OPTIMAL SECOND HARMONIC PITCH CONTROL FOR MINIMUM OSCILLATORY BLADE LIFT LOADS

by

L. Beiner, Visiting Scientist
Lehrstuhl für Flugmechanik und Flugregelung
Technische Universität München, Munich, Germany

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Abstract

The paper presents analytical solutions to the problem of finding the optimal 2/rev blade pitch control which minimizes the amplitude of the second blade lift harmonic for a given hinged rotor under a given flight condition. It is shown that this minimum is zero, thus suppressing the 2/rev hub axial force for a two-bladed rotor. The analysis assumes constant inflow ratio and blade lift slope, linear twist, and second harmonic flapping. Numerical examples were carried out with parameters covering the following ranges: μ = 0.1-0.3, C_T/σ = 0.06-0.10, γ = 5-15, θ_1 = 6-10 deg. and $X/qd^2\sigma$ = 0.08-0.12. Results indicate that up to 50% reduction of the peak-to-peak amplitude of total alternating blade lift (2P content canceled, 3P content strongly reduced) can be obtained with less than 1.5 deg of 2/rev blade pitch amplitude, in agreement with experimental results. The amplitude and phase angle of optimal 2/rev blade pitch increases, respectively decreases, with μ, C_T/σ, γ, and $X/qd^2\sigma$, while both decrease with θ_1.

Notation

a	blade lift-curve slope
a_0	rotor coning angle
a_n	coefficient of $\cos n\psi$ in expression for β
A_n	coefficient of $\cos n\psi$ in expression for θ
b	number of blades
b_n	coefficient of $\sin n\psi$ in expression for β
B_n	coefficient of $\sin n\psi$ in expression for θ
c	blade chord
C_T	rotor thrust coefficient, $T/\rho\pi R^2(\Omega R)^2$
d	rotor diameter
J	performance index
q	dynamic pressure
r	blade-element radius
R	rotor radius
t	total alternating blade lift
T	rotor thrust

†) "MINERVA"-Fellowship Visiting Scientist, on leave from the Mechanical Engineering Department, Ben Gurion University, Beer Sheva, Israel.

T_ψ blade thrust at azimuth

v induced velocity

V forward flight speed

x nondimensional radial coordinate, r/R

X rotor propulsive force

X_{nP} coefficient of $\cos n\psi$ in expression for T_ψ

Y_{nP} coefficient for $\sin n\psi$ in expression for

Z_{nP} amplitude of n-th harmonic blade lift

Z_t peak-to-peak amplitude of total alternating blade lift t

α_D rotor-disk angle of attack

α_r blade-element angle of attack

β blade flapping angle

γ blade inertia number, $\rho a c R^4/I$

ΔZ_t percentual reduction of Z_t, $[(Z_t - Z_{t_{opt}})/Z_t]\cdot 100$, %

$\Delta\phi$ phase angle of optimal 2/rev blade pitch

λ inflow ratio, $(V\sin\alpha_D - v)/\Omega R$

μ advance ratio, $V\cos\alpha_D/\Omega R$

Ω rotor angular velocity

ψ blade azimuth angle

ρ air mass density

σ rotor solidity, $bc/\pi R$

θ blade pitch angle

θ_0 collective pitch angle

θ_1 blade twist angle, positive when tip angle is smaller than root angle

θ_{2P} amplitude of optimal 2/rev blade pitch

1. Introduction

The application of a second harmonic cyclic pitch to the blades of a hingrd rotor as a mean to redistribute the loading over the disk and postpone the stalling of the retreating blade was firstly suggested and analyzed by Stewart (Ref.1). Early flight test results for 2P control of a two-bladed teetering rotor were described in Ref.2, while more recent wind-tunnel test results reported by McHugh and Shaw in Ref.3 show that 2/rev blade pitch applied to a two-bladed hingeless rotor can suppress 2P shaft axial force.

In this paper, Stewart's approach of Ref.1 is extended in the optimization sense and by including also the blade twist effect. Analytical expressions for the optimal 2/rev pitch control required to null the 2P hub axial force of a two-bladed hinged rotor are derived, allowing to discern the influence of various parameters. A comparison with experimental results is performed.

2. Review of Assumptions

The present analysis is based on the following assumptions:

(1) Constant inflow ratio
(2) Constant blade lift-curve slope
(3) Linear blade twist
(4) Untapered blades with zero flapping hinge offset
(5) Neglection of reversed flow and tip losses
(6) Second harmonic flapping
(7) Neglection of powers of μ^4 and higher

3. Rotor Dynamic Analysis

The investigation is carried out in tip-path plane axes in order to allow for deviations from the mean tip-path plane due to the second harmonic flapping motion. With $\mu\Omega R$ and $\lambda\Omega R$ being the components of the forward velocity parallel and perpendicular to the mean tip-path plane, respectively, the velocities at a blade element are as follows:

- parallel to the mean tip-path plane and perpendicular to the blade

$$U_T = \Omega R\,(x + \mu \sin\psi) \tag{1}$$

- perpendicular to the mean tip-path plane and to the blade

$$U_P = \lambda\Omega R - xR\dot{\beta} - \mu\Omega R\cos\psi\,\beta \tag{2}$$

The flapping angle expression includes terms as far as the second harmonic

$$\beta = a_0 - a_2\cos 2\psi - b_2\sin 2\psi \tag{3}$$

where the first harmonic terms are zero by definition of axes. Differentiating, one has

$$\dot{\beta} = 2\Omega\,(a_2\sin 2\psi - b_2\cos 2\psi) \tag{4}$$

$$\ddot{\beta} = 4\Omega^2\,(a_2\cos 2\psi + b_2\sin 2\psi) \tag{5}$$

Hence, the velocity through the disk is given by

$$U_P = \Omega R\left[\lambda - 2x\,(a_2\sin 2\psi - b_2\cos 2\psi) - \mu\cos\psi\,(a_0 - a_2\cos 2\psi - b_2\sin 2\psi)\right] \tag{6}$$

The blade pitch setting at any azimuth position can be expressed as

$$\theta = \theta_0 - \theta_1 x - A_1\cos\psi - B_1\sin\psi - A_2\cos 2\psi - B_2\sin 2\psi \tag{7}$$

The differential thrust acting on a blade element is

$$dT_\psi = \frac{1}{2}\rho U_T^2 a\,\alpha_r\,c\,dr = \frac{1}{2}\rho a c R U_T^2\left(\theta + \frac{U_P}{U_T}\right)dx \tag{8}$$

The corresponding differential thrust moment with respect to the flapping hinge is

$$dM_{T_\psi} = xR\,dT_\psi = \frac{1}{2}\rho a c R^2 U_T^2\left(\theta + \frac{U_P}{U_T}\right)x\,dx \tag{9}$$

The blade thrust at any azimuth position is obtained by integrating Eq.(8) in conjunction with Eqs.(1), (6), and (7) along the blade

$$T_\psi = \frac{1}{2}\rho a c R \int_0^1 U_T^2 \left(\theta + \frac{U_P}{U_T}\right) dx = \frac{1}{2}\rho a c \Omega^2 R^3 \left[\frac{\theta_0}{3}\left(1 + \frac{3}{2}\mu^2\right) + \frac{\lambda}{2} - \frac{\theta_1}{4}(1+\mu^2) - \frac{\mu}{2}B_1 + \frac{\mu^2}{4}A_2 + \frac{\mu^2}{4}b_2 + \sum_{n=1}^{4}\left(X_{nP}\cos n\psi + Y_{nP}\sin n\psi\right)\right] \tag{10}$$

where

$$X_{1P} = -\frac{1}{3}\left(1+\frac{3}{4}\mu^2\right)A_1 - \frac{\mu}{2}B_2 - \frac{\mu}{1}a_0 - \frac{\mu}{4}a_2 \tag{11}$$

$$Y_{1P} = \mu\theta_0 + \lambda\mu - \frac{2}{3}\mu\theta_1 - \frac{1}{3}\left(1+\frac{9}{4}\mu^2\right)B_1 + \frac{\mu}{2}A_2 - \frac{\mu}{4}b_2 \tag{12}$$

$$X_{2P} = -\frac{\mu^2\theta_0}{2} + \frac{\mu^2\theta_1}{4} + \frac{\mu}{2}B_1 - \frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)A_2 + \frac{2}{3}b_2 \tag{13}$$

$$Y_{2P} = -\frac{\mu}{2}A_1 - \frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)B_2 - \frac{\mu^2}{2}a_0 - \frac{2}{3}a_2 \tag{14}$$

$$X_{3P} = \frac{\mu^2}{4}A_1 + \frac{\mu}{2}B_2 + \frac{3\mu}{4}a_2 \tag{15}$$

$$Y_{3P} = \frac{\mu^2}{4}B_1 - \frac{\mu}{2}A_2 + \frac{3\mu}{4}b_2 \tag{16}$$

$$X_{4P} = \frac{\mu^2}{4}A_2 - \frac{\mu^2}{4}b_2 \tag{17}$$

$$Y_{4P} = \frac{\mu^2}{4}B_2 + \frac{\mu^2}{4}a_2 \tag{18}$$

The thrust moment at any azimuth position is obtained by integrating Eq.(9) in conjunction with Eqs.(1), (6), and (7) along the blade

