APPLICATION OF UDWADIA-KALABA METHOD FOR ROTORCRAFT ANALYSIS

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Abstract

Modeling a rotorcraft as a system of interconnected bodies requires a systematic procedure to assemble the equations of motion of the individual components and enforce the constraints between them. Such systematic treatment often involves the use of a redundant set of coordinates to describe the motion of the system so that the modeling process could easily be automated. The kinematic constraints relating these redundant coordinates are then enforced through Lagrange multipliers, resulting in a system of differential algebraic equations (DAEs). However, higher the index of the DAEs, more arduous methods need to be employed for their numerical integration. Udwadia-Kalaba (U-K) approach allows for the reduction of the equations of motion of a constrained multibody system to a set of ordinary differential equations (ODEs) even when redundant coordinates are employed. Thus, U-K approach improves the ease of implementation of the multibody simulation engine as a computer program and allows for the use of explicit time integration schemes. In this work, the U-K approach is first used to model the benchmark problem of a double four bar mechanism. The benchmarking results indicate that the U-K method provides a better accuracy than the augmented Lagrangian formulation. Subsequently, the U-K method is used to model a system of interest to the rotorcraft community, a rigid rotating flapping blade. In this case, the simulated results from the U-K method are shown to be in good agreement with those from the minimal coordinates approach. This proves that the U-K method could be used for automatic development of the equations of motion in a comprehensive rotorcraft analyses package.

1. INTRODUCTION

Modern rotorcraft dynamic analysis packages allow for the modelling of a rotorcraft system as a complex mechanism made up of multiple flexible and rigid bodies interconnected through various joints. This approach brings in modularity, flexibility, and expandability, allowing for various design complexities such as gimbal mounts and swash plates to be accounted for and enables modelling a vast array of rotorcraft configurations such as tiltrotors, quadrotors and eVTOLs used for urban air mobility [1, 2].

Figure 1: XV-15 Tiltrotor aircraft [3]

Existing multibody dynamics modelling software for rotorcraft analyses such as DYMORE [4] and MBDyn [5] rely on the Lagrange's multiplier method to derive the equations of motion as a set of index-3 Differential Algebraic Equations (DAEs). These software implement iterative implicit numerical integration schemes to perform forward dynamics simulation of these equations [6]. However, the DAE approach has certain limitations: the implicit methods used for numerical integration are often computationally heavy to implement and the linearized stability analysis of DAEs is not as straightforward as ordinary differential equations (ODEs) as the mass matrix of DAEs is singular [6].

Hence, this study looks at the complete elimination of Lagrange multipliers, thereby reformulating the equations of motion as a set of ODEs. This could allow the use of explicit numerical integration schemes, potentially reducing the computational effort, and also simplify the linearized stability analysis about a particular operating point.

Maggi's formulation, the null-space formulation and Udwadia-Kalaba (U-K) formulation are some of the techniques that enable a reformulation of the equations of motion of a constrained system in terms of ODEs. Amongst these, U-K formulation treats a wide range of problems such as those with redundant constraints and those involving contact surfaces such as rolling and slipping with ease [7]. The U-K formulation also treats both holonomic and nonholonomic constraints in the same way. These features make the U-K approach very attractive for implementation in a computer program.

Consequently, we intend to develop a multibody

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dynamics code for the analysis of rotorcraft systems based on the U-K approach. To this end, the U-K approach is first used to model the benchmark problem of a double four bar mechanism and is compared against other methods such as the minimal coordinates approach and the Augmented Lagrangian method. Subsequently, the U-K method is used to model a rigid rotating flapping blade.

This paper is organised as follows. A brief description of the U-K method is provided in Section 2. Subsequently, the two case studies have been described in detail in Section 3. The results from the case studies are discussed in Section 4. Thereafter, conclusions are drawn in Section 5.

2. UDWADIA – KALABA METHOD

Udwadia and Kalaba, in 1992, proposed a new formulation of the equations of motion for a system of constrained rigid bodies wherein the equations of motion could be reduced to a set of ordinary differential equations (ODEs) even when redundant coordinates are used [7]. The formulation utilises the concept of Moore-Penrose generalized inverse of non-square matrices to calculate constraint forces that account for the kinematic constraints present in the system. These constraint forces are calculated by expressing the constraints, both holonomic and non-holonomic, in their acceleration form as described below.

