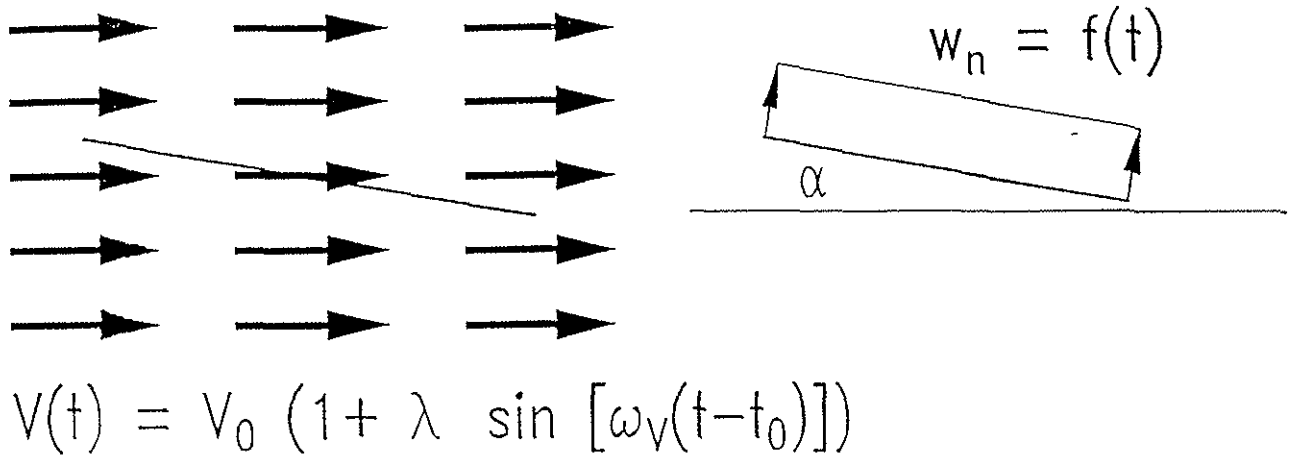
Figure 1: Flow environment of an airfoil in an unsteady freestream. Upper part: lead-lag type of motion; lower part: unsteady freestream as a gust problem. Right side: resulting normal velocity distributions.
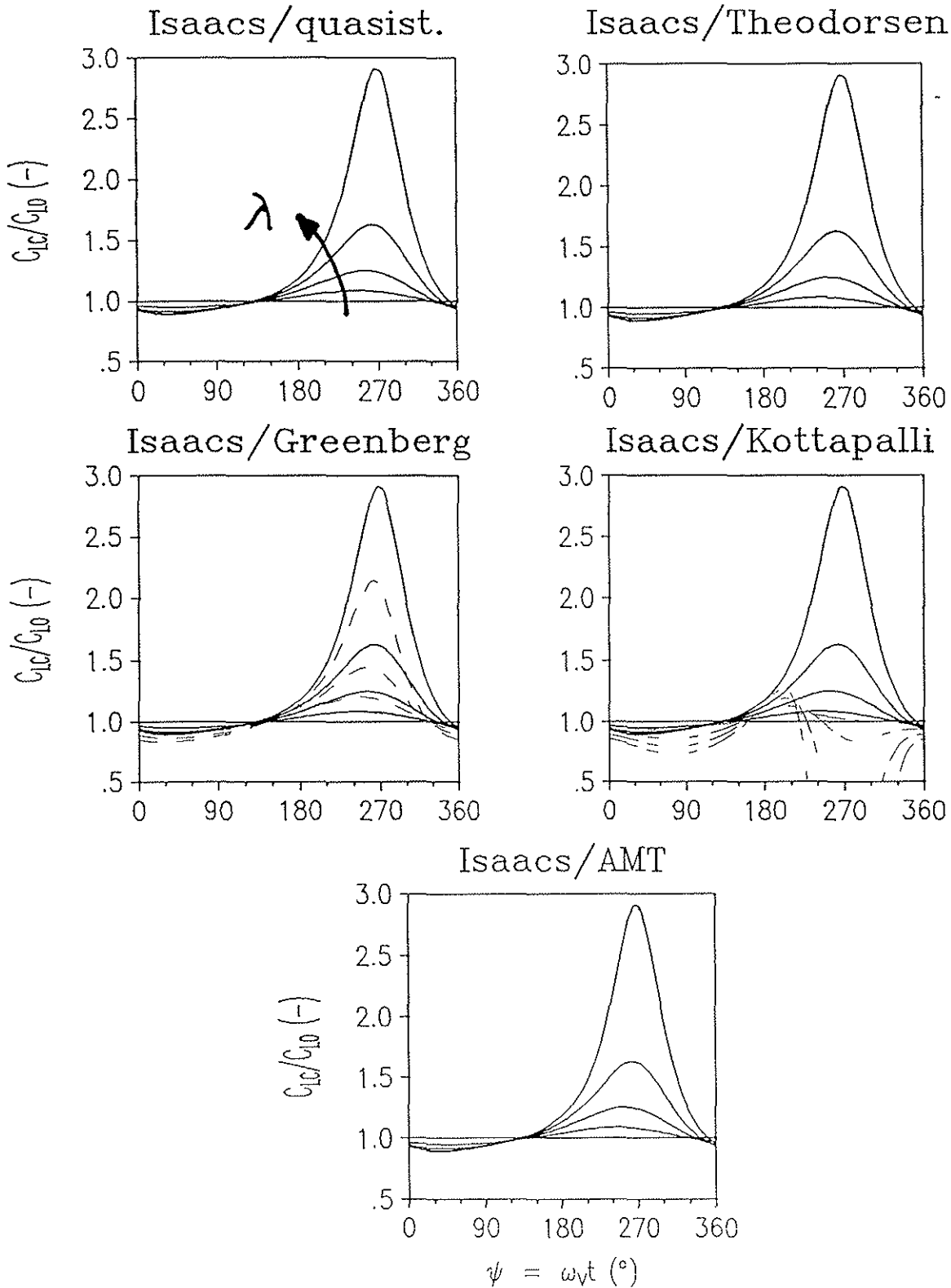
Circulatory lift coefficient