$$\begin{aligned} M_{T_\psi} &= \frac{1}{2}\rho a c R^2 \int_0^1 U_T^2\left(\theta + \frac{U_P}{U_T}\right)x\,dx = \frac{1}{2}\rho a c \Omega^2 R^4 \Big\{\theta_0\frac{1}{4}\left(1+\frac{1}{2}\mu^2\right) + \frac{\lambda}{3} \\ &- \frac{\theta_1}{5}\left(1+\frac{5}{6}\mu^2\right) - \frac{\mu}{3}B_1 + \frac{\mu^2}{8}A_2 + \frac{\mu^2}{8}b_2 + \cos\psi\left[-\frac{1}{4}\left(1+\frac{1}{2}\mu^2\right)A_1 - \frac{\mu}{3}B_2 - \frac{\mu}{3}a_0 \right. \\ &\left. - \frac{\mu}{6}a_2\right] + \sin\psi\left[\frac{2}{3}\mu\theta_0 + \frac{\lambda\mu}{2} - \frac{\mu}{2}\theta_1 - \frac{1}{4}\left(1+\frac{3}{2}\mu^2\right)B_1 + \frac{\mu}{3}A_2 - \frac{\mu}{6}b_2\right] + \cos 2\psi\left[-\frac{\mu^2}{4}\theta_0 \right. \\ &\left. + \frac{\mu^2}{6}\theta_1 + \frac{\mu}{3}B_1 - \frac{1}{4}(1+\mu^2)A_2 - \frac{b_2}{2}\right] + \sin 2\psi\left[-\frac{\mu}{3}A_1 - \frac{1}{4}(1+\mu^2)B_2 - \frac{\mu^2}{4}a_0 - \frac{a_2}{2}\right] \\ &+ \cos 3\psi\left(\frac{\mu^2}{8}A_1 + \frac{\mu}{3}B_2 + \frac{\mu}{2}a_2\right) + \sin 3\psi\left(\frac{\mu^2}{8}B_1 - \frac{\mu}{3}A_2 + \frac{\mu}{2}b_2\right) + \cos 4\psi\left(\frac{\mu^2}{8}A_2 - \frac{\mu^2}{8}b_2\right) \\ &+ \sin 4\psi\left(\frac{\mu^2}{8}B_2 + \frac{\mu^2}{8}a_2\right)\Big\} \end{aligned} \tag{19}$$

Assuming an articulated rotor, the equilibrium condition about the flapping hinge (neglecting the weight moment) is

$$M_{T_\psi} - M_{CF} - M_I = 0 \tag{20}$$

where the centrifugal and inertia force moments are $M_{CF} = I\Omega^2\beta$ and $M_I = I\ddot{\beta}$, respectively. In nondimensional form and substituting for β and $\ddot{\beta}$ from Eqs.(4) and (5), the equilibrium condition becomes

$$M_{T_\psi}/\left(\frac{1}{2}\rho a c \Omega^2 R^4\right) = (2/\gamma)\left(a_0 + 3a_2\cos 2\psi + 3b_2\sin 2\psi\right) \tag{21}$$

Comparing the equations (19) and (21) and equating the corresponding coefficients, the following equations are obtained

$$\frac{1}{4}(1+\mu^2)\theta_0 - \frac{1}{5}\left(1+\frac{5}{6}\mu^2\right)\theta_1 + \frac{\lambda}{3} - \frac{\mu}{3}B_1 + \frac{\mu^2}{8}A_2 - \frac{2}{\gamma}a_0 + \frac{\mu^2}{8}b_2 = 0 \tag{22}$$

$$-\frac{1}{4}\left(1+\frac{1}{2}\mu^2\right)A_1 - \frac{\mu}{3}B_2 - \frac{\mu}{3}a_0 - \frac{\mu}{6}a_2 = 0 \tag{23}$$

$$\frac{2}{3}\mu\theta_0 - \frac{1}{2}\mu\theta_1 + \frac{\lambda\mu}{2} - \frac{1}{4}\left(1+\frac{3}{2}\mu^2\right)B_1 + \frac{\mu}{3}A_2 - \frac{\mu}{6}b_2 = 0 \tag{24}$$

$$-\frac{\mu^2}{4}\theta_0 + \frac{\mu^2}{6}\theta_1 + \frac{\mu}{3}B_1 - \frac{1}{4}(1+\mu^2)A_2 - \frac{6}{\gamma}a_2 + \frac{b_2}{2} = 0 \tag{25}$$

$$-\frac{\mu}{3}A_1 - \frac{1}{4}(1+\mu^2)B_2 - \frac{\mu^2}{4}a_0 - \frac{a_2}{2} - \frac{6}{\gamma}b_2 = 0 \tag{26}$$

λ can be eliminated from Eqs.(22) and (24). From Eq.(22) the value of λ is

$$\lambda = 3\left[-\frac{1}{4}(1+\mu^2)\theta_0 + \frac{1}{5}\left(1+\frac{5}{6}\mu^2\right)\theta_1 + \frac{\mu}{3}B_1 - \frac{\mu^2}{8}A_2 + \frac{2}{\gamma}a_0 - \frac{\mu^2}{8}b_2\right] \tag{27}$$

Substituting in Eq.(24)

$$\frac{\mu}{24}(7-9\mu^2)\theta_0 - \frac{\mu}{5}\left(1-\frac{5}{4}\mu^2\right)\theta_1 - \frac{1}{4}\left(1-\frac{1}{2}\mu^2\right)B_1 + \frac{\mu}{3}\left(1-\frac{9}{16}\mu^2\right)A_2 + \frac{3\mu}{\gamma}a_0 - \frac{\mu}{6}\left(1+\frac{9}{8}\mu^2\right)b_2 = 0 \tag{28}$$

From Eq.(28) and (23) one obtains the evaluation of the first harmonic of control

$$B_1 = \frac{\mu}{6}\left(\frac{7-9\mu^2}{1-\frac{1}{2}\mu^2}\right)\theta_0 - \frac{4\mu}{5}\left(\frac{1-\frac{5}{4}\mu^2}{1-\frac{1}{2}\mu^2}\right)\theta_1 + \frac{4\mu}{3}\left(\frac{1-\frac{9}{16}\mu^2}{1-\frac{1}{2}\mu^2}\right)A_2 + \frac{12}{\gamma\left(1-\frac{1}{2}\mu^2\right)}a_0 - \frac{2\mu}{3}\left(\frac{1+\frac{9}{8}\mu^2}{1-\frac{1}{2}\mu^2}\right)b_2 \tag{29}$$

$$-A_1 = \frac{4}{3}\frac{\mu}{1+\frac{1}{2}\mu^2}B_2 + \frac{4}{3}\frac{\mu}{1+\frac{1}{2}\mu^2}a_0 + \frac{2}{3}\frac{\mu}{1+\frac{1}{2}\mu^2}a_2 \tag{30}$$

Substituting the above values in Eqs.(25) and (26), respectively,

$$\frac{6}{\gamma}a_2 - \frac{1}{2}\left(\frac{1-\frac{17}{18}\mu^2-\frac{1}{2}\mu^4}{1-\frac{1}{2}\mu^2}\right)b_2 = -\frac{1}{4}\left(\frac{1-\frac{23}{18}\mu^2+\frac{1}{2}\mu^4}{1-\frac{1}{2}\mu^2}\right)A_2 + \frac{\mu^2}{10\left(1-\frac{1}{2}\mu^2\right)}\left(-1+\frac{5}{2}\mu^2\right)\theta_1 + \frac{\mu^2}{1-\frac{1}{2}\mu^2}\left(\frac{5}{36}-\frac{3}{8}\mu^2\right)\theta_0 + \frac{4\mu^2}{\gamma\left(1-\frac{1}{2}\mu^2\right)}a_0 \tag{31}$$

$$\frac{1}{2}\left(\frac{1+\frac{1}{18}\mu^2}{1+\frac{1}{2}\mu^2}\right)a_2 + \frac{6}{\gamma}b_2 = -\frac{1}{4}\left(\frac{1-\frac{5}{18}\mu^2+\frac{1}{2}\mu^4}{1+\frac{1}{2}\mu^2}\right)B_2 + \frac{\mu^2}{1+\frac{1}{2}\mu^2}\left(\frac{7}{36}-\frac{1}{8}\mu^2\right)a_0 \tag{32}$$

Equations (31) and (32) yield the second harmonic flapping coefficients a_2 and b_2 as linear functions of A_2 , B_2 , θ_1 , a_0 , and θ_0 . Dividing out fractions and neglecting powers of μ^4 and higher (this will be consistently done in all subsequent calculations without special mention), the following expressions are obtained after some lengthy manipulations

$$a_2 = -c_1A_2 - c_2B_2 + c_3\theta_1 + c_4 \tag{33}$$

$$b_2 = c_2A_2 - c_1B_2 + c_5\theta_1 + c_6 \tag{34}$$

where

$$c_1 = \frac{1}{2\Delta}\left(1-\frac{1}{3}\mu^2\right) \tag{35}$$

$$c_2 = \frac{\gamma}{24\Delta}\left(1-\frac{7}{9}\mu^2\right) \tag{36}$$

$$c_3 = -\frac{\mu^2}{5\Delta}\left(1-\frac{14}{9}\mu^2\right) \tag{37}$$

$$C_4 = k_1 a_o + k_2 \theta_o \tag{38}$$

$$C_5 = \frac{\mu^2}{10\Delta}(1-2\mu^2) \tag{39}$$

$$C_6 = k_3 a_o + k_4 \theta_o \tag{40}$$

$$\Delta = \frac{12}{\gamma}\left(1+\frac{4}{9}\mu^2\right) + \frac{\gamma}{12}\left(1-\frac{4}{9}\mu^2\right) \tag{41}$$

$$k_1 = \frac{\mu^2}{\Delta}\left[\frac{7\gamma}{216}\left(1-\frac{8}{7}\mu^2\right) + \frac{8}{\gamma}\left(1+\frac{17}{18}\mu^2\right)\right] \tag{42}$$

$$k_2 = \frac{5\mu^2}{18\Delta}\left(1-\frac{79}{45}\mu^2\right) \tag{43}$$

$$k_3 = \frac{\mu^2}{\Delta}\left[\frac{7}{18}\left(1-\frac{44}{63}\mu^2\right) - \frac{2}{3}\left(1+\frac{1}{2}\mu^2\right)\right] \tag{44}$$

$$k_4 = -\frac{5\mu^2}{216\Delta}\gamma\left(1-\frac{11}{5}\mu^2\right) \tag{45}$$