Let the redundant coordinates used to describe the system be denoted as $\boldsymbol{q}$. The equations of motion of the unconstrained system can then be represented as,

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{1}$$

where t denotes time, the overdot $(\dot{\ })$ denotes differentiation with respect to time, $\boldsymbol{M}$ denotes the mass matrix and $\boldsymbol{Q}$ denotes the resultant of stiffness, damping and external forces,

$$\boldsymbol{Q} = \boldsymbol{Q}_{ext} - \boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{K}\boldsymbol{q} \tag{2}$$

where $\boldsymbol{Q}_{ext}$ denotes the resultant of external forces, and $\boldsymbol{C}$ and $\boldsymbol{K}$ denote damping and stiffness matrices respectively.

Let the holonomic constraints that depend only on the coordinates $\boldsymbol{q}$, and time t be expressed as,

$$\boldsymbol{\Phi}_q(\boldsymbol{q}, t) = 0 \tag{3}$$

The constraints that cannot be expressed in the above form are said to be non-holonomic and are expressed as,

$$\boldsymbol{\Phi}_v(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0 \tag{4}$$

The holonomic constraints, given in Eq. (3), are differentiated twice with respect to time and represented in their acceleration form as,

$$\boldsymbol{A}_q(\boldsymbol{q}, t)\ddot{\boldsymbol{q}} = \boldsymbol{b}_{qv}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{5}$$

where,

$$\boldsymbol{A}_q = \frac{\partial \boldsymbol{\Phi}_q}{\partial \boldsymbol{q}} \tag{6}$$

and

$$\boldsymbol{b}_{qv} = -\frac{\partial}{\partial \boldsymbol{q}}\left(\frac{\partial \boldsymbol{\Phi}_q}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}\right)\dot{\boldsymbol{q}} - 2\frac{\partial^2 \boldsymbol{\Phi}_q}{\partial t\, \partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \frac{\partial^2 \boldsymbol{\Phi}_q}{\partial t^2} \tag{7}$$

Similarly, the non-holonomic constraints, given in Eq. (4), are differentiated once and expressed in their acceleration form as,

$$\boldsymbol{A}_{vv}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ddot{\boldsymbol{q}} = \boldsymbol{b}_{vv}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{8}$$

where,

$$\boldsymbol{A}_{vv} = \frac{\partial \boldsymbol{\Phi}_v}{\partial \dot{\boldsymbol{q}}} \tag{9}$$

and

$$\boldsymbol{b}_{vv} = -\frac{\partial \boldsymbol{\Phi}_v}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} - \frac{\partial \boldsymbol{\Phi}_v}{\partial t} \tag{10}$$

Combining Eqs. (5) and (8) yields,

$$\boldsymbol{A}_v(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ddot{\boldsymbol{q}} = \boldsymbol{b}_v(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{11}$$

where, $\boldsymbol{A}_v = \left[\boldsymbol{A}_q^T, \boldsymbol{A}_{vv}^T\right]^T$ and $\boldsymbol{b}_v = \left[\boldsymbol{b}_{qv}^T, \boldsymbol{b}_{vv}^T\right]^T$

Then, the equation of motion of the constrained system can be formulated based on the U-K approach as follows [8],

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{Q} + \boldsymbol{M}^{0.5}(\boldsymbol{A}_v\boldsymbol{M}^{-0.5})^{+}(\boldsymbol{b}_v - \boldsymbol{A}_v\boldsymbol{M}^{-1}\boldsymbol{Q}) \tag{12}$$

where, the operation $(\ )^{+}$ denotes the Moore-Penrose generalized inverse of a matrix [8].

Using the variable $\boldsymbol{v}$ to denote the velocities, the equations of motion can be rewritten in the state-space form as,

$$\begin{Bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{v}}\end{Bmatrix} = \begin{Bmatrix}\boldsymbol{v}\\ \boldsymbol{M}^{-1}\boldsymbol{Q} + \boldsymbol{M}^{-0.5}(\boldsymbol{A}_v\boldsymbol{M}^{-0.5})^{+}(\boldsymbol{b}_v - \boldsymbol{A}_v\boldsymbol{M}^{-1}\boldsymbol{Q})\end{Bmatrix} \tag{13}$$