Isaacs/quasist.

Isaacs/Theodorsen

Isaacs/Greenberg

Isaacs/Kottapalli

Isaacs/AMT

Figure 2: Unsteady circulatory lift development for constant angle of attack in an oscillating flow, $k_V = 0.2$, $\lambda = 0...0.8$.

Circulatory lift coefficient

Isaacs/quasist.

Isaacs/Theodorsen

Isaacs/Greenberg

Isaacs/Kottapalli

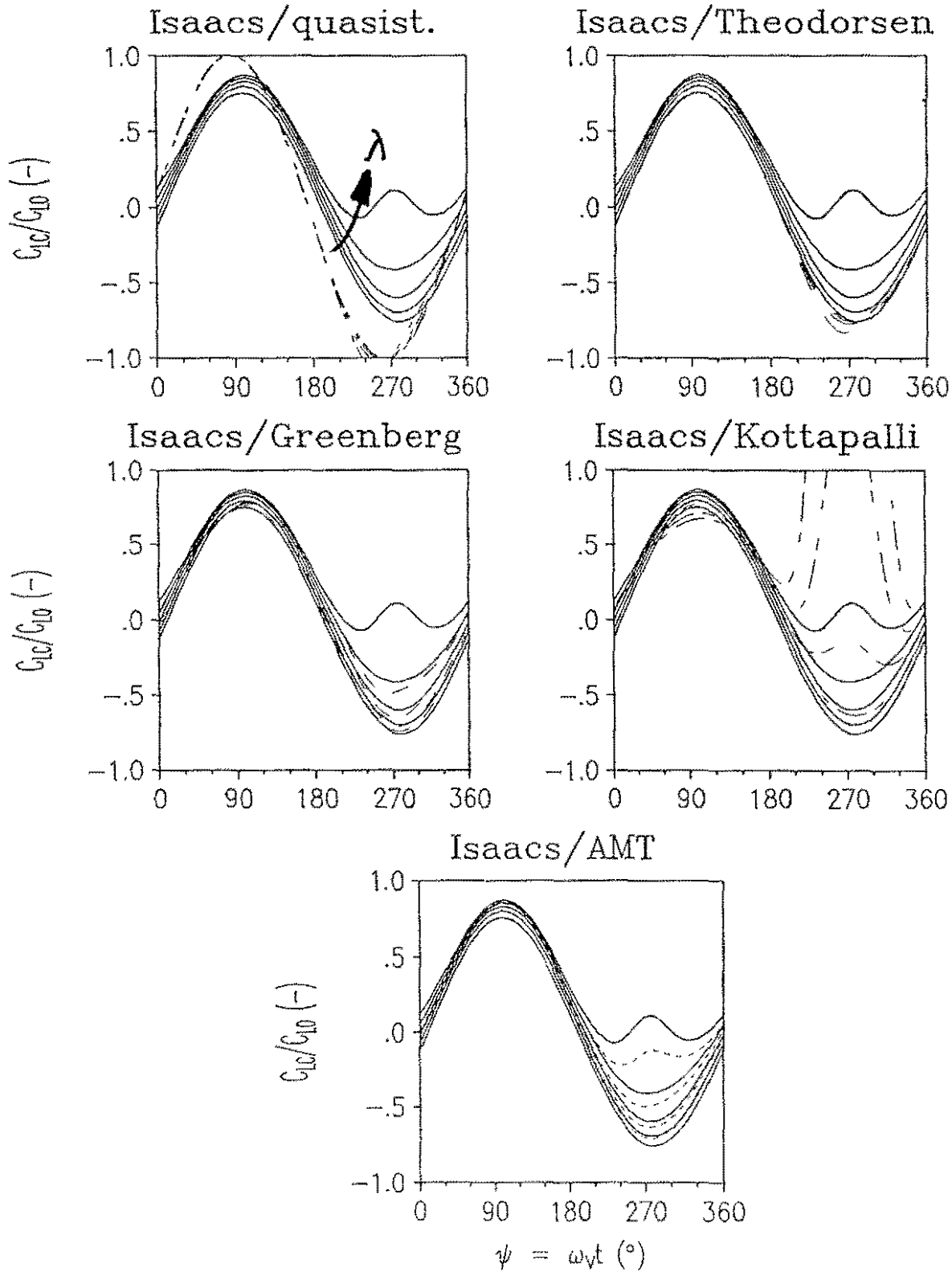### Isaacs/AMT

$\psi = \omega_v t \; (°)$

Figure 3: Unsteady circulatory lift development for in-phase oscillating angle of attack in an oscillating flow. $k_v = 0.2$, $\lambda = 0..0.8$.
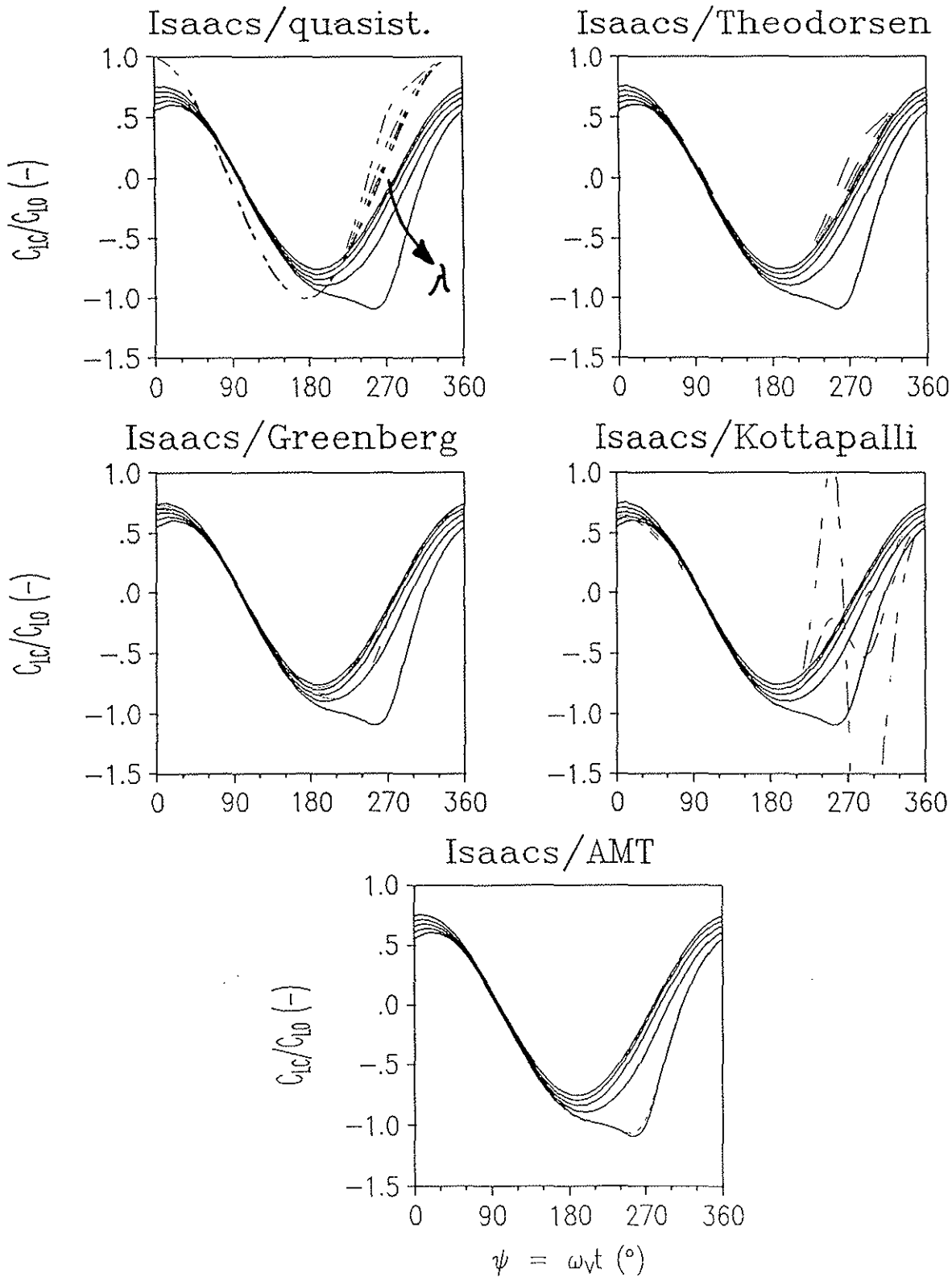
Circulatory lift coefficient

Isaacs/quasist.