Substituting Eqs.(33) and (34) into Eqs.(30) and (29), respectively, the first harmonic of control takes the form

$$A_1 = c_7 A_2 + c_8 B_2 + c_9 \theta_1 + c_{10} \tag{46}$$

$$B_1 = c_{11} A_2 + c_{12} B_2 + c_{13} \theta_1 + c_{14} \tag{47}$$

where

$$c_7 = \frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_1 \tag{48}$$

$$c_8 = \frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)(c_2-2) \tag{49}$$

$$c_9 = -\frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)c_3 \tag{50}$$

$$c_{10} = -\frac{2\mu}{3}\left(1-\frac{1}{2}\mu^2\right)(c_4+2a_o) \tag{51}$$

$$c_{11} = \frac{4\mu}{3}\left(1-\frac{1}{16}\mu^2\right) - \frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_2 \tag{52}$$

$$c_{12} = \frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_1 \tag{53}$$

$$c_{13} = -\frac{4\mu}{5}\left(1-\frac{3}{4}\mu^2\right) - \frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_5 \tag{54}$$

$$c_{14} = -\frac{2\mu}{3}\left(1+\frac{13}{8}\mu^2\right)c_6 + \frac{\mu}{6}\left(7-\frac{11}{2}\mu^2\right)\theta_o + \frac{12\mu}{\gamma}\left(1+\frac{1}{2}\mu^2\right)a_o \tag{55}$$

The average rotor thrust is found by integrating Eq.(10) around the azimuth

$$T = \frac{b}{2\pi}\int_0^{2\pi} T_\psi \, d\psi = \frac{1}{2}\rho a b c \Omega^2 R^3 \left[\frac{1}{3}\left(1+\frac{3}{2}\mu^2\right)\theta_o - \frac{1}{4}(1+\mu^2)\theta_1 + \frac{\lambda}{2} - \frac{\mu}{2}B_1 + \frac{\mu^2}{4}A_2 + \frac{\mu^2}{4}b_2\right] \tag{56}$$

or, in nondimensional form and by substituting Eqs.(27), (34), and (47)

$$\frac{2C_T}{\sigma a} = \frac{\mu^2}{16}(c_2+1)A_2 - \frac{\mu^2}{16}c_1 B_2 + \left(\frac{\mu^2}{16}c_5 + \frac{1}{20}\right)\theta_1 + \frac{\mu^2}{16}c_6 - \frac{\theta_o}{24}(1-3\mu^2) + \frac{3a_o}{\gamma} \tag{57}$$

Equation (10) can be now rewritten in nondimensional form as follows

$$\frac{2bC_{T\psi}}{\sigma a} = \frac{2C_T}{\sigma a} + t(\psi) \tag{58}$$

where

$$t(\psi) = \sum_{n=1}^{4} (X_{nP} \cos n\psi + Y_{nP} \sin n\psi) \tag{59}$$

is the total alternating blade lift and $C_{T_\psi} = T_\psi / \rho \pi R^2 (\Omega R)^2$.

3. The Optimization Problem

The parameter optimization problem to be solved can be formulated as follows: for a given hinged rotor under a given flight condition, find the second harmonic pitch control A_2 and B_2 which minimizes the square of the the second harmonic blade lift amplitude, while keeping the rotor thrust at a prescribed value. That is, for given μ , γ , θ_1 , and λ (i.e., $X/qd^2\sigma$, see Section 5), find A_2 and B_2 which minimize the performance index

$$J = Z_{2P}^2 = X_{2P}^2 + Y_{2P}^2 \tag{60}$$

by yielding a prescribed C_T/σ value. Then, show that $J_{min} = 0$.

Upon substitution of Eqs.(33), (34), (46), and (47), Eqs.(13) and (14) become

$$X_2 = c_{15} A_2 + c_{16} B_2 + c_{17} \theta_1 + c_{18} \tag{61}$$

$$Y_2 = c_{19} A_2 + c_{20} B_2 + c_{21} \theta_1 + c_{22} \tag{62}$$

where the coefficients $c_{15} \div c_{22}$ expressed as functions of the basic coefficients $c_1 \div c_6$ are

$$c_{15} = \frac{1}{3} (1 - \frac{1}{2}\mu^2)(2c_2 - 1) \tag{63}$$

$$c_{16} = -\frac{2}{3} (1 - \frac{1}{2}\mu^2) c_1 \tag{64}$$

$$c_{17} = \frac{2}{3} (1 - \frac{1}{2}\mu^2) c_5 - \frac{3\mu^2}{20} (1 - 2\mu^2) \tag{65}$$

$$c_{18} = \frac{2}{3} (1 - \frac{1}{2}\mu^2) c_6 + \frac{\mu^2}{12} (1 - \frac{11}{2}\mu^2)\theta_o + \frac{6\mu^2}{\gamma} (1 + \frac{1}{2}\mu^2) a_o \tag{66}$$

$$c_{19} = - c_{16} \tag{67}$$

$$c_{20} = c_{15} \tag{68}$$

$$c_{21} = - \frac{2}{3} (1 - \frac{1}{2}\mu^2) c_3 \tag{69}$$

$$c_{22} = - \frac{2}{3} (1 - \frac{1}{2}\mu^2) c_4 + \frac{\mu^2}{6} (1 - 2\mu^2) a_o \tag{70}$$

Thus, the performance index becomes

$$J = (c_{15} A_2 + c_{16} B_2 + c_{17}\theta_1 + c_{18})^2 + (- c_{16} A_2 + c_{15} B_2 + c_{21}\theta_1 + c_{22})^2 \tag{71}$$

The sufficient conditions for a minimum are

$$\frac{\partial J}{\partial A_2} = 0, \qquad \frac{\partial J}{\partial B_2} = 0 \tag{72}$$

leading to

$$(c_{15}^2 + c_{16}^2) A_2 + (c_{15} c_{17} - c_{16} c_{21}) \theta_1 + c_{15} c_{18} - c_{16} c_{22} = 0 \tag{73}$$

$$(c_{15}^2 + c_{16}^2) B_2 + (c_{16} c_{17} + c_{15} c_{21}) \theta_1 + c_{16} c_{18} + c_{15} c_{22} = 0 \tag{74}$$

and

$$\frac{\partial^2 J}{\partial A^2} = \frac{\partial^2 J}{\partial B^2} = c_{15}^2 + c_{16}^2 > 0 \tag{75}$$

both being satisfied.

Furthermore, by evaluating the optimal values of A_2 and B_2 from Eqs.(73) and (74), respectively,

$$A_2 = \frac{1}{e_1}\left(-e_2\theta_1 - e_3\right) \tag{76}$$

$$B_2 = \frac{1}{e_1}\left(-e_4\theta_1 - e_5\right) \tag{77}$$

where

$$e_1 = c_{15}^2 + c_{16}^2 \tag{78}$$

$$e_2 = c_{15}c_{17} - c_{16}c_{21} \tag{79}$$

$$e_3 = c_{15}c_{18} - c_{16}c_{22} \tag{80}$$

$$e_4 = c_{16}c_{17} + c_{15}c_{21} \tag{81}$$

$$e_5 = c_{16}c_{18} + c_{15}c_{22} \tag{82}$$

and replacing them into the performance index (71), one gets

$$J_{min} = 0 \tag{83}$$

which proves that by applying the optimal blade pitch control (76)-(77), the second harmonic of the alternating blade lift is reduced to zero, thus cancelling the 2/rev hub axial force of a two-bladed hinged rotor.