Using $\boldsymbol{a} = \boldsymbol{M}^{-1}\boldsymbol{Q}$ to denote the unconstrained acceleration, Eq. (13) can be rewritten as,

$$\begin{Bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{v}}\end{Bmatrix} = \begin{Bmatrix}\boldsymbol{v}\\ \boldsymbol{a} + \boldsymbol{M}^{-0.5}(\boldsymbol{A}_v\boldsymbol{M}^{-0.5})^{+}(\boldsymbol{b}_v - \boldsymbol{A}_v\boldsymbol{a})\end{Bmatrix} \tag{14}$$

It could be seen that the Udwadia - Kalaba (U-K) method enables the treatment of both holonomic and non-holonomic constraints in a similar manner, making it very attractive for implementation in a computer program. The constraints enforced in the acceleration form as in Eq. (11) together with the initial conditions is equivalent to algebraic constraints represented in Eqs. (3) and (4) provided the initial conditions satisfy these constraints.

However, the above equivalence may be affected when the equations are implemented as a computer program as the round-off errors accumulated from the numerical integration of the acceleration-level constraints can lead to a drifting of the solutions [9]. This could result in the numerically obtained solutions not satisfying the constraints in their velocity and displacement forms. This is called the drift phenomenon and is a major limitation in implementing Udwadia-Kalaba equations as a computer program.

2.1. Mitigating the drift phenomenon

To mitigate the drift phenomenon in constrained dynamical systems, various techniques have been explored. For instance, constraint violation stabilization techniques were pursued by Baumgarte [10] and Gear [11], projection based methods were pursued by Eich [12] and state-space based techniques were pursued by ten Dam [13]. However, some of these techniques require iterative constraint stabilization, defeating the purpose of moving to a ODE-based formulation from a DAE based formulation.

Braun and co-workers [14] proposed the inclusion of a displacement constraint correction term and a velocity constraint correction term to the equations of motion obtained using the U-K method as it enforces the constraints at the acceleration level only. The mitigation strategy proposed in [14] allows the use of explicit ODE solvers such as the fourth order Runge-Kutta method, avoiding the need for iterative constraint stabilization. Further, the strategy also takes into account the finite word-length of the computational environment, and also accommodates possibly inconsistent initial conditions. Further, it was also shown in [14] that the proposed technique provides a better accuracy than conventional methods such as the Baumgarte's method. Hence, the mitigation strategy proposed in [14] has adopted in the U-K formulation used in this work.

The modified equations of motion with the constraint correction terms from [14] being included are shown below,

$$\left\{\begin{matrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{v}}\end{matrix}\right\} = \left\{\begin{matrix}\boldsymbol{v} + \boldsymbol{M}^{-0.5}\left(\boldsymbol{A}_q\boldsymbol{M}^{-0.5}\right)^{+}\left(\boldsymbol{b}_q - \boldsymbol{A}_q\boldsymbol{v} - h^{-1}\boldsymbol{\Phi}_q\right)\\ \boldsymbol{a} + \boldsymbol{M}^{-0.5}\left(\boldsymbol{A}_v\boldsymbol{M}^{-0.5}\right)^{+}\left(\boldsymbol{b}_v - \boldsymbol{A}_v\boldsymbol{a} - h^{-1}\boldsymbol{\Phi}_{qv}\right)\end{matrix}\right\} \tag{15}$$

where, h is the step size for integration, $\boldsymbol{b}_q = {\partial\boldsymbol{\Phi}_q}/{\partial t}$ and $\boldsymbol{\Phi}_{qv} = \left[\dot{\boldsymbol{\Phi}}_q^T, \boldsymbol{\Phi}_v^T\right]^T$

Equation (14) has been used to formulate and numerically solve the equations of motion of all multibody systems described in this paper.

3. CASE STUDIES

In this section, the multibody dynamics formulation given in Eq. (15) is first tested on the standard benchmark problem, a double four bar mechanism. This is done to validate the formulation and assess its accuracy. Subsequently, the formulation has been used to model a rigid rotating-flapping blade which is of interest in rotorcraft applications.

3.1. Double four bar mechanism

Amongst the collection of problems provided at the Library of Computational Benchmark Problems [15], a double four bar mechanism has been chosen to validate the formulation proposed in this study. The double four bar mechanism is a single degree of freedom system and while automating the equations of motion, fifteen redundant coordinates need to be used. This makes the problem more prone to numerical drift. Furthermore, the mechanism also has a singularity at its vertical upright position. Furthermore, multiple published results based on different formulations such as the augmented Lagrangian and minimal coordinates approach are available for the system, against which the results from this study could be compared. Consequently, this benchmark problem has been chosen for this work.