Isaacs/Theodorsen

Isaacs/Greenberg

Isaacs/Kottapalli

Isaacs/AMT

$$\psi = \omega_V t \ (°)$$

Figure 4: Unsteady circulatory lift development for out-of-phase oscillating angle of attack in an oscillating flow, $k_V = 0.2$, $\lambda = 0...0.8$.
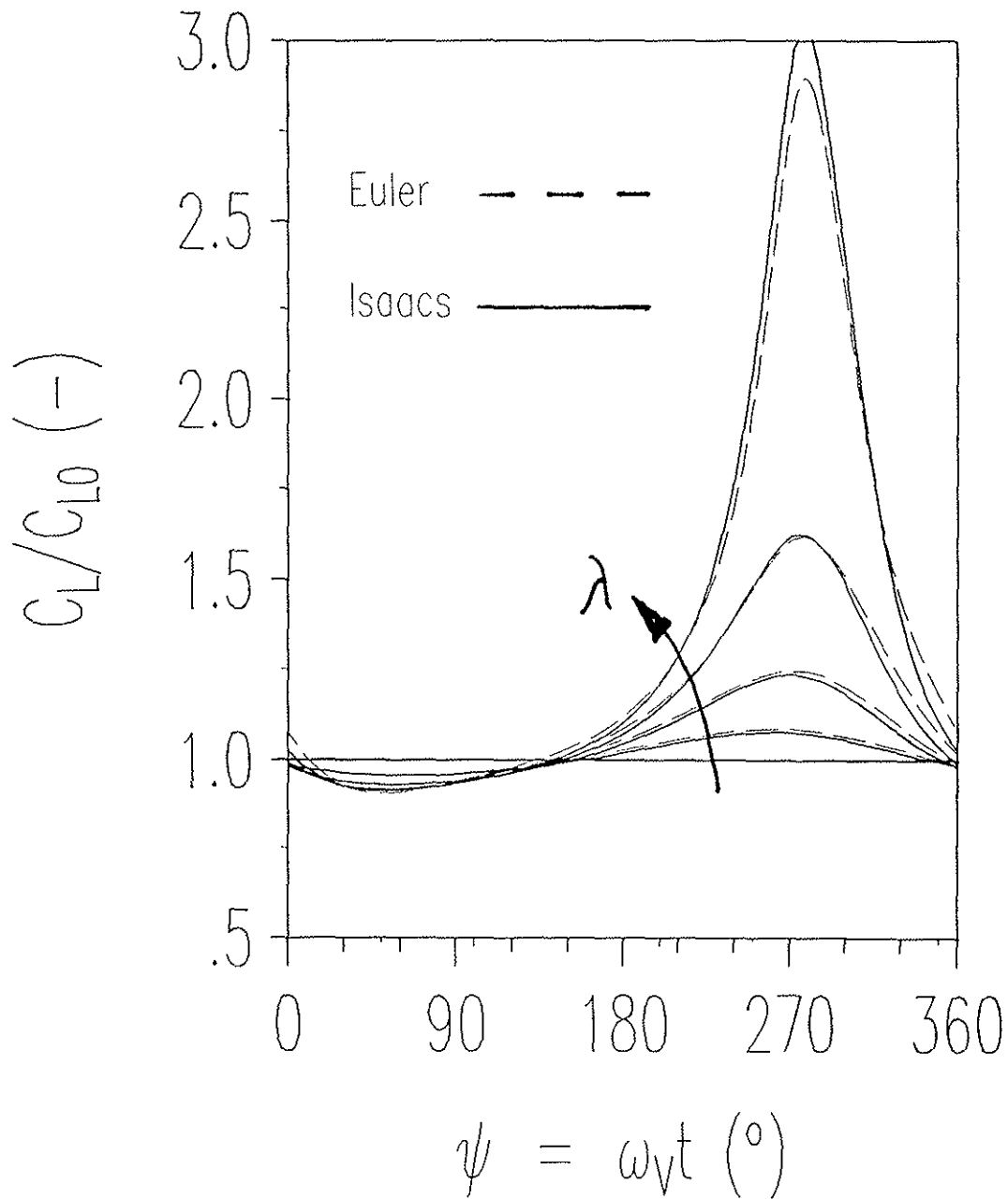
Lift Coefficient

Figure 5: Comparison of Isaacs' theory with Euler results for constant angle of attack in an oscillating flow, $k_V = 0.2$, $\lambda = 0...0.8$.