Expressing $e_1 \div e_5$ as functions of $c_1 \div c_6$ and putting into evidence a_o and θ_o, Eqs.(76) and (77) can be rewritten as

$$A_2 = \frac{1}{e_1(\mu,\gamma)}\left[-e_2(\mu,\gamma)\theta_1 - e_3'(\mu,\gamma)a_o - e_3''(\mu,\gamma)\theta_o\right] \tag{84}$$

$$B_2 = \frac{1}{e_1(\mu,\gamma)}\left[-e_4(\mu,\gamma)\theta_1 - e_5'(\mu,\gamma)a_o - e_5''(\mu,\gamma)\theta_o\right] \tag{85}$$

where

$$e_1(\mu,\gamma) = \frac{1-\mu^2}{9}\left[(2c_2-1)^2 + 4c_1^2\right] \tag{86}$$

$$e_2(\mu,\gamma) = \left[\frac{2(1-\mu^2)}{9}c_5 - \frac{\mu^2(1-\frac{5}{2}\mu^2)}{20}\right](2c_2-1) - \frac{4(1-\mu^2)}{9}c_1c_3 \tag{87}$$

$$e_3'(\mu,\gamma) = \left[\frac{2(1-\mu^2)}{9}k_3 + \frac{2\mu^2}{\gamma}\right](2c_2-1) + \left[\frac{\mu^2}{9}\left(1-\frac{5}{2}\mu^2\right) - \frac{4(1-\mu^2)}{9}k_1\right]c_1 \tag{88}$$

$$e_3''(\mu,\gamma) = \left[\frac{2(1-\mu^2)}{9}k_4 + \frac{\mu^2}{36}(1-3\mu^2)\right](2c_2-1) - \frac{4(1-\mu^2)}{9}k_2c_1 \tag{89}$$

$$e_4(\mu,\gamma) = \left[\frac{\mu^2}{10}\left(1-\frac{5}{2}\mu^2\right) - \frac{4}{9}(1-\mu^2)c_5\right]c_1 - \frac{2(1-\mu^2)}{9}c_3(2c_2-1) \tag{90}$$

$$e_5'(\mu,\gamma) = \left[\frac{\mu^2}{18}\left(1-\frac{5}{2}\mu^2\right) - \frac{2}{9}(1-\mu^2)k_1\right](2c_2-1) - 4\left(\frac{1-\mu^2}{9}k_3 + \frac{\mu^2}{\gamma}\right)c_1 \tag{91}$$

$$e_5''(\mu,\gamma) = \left[-\frac{\mu^2}{18}(1-6\mu^2) - \frac{4}{9}(1-\mu^2)k_4\right]c_1 - \frac{2}{9}(1-\mu^2)k_2(2c_2-1) \tag{92}$$

4. Evaluation of Coning and Collective Pitch Angles

Coning angle and collective pitch can be now determined from inflow ratio equation (27) and rotor thrust equation (57), where λ and C_T/σ have prescribed values. Replacing a_2, B_1, A_2, and B_2 with their expressions (33), (47), (84), and (85), respectively, Eqs.(27) and (57) are brought after some lengthy manipulations to the following simple form

$$d_1 a_o + d_2 \theta_o = d_3 \tag{93}$$

$$d_4 a_o + d_5 \theta_o = d_6 \tag{94}$$

where

$$d_1 = \frac{2}{\gamma}(1 + 2\mu^2) \tag{95}$$

$$d_2 = -\frac{1}{4}(1 - \frac{5}{9}\mu^2) \tag{96}$$

$$d_3 = -\frac{1}{5}(1 - \frac{1}{2}\mu^2)\theta_1 + \frac{\lambda}{3} \tag{97}$$

$$d_4 = \frac{3}{\gamma} \tag{98}$$

$$d_5 = -\frac{1}{24}(1 - 3\mu^2) \tag{99}$$

$$d_6 = -\frac{1}{20}\theta_1 + \frac{2C_T}{\sigma a} \tag{100}$$

Then, solving for a_o and θ_o yields finally

$$a_o = -\frac{\gamma}{160}(1 + \frac{35}{6}\mu^2)\theta_1 - \frac{\gamma}{48}(1 - \frac{5}{2}\mu^2)\lambda + \frac{3\gamma}{4}(1 - \frac{1}{18}\mu^2)\frac{C_T}{\sigma a} \tag{101}$$

$$\theta_o = \frac{3}{4}(1 - \frac{1}{2}\mu^2)\theta_1 - \frac{3}{2}(1 + \frac{1}{2}\mu^2)\lambda + 6(1 + \frac{5}{2}\mu^2)\frac{C_T}{\sigma a} \tag{102}$$

Substituting now the above expressions of a_o and θ_o into Eqs.(84) and (85), the optimal second harmonic pitch control takes the final form

$$A_2 = m_1(\mu,\gamma)\theta_1 + m_2(\mu,\gamma)\lambda + m_3(\mu,\gamma)\frac{C_T}{\sigma a} \tag{103}$$

$$B_2 = m_4(\mu,\gamma)\theta_1 + m_5(\mu,\gamma)\lambda + m_6(\mu,\gamma)\frac{C_T}{\sigma a} \tag{104}$$

where

$$m_1(\mu,\gamma) = \frac{1}{e_1}\left[-e_2 + \frac{\gamma e_3'}{160}(1 + \frac{35}{6}\mu^2) - \frac{3e_3''}{4}(1 - \frac{1}{2}\mu^2)\right] \tag{105}$$

$$m_2(\mu,\gamma) = \frac{1}{e_1}\left[\frac{\gamma e_3'}{48}(1 - \frac{5}{2}\mu^2) + \frac{3e_3''}{2}(1 + \frac{1}{2}\mu^2)\right] \tag{106}$$

$$m_3(\mu,\gamma) = \frac{1}{e_1}\left[-\frac{3\gamma e_3'}{4}(1 - \frac{1}{18}\mu^2) - 6e_3''(1 + \frac{5}{2}\mu^2)\right] \tag{107}$$

$$m_4(\mu,\gamma) = \frac{1}{e_1}\left[-e_4 + \frac{\gamma e_5'}{160}(1 + \frac{35}{6}\mu^2) - \frac{3e_5''}{4}(1 - \frac{1}{2}\mu^2)\right] \tag{108}$$

$$m_5(\mu,\gamma) = \frac{1}{e_1}\left[\frac{\gamma e_5'}{48}(1 - \frac{5}{2}\mu^2) + \frac{3e_5''}{2}(1 + \frac{1}{2}\mu^2)\right] \tag{109}$$

$$m_6(\mu,\gamma) = \frac{1}{e_1}\left[-\frac{3\gamma e_5'}{4}(1 - \frac{1}{18}\mu^2) - 6e_5''(1 + \frac{5}{2}\mu^2)\right] \tag{110}$$

Equations (103) and (104) show that A_2 and B_2 vary linearly with θ_1, λ, and C_T/σ, and nonlinearly with μ and γ. The amplitude and phase angle of the optimal 2/rev pitch control are given by

$$\theta_{2P} = \sqrt{A_2^2 + B_2^2} \tag{111}$$

and

$$\Delta\phi = \tan^{-1}\left(\frac{B_2}{A_2}\right) \tag{112}$$

respectively.

5. Numerical Examples and Discussion of Results

In order to gain an appreciation of the parameter influence, several numerical applications have been carried out with data covering the following ranges: $\mu = 0.1-0.3$, $C_T/\sigma = 0.06-0.10$, $\gamma = 5-15$, $\theta_1 = 6-10$ deg, and $X/qd^2\sigma = 0.08-0.12$. The connection between λ, which actually appears in the formulas, and $X/qd^2\sigma$ is established as follows: assuming small α_D, one has $\sin\alpha_D \cong \alpha_D$, $\cos\alpha_D \cong 1$, $\mu = V/\Omega R$, and $X = T\alpha_D$, then

$$\frac{C_T/\sigma}{X/qd^2\sigma} = \frac{[T/\rho\pi R^2(\Omega R)^2](\rho V^2/2)(2R)^2}{T\alpha_D} = \frac{2\mu^2}{\pi\alpha_D} \tag{113}$$

wherefrom

$$\alpha_D = -\frac{2\mu^2}{\pi}\frac{(X/qd^2\sigma)}{(C_T/\sigma)} \tag{114}$$

and by using the rotor-disk angle of attack expression

$$\alpha_D = \frac{\lambda}{\mu} + \frac{C_T}{2\mu\sqrt{\mu^2+\lambda^2}} \tag{115}$$

one gets for $\lambda \ll \mu$

$$\lambda = \mu\alpha_D - \frac{C_T}{2\mu} = -\frac{2\mu^3}{\pi}\frac{(X/qd^2\sigma)}{(C_T/\sigma)} - \frac{C_T}{2\mu} \tag{116}$$

The results are plotted on Figures 1-9. Figure 1 shows the effect of optimal 2/rev blade pitch on the harmonic components of alternating blade lift: 1P content remains unchanged, 2P content is reduced to zero (as predicted analytically), 3P content is strongly reduced, while 4P content is somewhat increased. This results in a substantial flattening of the total alternating blade lift variation around the azimuth, as it can be seen by comparing the peak-to-peak amplitudes Z_t and $Z_{t_{opt}}$ in Figure 2. As indicated by Figure 3, the percentual reduction of the peak-to-peak amplitude $\Delta Z_t = (Z_t - Z_{t_{opt}})/Z_t \cdot 100$ increases with μ and C_T/σ and decreases with γ, reaching values of 50% for $\mu = 0.3$. As expected on physical grounds, Figures 4-8 show that the optimal 2/rev blade pitch amplitude increases with μ, C_T/σ, γ, and $X/qd^2\sigma$ (i.e., with aerodynamic loading), and decreases with θ_1 (blade twist has a favourable effect), while not exceeding 1.5 deg for the given range of parameters. For the same range, Figure 9 indicates that the optimal phase angle decreases with μ, C_T/σ, and γ (and also with θ_1 and $X/qd^2\sigma$, not represented), while remaining essentially below 15 deg in absolute value. This shows that the 2/rev pitch amplitude is dominated by A_2, thus explaining the linear variation of θ_{2P} with C_T/σ, θ_1, and $X/qd^2\sigma$ in Figures 5, 7, and 8, respectively, as predicted by Eq.(103).

In order to check the validity of the present approach, a comparison with experiment was sought. Since no test data pertaining to a two-bladed hinged rotor were available, the comparison was done using the wind-tunnel test results of McHugh and Shaw (Ref.3) which were obtained on a hingeless rotor. Accordingly, the comparative values listed in

Table 1 should be regarded rather as an order-of-magnitude check ensuring that no gross errors have occured during the lengthy manipulations required by the present analysis.