The initial configuration of the double four bar mechanism, with all the five links having an equal length of 1 m and mass of 1 kg, is indicated in Fig. 2. For each link, three coordinates were assigned, two corresponding to the global $\boldsymbol{x}$ and $\boldsymbol{y}$ coordinates of the centre of mass of the link and the other corresponding to the angle subtended by the link with respect to the global $\boldsymbol{x}$-axis. Consequently, a total of fifteen redundant coordinates and fourteen constraints were for the formulation. The mechanism is modelled as a conservative system, with gravity acting along the y-direction.

The equations of motion of the unconstrained system are given as follows,

$$\begin{bmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & {ml^2}/{12}\end{bmatrix}\left\{\begin{matrix}\ddot{x}_i\\ \ddot{y}_i\\ \ddot{\theta}_i\end{matrix}\right\} = \left\{\begin{matrix}0\\ mg\\ 0\end{matrix}\right\} \tag{16}$$

where, $i = 1$ to 5 and $g = -9.81$ ms^{-2} denotes the acceleration due to gravity.

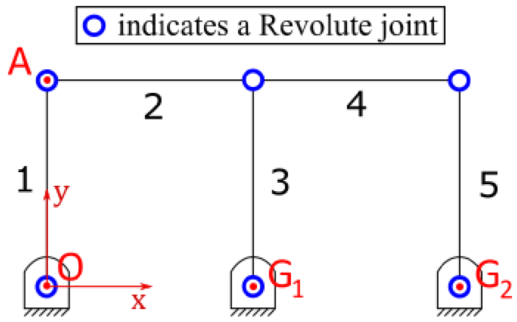

Figure 2: Schematic representation of the double fourbar mechanism

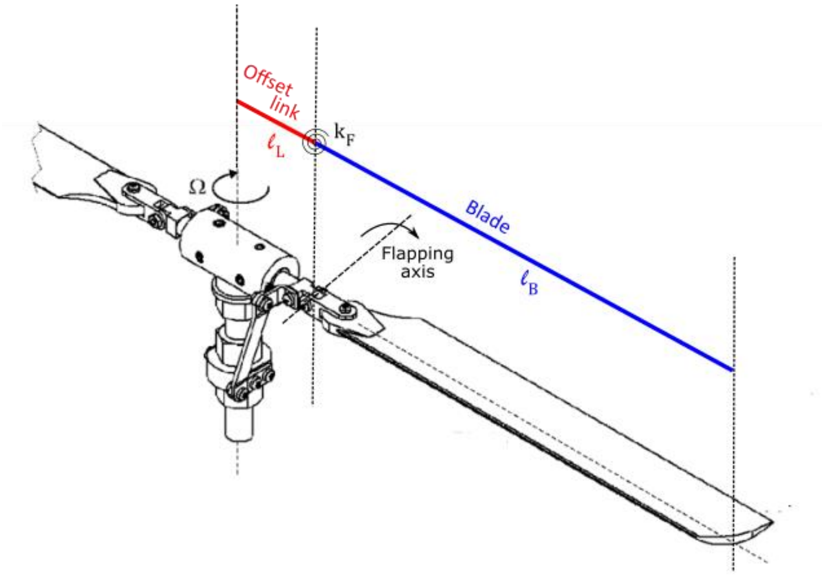

Figure 3: Multibody representation of a helicopter blade with flapping hinge

The kinematic constraints are given as follows,

$$\begin{Bmatrix} x_1 - 0.5\,l\cos\theta_1 \\ y_1 - 0.5\,l\sin\theta_1 \\ x_2 - 2x_1 - 0.5\,l\cos\theta_2 \\ y_2 - 2y_1 - 0.5\,l\sin\theta_2 \\ x_3 - 0.5\,l\cos\theta_3 \\ y_3 - 0.5\,l\sin\theta_3 \\ x_2 - 2x_3 + 0.5\,l\cos\theta_2 \\ y_2 - 2y_3 + 0.5\,l\sin\theta_2 \\ x_4 - 2x_3 - 0.5\,l\cos\theta_4 \\ y_4 - 2y_3 - 0.5\,l\sin\theta_4 \\ x_4 - 2x_5 + 0.5\,l\cos\theta_4 \\ y_4 - 2y_5 + 0.5\,l\sin\theta_4 \\ x_5 - 0.5\,l\cos\theta_5 \\ y_5 - 0.5\,l\sin\theta_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

In this study, the fourteen constraints given in Eq. (17) are enforced with the help of the U-K method as given in Eq. (15). This leads to a set of fifteen second order ODEs as the equations of motion of the constrained system.

On the other hand, if the Lagrange multipliers method were to be employed, the equations of motion of the system would be a system of DAEs, consisting of fifteen second order ODEs and fourteen algebraic constraints, a total of 29 equations.

In the minimal coordinates approach, which is the simplest possible representation of the system, the fourteen constraints are used to eliminate fourteen degrees of freedom, and the motion of the system is described by a single second order ODE,

$$\ddot{\theta}_1 - \frac{7g}{6l}\cos\theta_1 = 0 \tag{18}$$

The results pertaining to the simulation of the equations of motion using the different strategies are discussed in Section 4.1.

3.2. Rotating flapping rigid blade

As an example pertaining to rotorcraft applications, a rotating flapping blade is to be modelled using the Udwadia-Kalaba method. This case study is formulated in such a way that the imposition of both holonomic and non-holonomic constraints using the U-K method could be demonstrated.

The system is considered to be made of two rigid bodies, the blade itself and another link of negligible mass to represent the offset between the flapping hinge and the hub as shown in Fig. 3.

The blade chosen for this study has a length l_B of 8.9 m and mass m_B of 100 kg as shown in Fig. 3. The offset link has a length l_L of 0.4 m and mass m_L of 2 kg. The torsional stiffness of the system for flapping is 2900 Nm/rad and the blade is assumed to rotate at 22 rad/s. The effect of gravity is neglected in this analysis.

U-K method was used to automate the generation of equations of motion of this system, with each body represented using four degrees of freedom, three representing the position of centre of mass of the body, and one representing the rotational coordinate of interest.

Accordingly, the blade is modelled using four DOFs, the first three representing the position of its centre of mass (x_B, y_B, z_B) and the last one representing the flapping angle β. Similarly, the link is modelled using four DOFs, the first three representing the position of its centre of mass (x_L, y_L, z_L) and the last one representing its orientation around the hub θ. The angular velocity of rotation of the hub is related to this orientation as, $\Omega = \dot{\theta}$.

The unconstrained equations of motion of the blade are given as,

$$\begin{bmatrix} m_B & 0 & 0 & 0 \\ 0 & m_B & 0 & 0 \\ 0 & 0 & m_B & 0 \\ 0 & 0 & 0 & I_B \end{bmatrix} \begin{Bmatrix} \ddot{x}_B \\ \ddot{y}_B \\ \ddot{z}_B \\ \ddot{\beta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

Similarly, the unconstrained equations of motion of the link are given as,

$$\begin{bmatrix} m_L & 0 & 0 & 0 \\ 0 & m_L & 0 & 0 \\ 0 & 0 & m_L & 0 \\ 0 & 0 & 0 & I_L \end{bmatrix} \begin{Bmatrix} \ddot{x}_L \\ \ddot{y}_L \\ \ddot{z}_L \\ \ddot{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

The blade is connected to the link with the help of a revolute joint along the global y-axis. The link in turn is attached to the hub with the help of a revolute joint along the global z-axis. Each revolute joint enforces three constraints. Hence, a total of six holonomic constraints are imposed as follows,

$$\begin{Bmatrix} x_B - 2x_L + 0.5 l_B \cos\beta \cos\theta \\ y_B - 2y_L + 0.5 l_B \cos\beta \sin\theta \\ z_B - 2z_L - 0.5 l_B \sin\beta \\ x_L - 0.5 l_L \cos\theta \\ y_L - 0.5 l_L \sin\theta \\ z_L \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{21}$$

Furthermore, the angular velocity of rotation of the hub is imposed as a non-holonomic constraint, as the angular velocity Ω is considered as a system state,

$$\Omega = 22 \tag{22}$$

Incorporating the constraints in Eqs. (21) and (22) using the Udwadia-Kalaba method into the unconstrained equations in Eq. (20), leads to eight second order ODEs representing the constrained system.

On the other hand, the six constraints could be used to eliminate six DOFs and the system could be described using two second order ODEs,

$$\begin{Bmatrix} \ddot{\theta} \\ \ddot{\beta} + \omega_{F1}^2 \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{23}$$

where ω_{F1} is the first flapping frequency, given as,

$$\omega_{F1} = \Omega \left(1 + \frac{3\, e_F}{2\,(1 - 3\, e_F)} + \frac{k_F}{I_B \Omega^2} \right)^{0.5} \tag{24}$$

where $e_F = l_L / (l_L + l_B)$.