Table 1. Comparison with wind-tunnel test results of a hingeless rotor
($\gamma = 12.4$, $\theta_1 = 9$ deg)

Flight condition	2/rev pitch amplitude required to null 2/rev shaft axial force, deg	
	Wind-tunnel tests (Ref.3, hingeless rotor)	Present method (hinged rotor)
$\mu = 0.3$, $C_T/\sigma = 0.066$, $X/qd^2\sigma = 0.13$	0.60	0.77
$\mu = 0.3$, $C_T/\sigma = 0.122$, $X/qd^2\sigma = 0.10$	1.80†)	1.63
$\mu = 0.5$††) $C_T/\sigma = 0.063$ $X/qd^2\sigma = -0.05$	0.80	1.25

†) test result obtained by extrapolation
††) value somewhat high for the assumption of negligible μ^4

6. Concluding Remarks

Analytical expressions for the optimal 2/rev blade pitch required to reduce to zero the 2P hub axial force of a two-bladed hinged rotor are presented in a form allowing to discern the influence of various parameters. The 2/rev pitch amplitude requirements increase with μ, C_T/σ, γ, and $X/qd^2\sigma$, and decrease with θ_1, while not exceeding 1.5 deg for rotor characteristics and flight conditions in the usual range. Up to 50% reduction of the peak-to-peak amplitude of total alternating blade lift is obtained at $\mu = 0.3$. Charts for rapid estimation of the optimal 2/rev pitch amplitude and phase angle are provided. A comparison with wind-tunnel test results of a hingeless rotor is performed as an order-of-magnitude check of the formulas.

7. References

1) W. Stewart, Second Harmonic Control on the Helicopter Rotor, Aeronautical Research Council R. & M. No. 2997, August 1952.

2) Bell Helicopter Company, An Experimental Investigation of a Second Harmonic Feathering Device on the UH-1A Helicopter, USATRECOM TR 62-109, June 1963.

3) F.J. McHugh and J. Shaw, Jr., Benefits of Higher Harmonic Blade Pitch: Vibration Reduction, Blade-Load Reduction, and Performance Improvement, Proceedings of the American Helicopter Society Mideast Region Symposium on Rotor Technology, August 1976.