Further, considering Ω as a parameter instead of a system state eliminates one more degree of freedom. Consequently, in the minimal coordinates approach, the system is modelled using only one second order ODE,

$$\ddot{\beta} + \omega_{F1}^2 \beta = \mathbf{0} \tag{25}$$

Eq. (25) can also be obtained by appealing to the first principles in mechanics and deriving it from the free body diagram of the system.

To sum up, the rotating flapping blade has been modelled using both the Udwadia-Kalaba method as eight second order ODEs and using the minimal coordinates method as a single second order ODE and the results are discussed in Section 4.2.

4. RESULTS AND DISCUSSION

The results from the case studies elucidated in Section 3 are described in this section, starting with the benchmark problem of the double slider mechanism.

4.1. Double four bar mechanism

The Udwadia-Kalaba equations for the double four bar mechanism are obtained by substituting Eqs. (16) and (17) into Eq. (15). Subsequently, the U-K equations are integrated numerically using a fourth order Runge-Kutta solver with a time step of 0.001 s. The time histories of the displacement and velocity of node A as indicated in Fig. 2 are shown in Fig. 4. The results obtained from the numerical integration of the minimal coordinates equation (Eq. (18)) are also indicated therein. It can be seen the two results are in good agreement with each other.

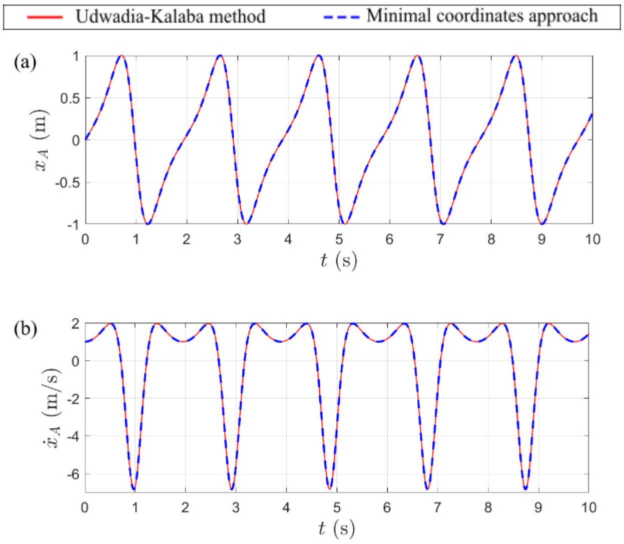

Figure 4: Time histories of the motion of node A along x-direction: (a) displacement and (b) velocity

Furthermore, since the double four bar mechanism considered here is a conservative system, the drift in the total energy over time can be considered as a

Table 1: Comparison of the drift while using different multibody dynamics formulations.

Researcher	Formulation adopted	Solver	Time step (s)	Max. drift in total energy (J)
Burkhardt (Neweul-M2)	Minimal coordinates approach	Explicit ode45 (MATLAB)	Variable	0.0015
This work	**Udwadia-Kalaba method**	**Explicit fourth order Runge-Kutta**	**0.001**	**0.0072**
Mouzo	Augmented Lagrangian (Penalty: 1×10^{15})	Implicit Trapezoidal rule	0.01	0.029
Pastorino and Cosco	Augmented Lagrangian (Penalty: 8×10^{8})	Implicit Trapezoidal rule	0.01	0.0877
Masarati (MBDyn)	Lagrange Multipliers	Implicit	0.008	0.09
Cuadrado	Augmented Lagrangian (Penalty: 1×10^{9})	Implicit Trapezoidal rule	0.01	0.0917

measure of accuracy. A higher drift indicates a higher deviation from the conservation of total energy and hence, a lower accuracy.

From the Library of Computational Benchmark Problems, the drift in total energy for different methods such as Lagrange multipliers method and the Augmented Lagrangian method have been obtained and are compared against the results from the U-K method in Table 1.

From Table 1, it could be seen the lowest drift is achieved when using a minimal coordinates approach. However, if the generation of equations of motion are to be automated, redundant coordinates are necessary. It could be seen that when employing redundant coordinates, the U-K method performs better than the widely used Augmented Lagrangian and Lagrange multipliers method.

This proves that the U-K method offers a two-fold advantage for deployment in a multibody dynamics-based computer program - providing both ease of implementation and better accuracy.