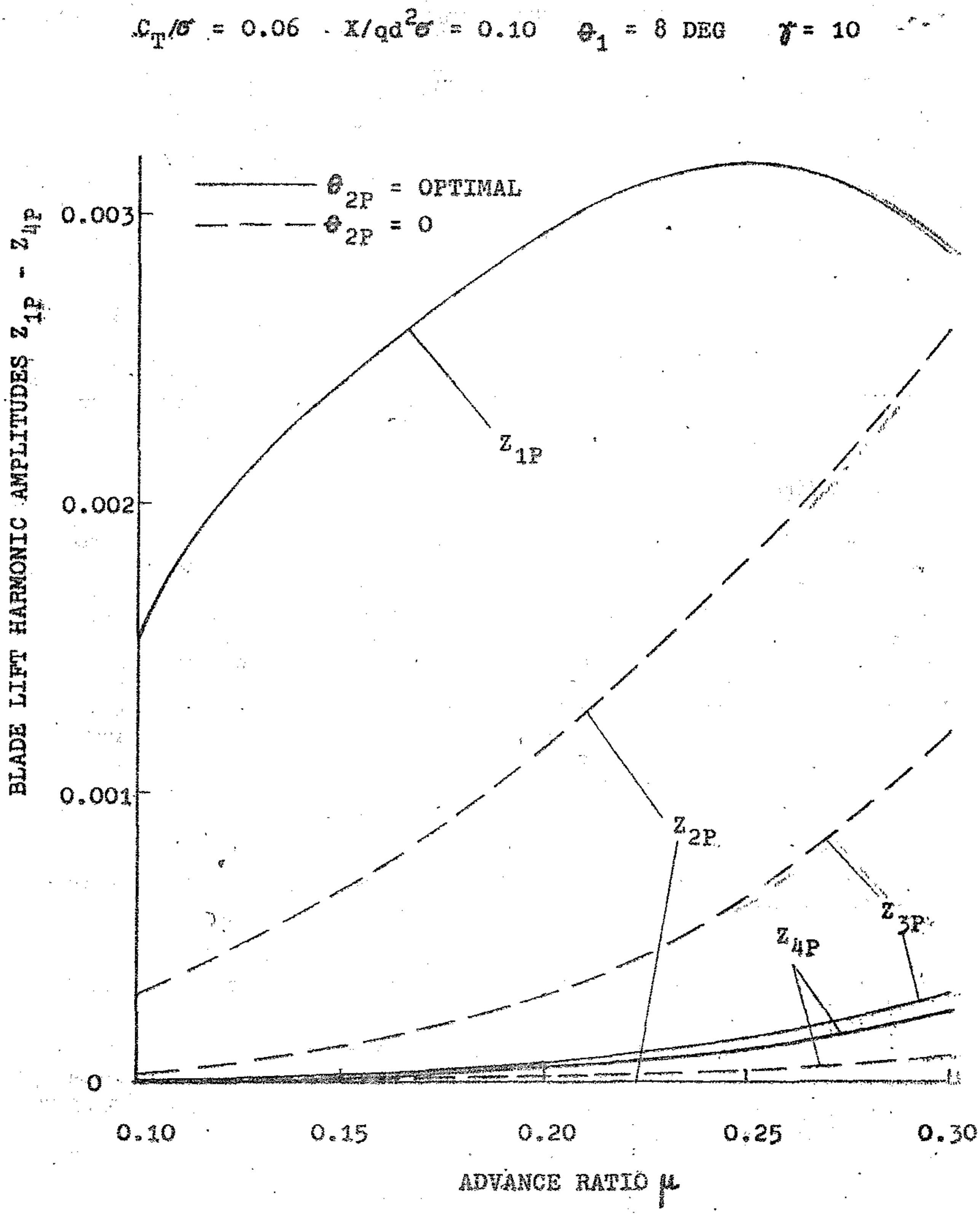

FIGURE 1. OPTIMAL 2/REV BLADE PITCH EFFECTS ON BLADE LIFT HARMONIC AMPLITUDES VERSUS FORWARD SPEED

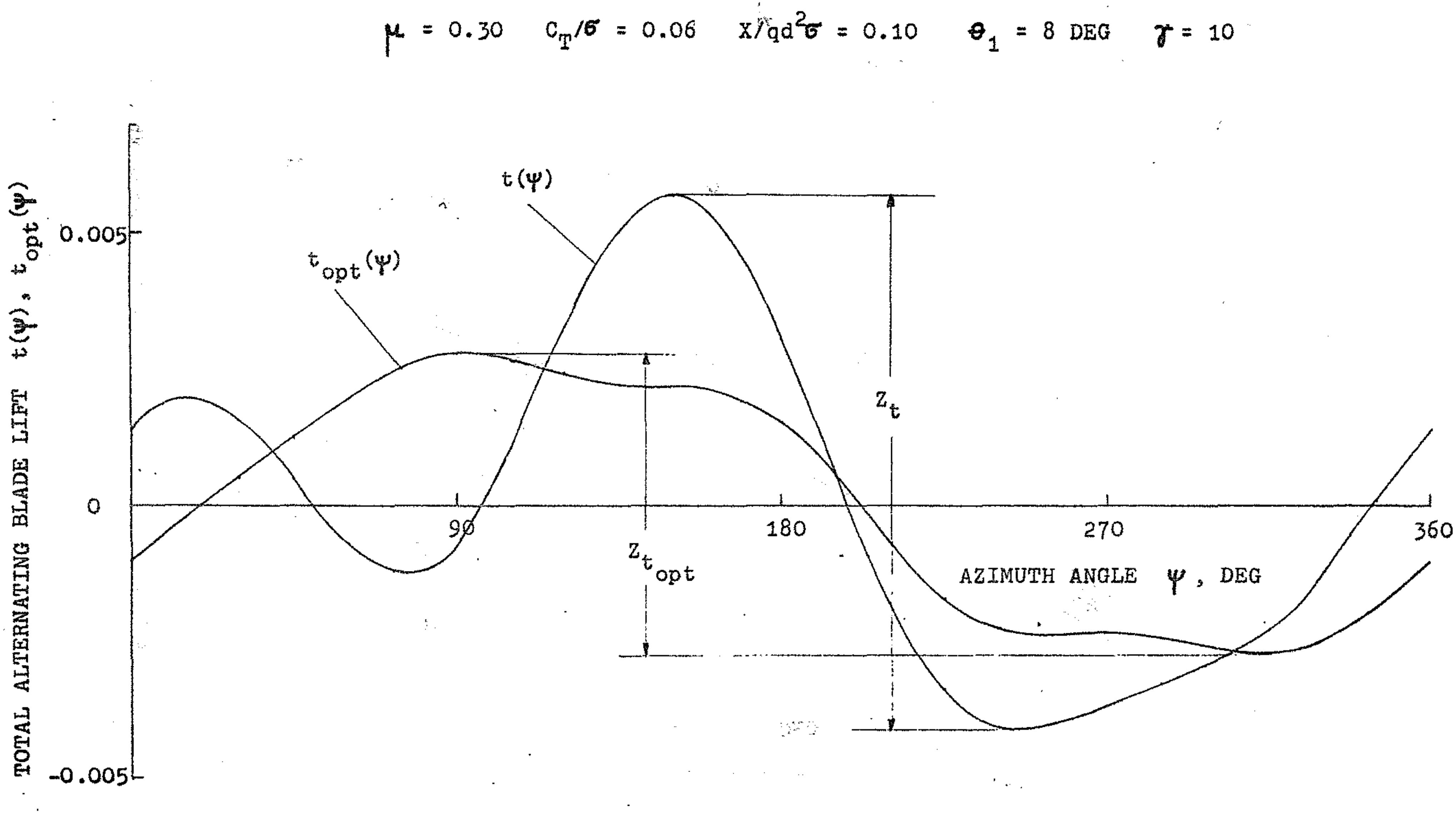

FIGURE 2. OPTIMAL 2/REV BLADE PITCH EFFECTS ON TOTAL ALTERNATING BLADE LIFT

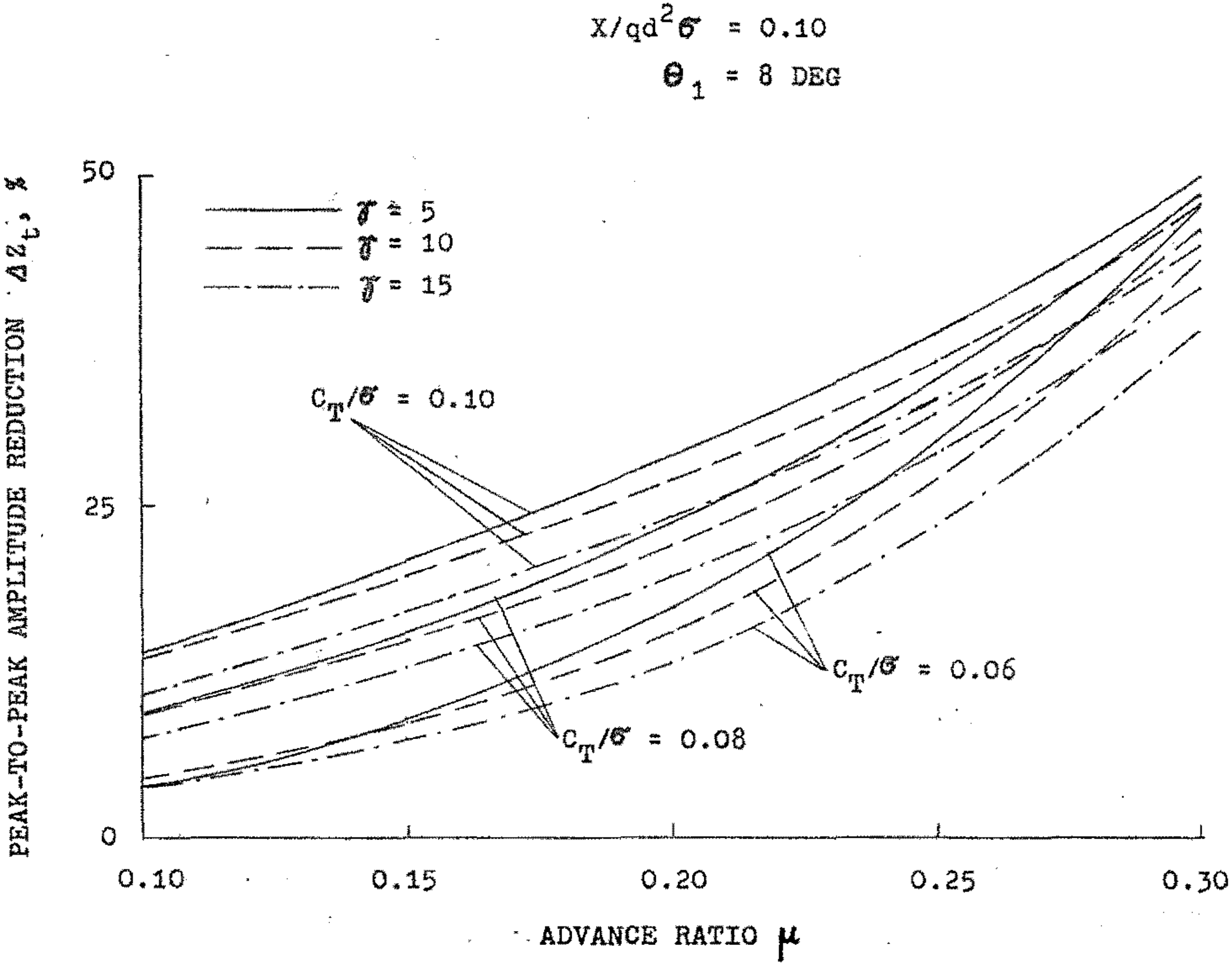

FIGURE 3. PERCENTUAL REDUCTION OF THE PEAK-TO-PEAK AMPLITUDE OF TOTAL ALTERNATING BLADE LIFT DUE TO OPTIMAL 2/REV BLADE PITCH VERSUS FORWARD SPEED

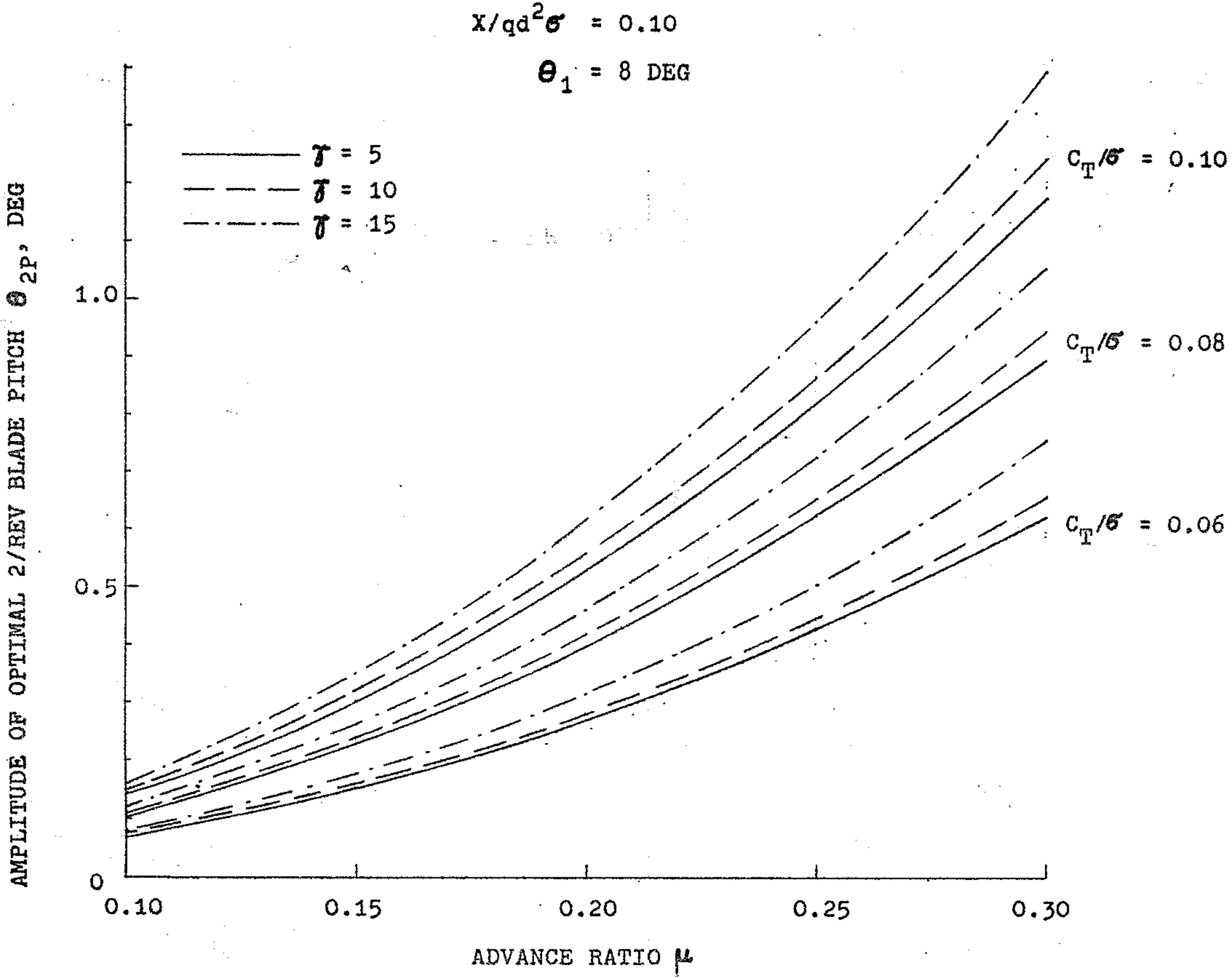

FIGURE 4. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VERSUS FORWARD SPEED

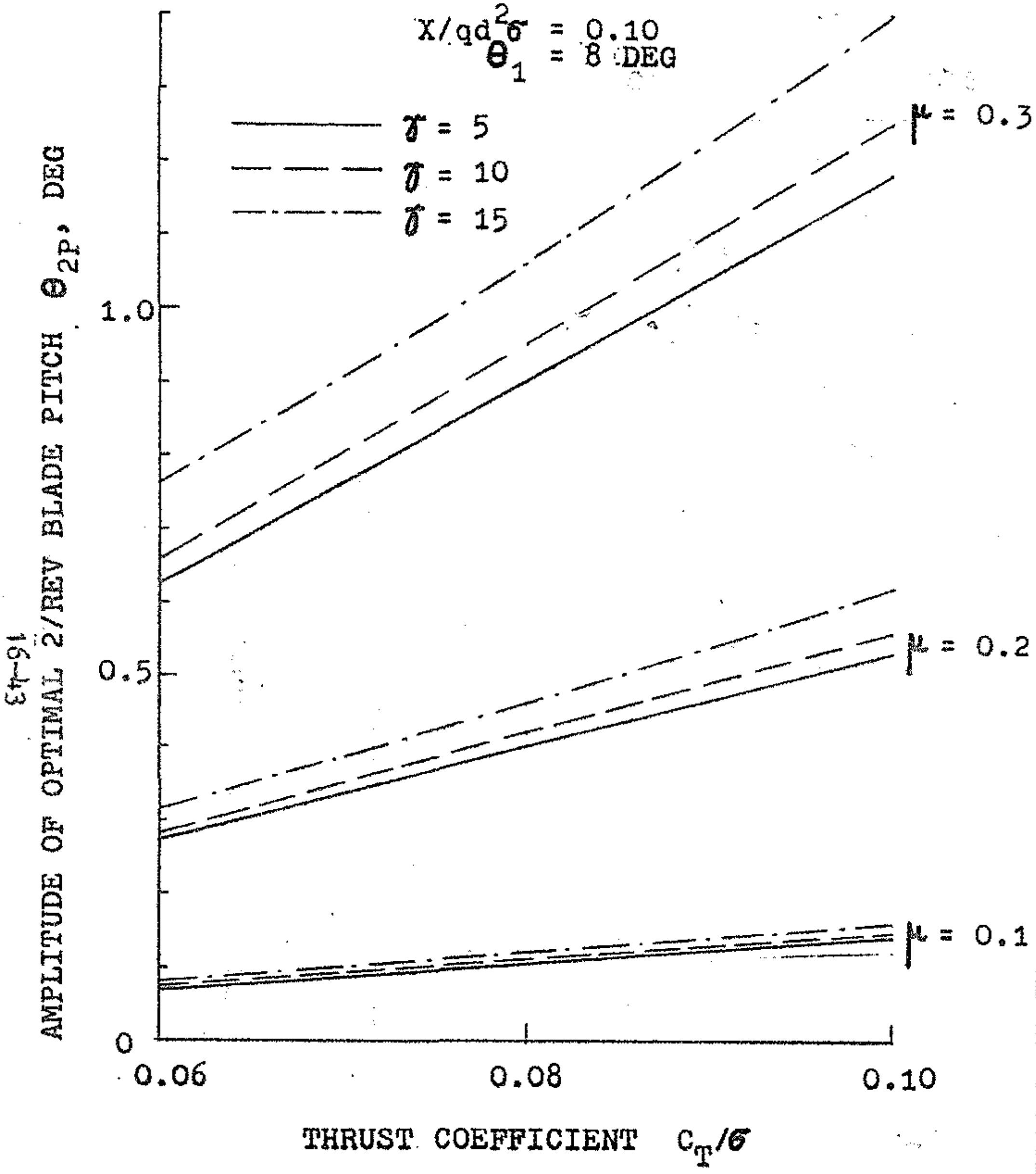

FIGURE 5. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VERSUS THRUST COEFFICIENT

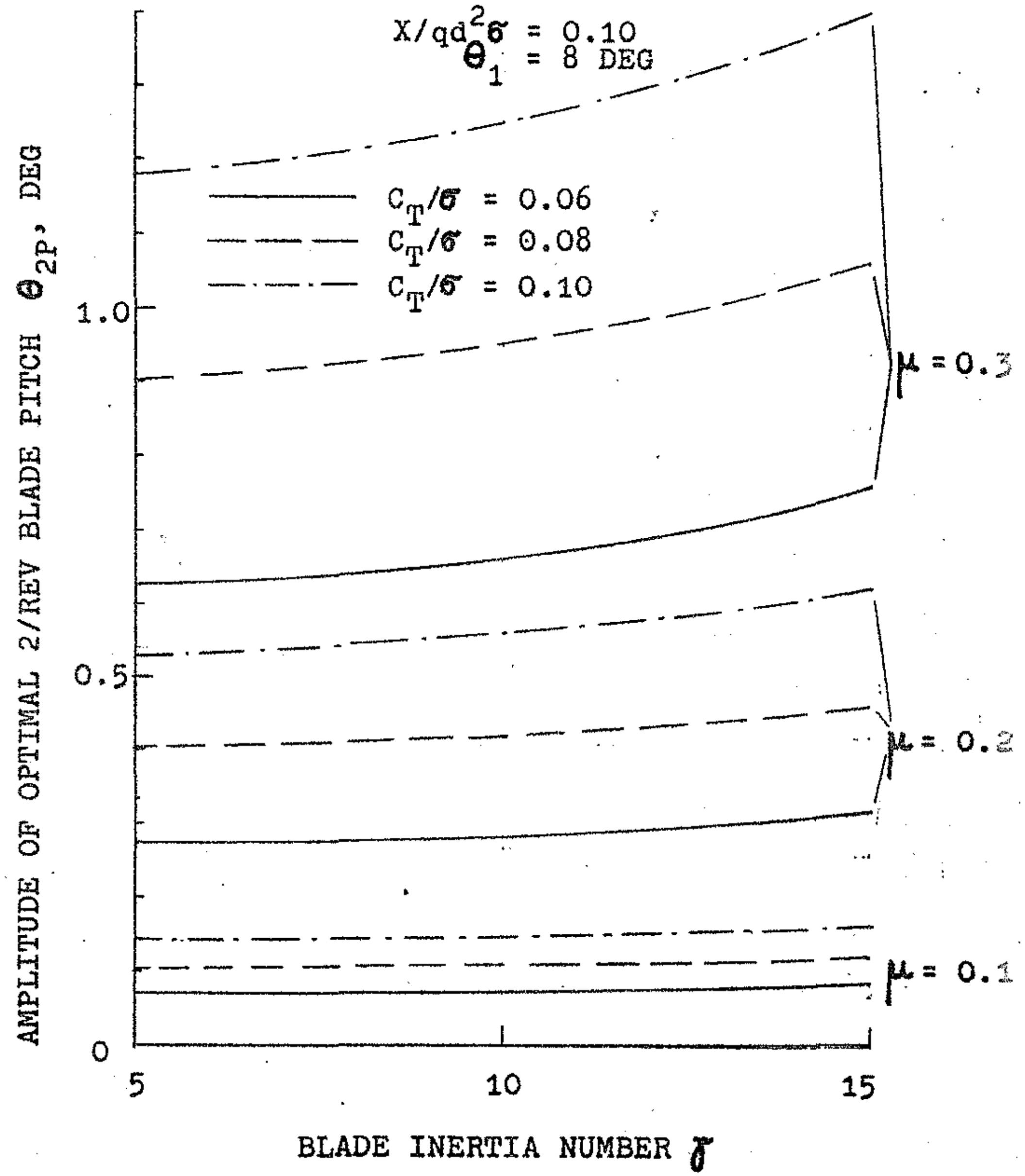

FIGURE 6. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VERSUS BLADE INERTIA NUMBER