4.2. Rotating flapping rigid blade

As mentioned in Section 3.2, the rotating flapping rigid blade is modelled as an eight DOF system using the Udwadia - Kalaba method and a single DOF system using the minimal coordinates approach. In both cases, the equations of motion are numerically integrated using a fourth order Runge-Kutta method with a time step of 0.001 s. The simulation is carried out with an initial flap angle of 10º and rest of the states being calculated according to the constraints.

The flapping angle β obtained using the two methods are shown in Fig. 5. It could be observed that the results obtained from the two methods are in good agreement with each other.

Furthermore, the error in the holonomic constraint, $\boldsymbol{\Phi}_{q1} = \boldsymbol{x_B} - 2\boldsymbol{x_L} + 0.5\boldsymbol{l_B}\cos\boldsymbol{\beta}\cos\boldsymbol{\theta}$ when enforced through the U-K method is shown in Fig. 6. It could be seen that the maximum error is about six orders of magnitude lower than the blade length.

Similarly, the error in the non-holonomic constraint, $\boldsymbol{\Phi}_v = \boldsymbol{\Omega} - \mathbf{22}$, is shown in Fig. 7. The error is of the order of $\mathbf{10^{-13}}$, indicating that the constraint is satisfied well when enforced through the U-K method.

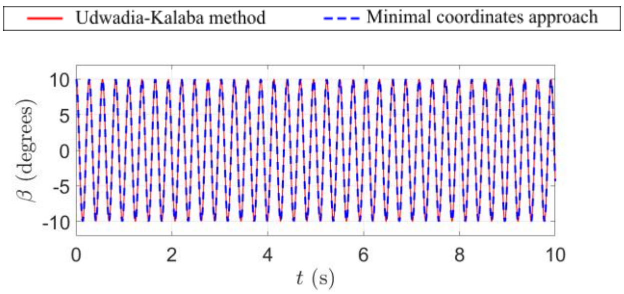

Figure 5: Flapping angle of the blade modelled using U-K and minimal coordinates methods

5. CONCLUSION AND FUTURE WORK

In this paper, an alternative ODE-based formulation for multibody systems, known as the Udwadia-Kalaba method, is evaluated in the context of rotorcraft problems. The U-K formulation is first tested against the commonly used Augmented Lagrangian Formulation in the benchmark problem of a double four bar mechanism. Simulated results

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indicate that the drift in total energy is significantly less for the U-K formulation compared to that the Augmented Lagrangian formulations.

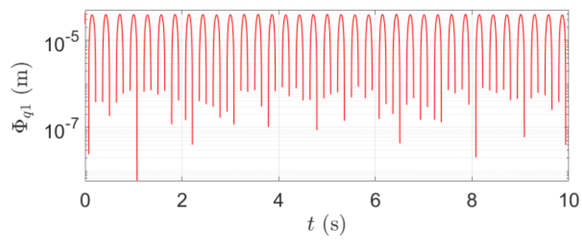

Figure 6: Error in the holonomic constraint enforced through U-K method

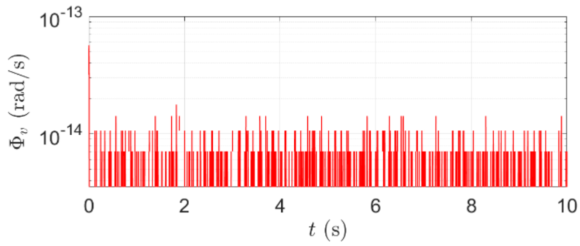

Figure 7: Error in the non-holonomic constraint enforced through U-K method

Subsequently, the U-K formulation is used to model a rigid rotating flapping blade as a multibody system with eight degrees of freedom. The results from this 8 DOF model are in good agreement with the minimal coordinates approach, represented using a single degree of freedom. This proves that U-K method performs well even in the presence of a large number of redundant coordinates and is suited for deployment in a computer program for automating multibody formulations for rotorcraft systems.

Currently, we are developing a multibody dynamics framework based on Udwadia – Kalaba method that allows for bottom-up modelling of any multibody system. The framework is capable of automatically generating the equations of the motion at once the inertia, stiffness and damping properties of the individual bodies and the kinematic constraints between the bodies are specified. The framework is intended towards modelling rotorcrafts as multibody systems. The framework would bring about modularity and flexiblity as different configurations of rotorcraft could be modelled easily.

6. ACKNOWLEDGEMENT

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