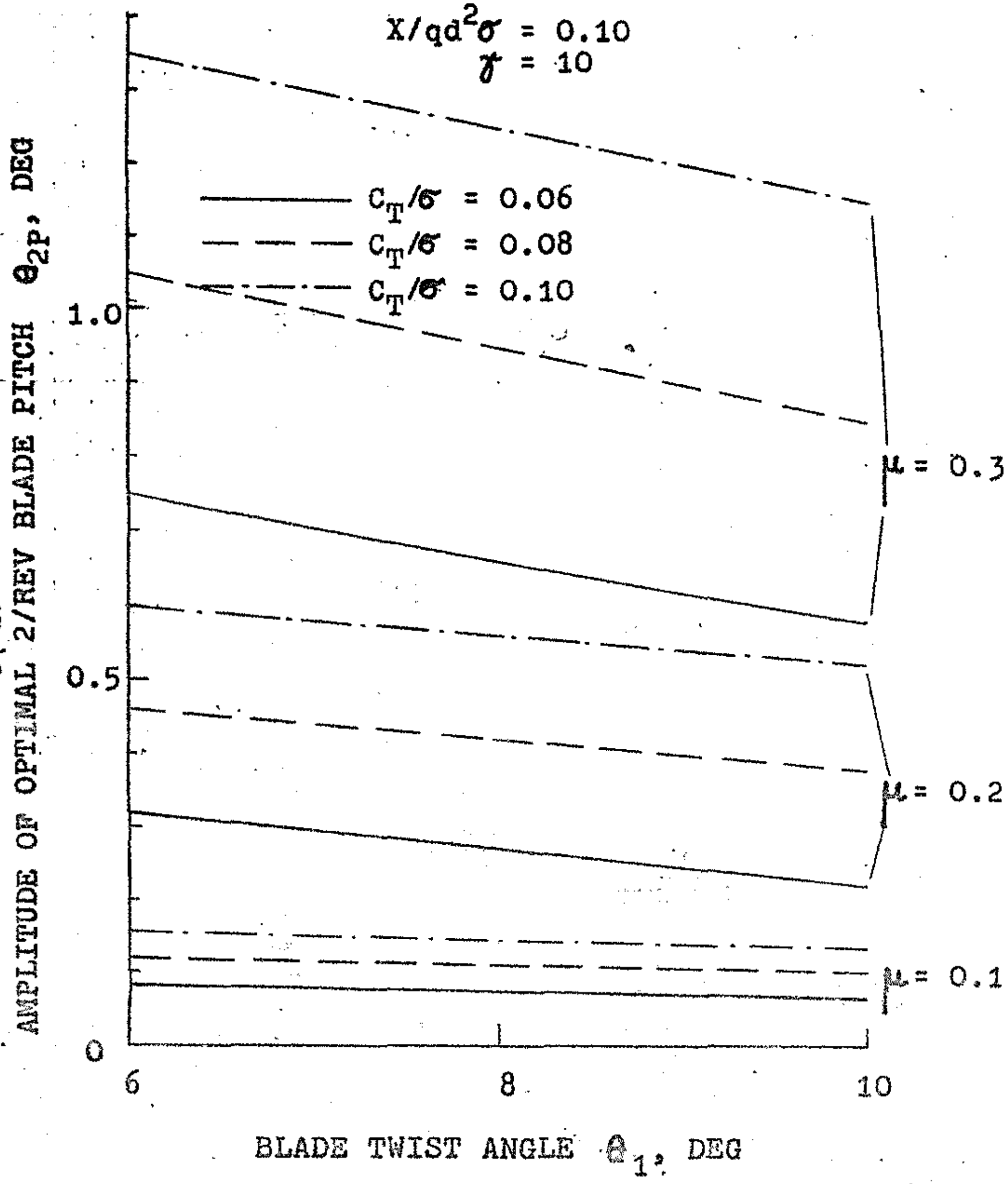

FIGURE 7. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VERSUS BLADE TWIST ANGLE

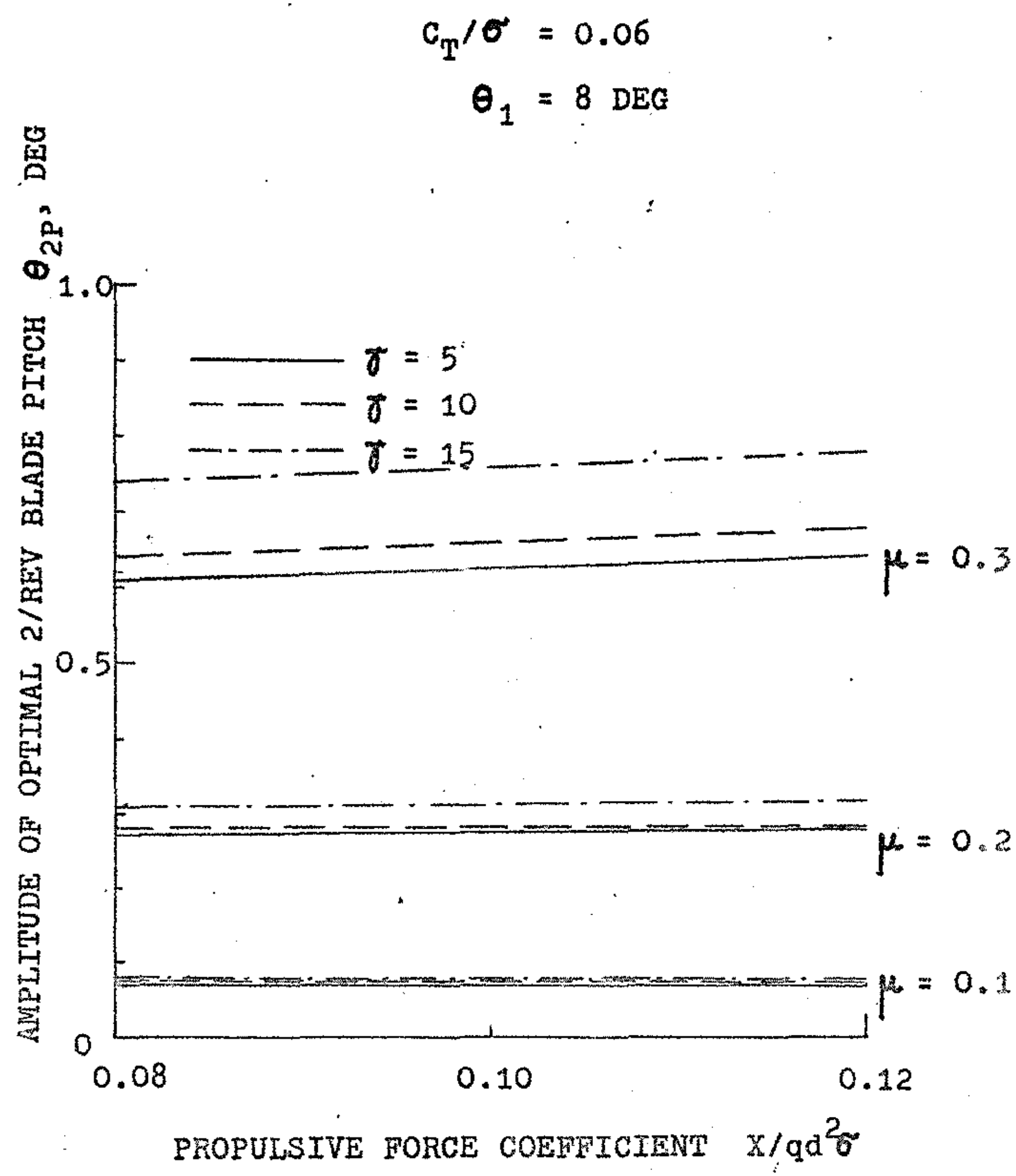

FIGURE 8. AMPLITUDE OF OPTIMAL 2/REV BLADE PITCH VERSUS PROPULSIVE FORCE COEFFICIENT

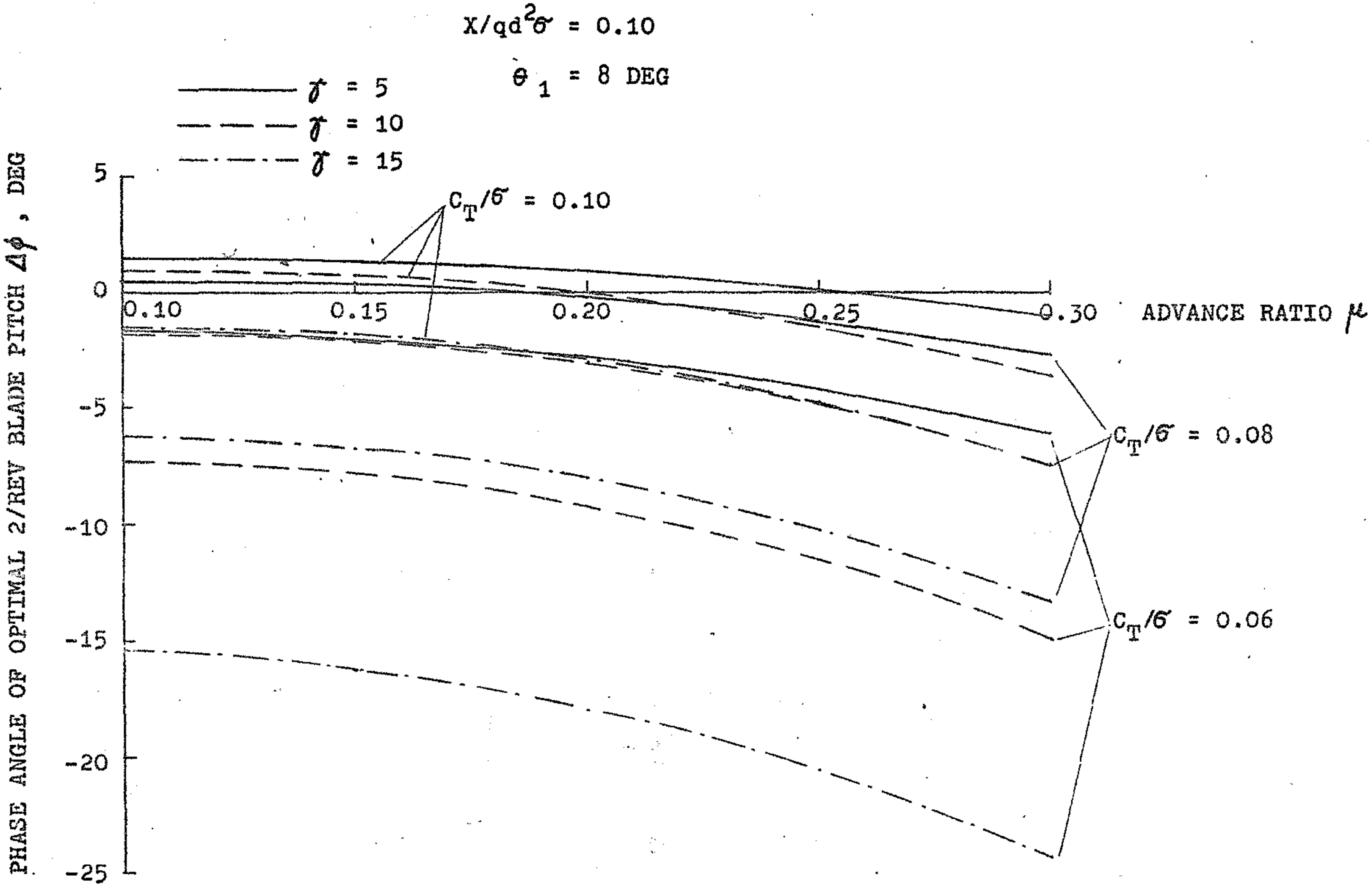

FIGURE 9. PHASE ANGLE OF OPTIMAL 2/REV BLADE PITCH VERSUS FORWARD SPEED