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A GENERAL ALGORITHM FOR QUASILINEARIZATION IN HELICOPTER DYNAMICS

H. ENGMANN

MESSERSCHMITT-BÖLKOW-BLOHM GMBH
MÜNCHEN, F.R.G.

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Abstract

Starting from a representation of the helicopter as a multi body system a rigorous linearization procedure is presented. Using this procedure exact solutions of the complete nonlinear equations of motion may be obtained by combining Floquet theory and the method of quasilinearization.

1. Introduction

By far reaching agreement a today's helicopter model for aeroelastic problems has to include an elastic fuselage and various rotor systems with elastic blades, geometric and aerodynamic nonlinearities, provisions for stability analysis, automatic synthesis of equations of motion, and frequency response. At the same time any numerical helicopter model must be sufficiently general, consistent, verifiable and manageable (Ref. [1], [2]). The same conditions are to be satisfied by models of wind turbines with elastic tower. To meet the above requirements more closely, MBB is developing the multi body system program Vibrat in addition to well approved programs established at MBB, (Ref.[3]). The new program is a general aeroelastic stability and dynamic response program designed to model helicopters as well as wind turbines.

The multi body system has tree structure with a maximum of six branches to model the blades. The main components of the aeroelastic model are listed below, see Fig. 1:

I. Helicopter

- Fuselage (elastic body, finite element model)
- Gearbox (rigid body)
- Rotorshaft (elastic body, massless)
- Multibladed rotor (elastic blades, continuum beam model)

II. Wind Turbine

- Tower (elastic body, finite element model)
- Nacelle (rigid body)
- Rotorshaft (elastic body, massless)
- Single or multibladed rotor (elastic blades, continuum beam model)

The rigid or elastic bodies are interconnected by "hinges", every branch has twenty-one hinges at most which define the coupling between the bodies. The formulation and solution of the equations of motion are further discussed in the following sections with emphasis on the derivation of the inertial terms. The derivation of the structural, aerodynamical and gravitational terms in the equations of motion are beyond the scope of this paper, but nevertheless some additional remarks are given.

Special attention is given to the modelling of modern bearingless rotor systems with redundant load paths and complex control system kinematics. Geometric nonlinearities in the structural terms due to moderate deformations of the blades are accounted for, see Ref. [4], [5].

The aerodynamic blade loads are calculated by a combined quasi-static blade element and vortex disc theory, which is adequate for both helicopter rotors, propellers and wind turbines. The aerodynamic section loads are found from measured nonlinear airfoil data including static stall, compressibility and reverse-flow effects. The aerodynamic coefficients are functions of the local angle of attack and Mach-number. They are calculated by a table-look-up followed by a two-dimensional spline interpolation and differentation. The vortex disc theory for lifting airscrews in oblique flow by G.I. Maykapar (see Ref. [6]) is used for the calculation of the averaged rotor induced velocities by an iterative scheme. The integral representation of the induced velocity harmonics is taken from the same reference to calculate variable inflow effects. For wind turbine problems special attention is given to shear flow and tower shadow effects.

2. Taylor expansion

It is well known that many of the helicopter instabilities are due to geometric or aerodynamic nonlinearities. Hence general helicopter equations are necessarily nonlinear. To solve these the following procedures have been conveyed or suggested:

- direct numerical integration of the nonlinear equations (e.g. Ref. [3])
- linearization:

  physical (ordering scheme)

  mathematical (Taylor expansion).

The main advantage of the methods using linearization compared to direct solution procedures is the availability of

- the full power of the theory of linear differential equations, in particular
- the theory of linear differential equations with periodic coefficients (Floquet-Liapunoff)
- eigen-solutions for stability analysis
- a high degree of verifiability by step-by-step comparison of known linear analytical models with numerical results.

The inclusion of elastic bodies also demands for a consistent expansion of the strains.

Linearization, mathematical as well as physical amounts to neglecting second and higher order terms. The suppressed terms need not be necessarily identical for the engineer and the mathematician. If for instance the blade flap angle β is considered small together with its time derivatives, then a formal Taylor expansion would neglect the Coriolis term $\beta\dot{\beta}$, although it is known to play a prominent role in rotor dynamics (Ref. [7]). Any model obtained by mathematical linearization therefore has to be sufficiently general to include all known physically important terms. Vibrat for example admits an expansion $\beta = \bar{\beta} + \Delta\beta$. The above Coriolis term then is recovered in the summands of the first order expansion

$$\beta\dot{\beta} = \bar{\beta} \cdot \dot{\bar{\beta}} + \bar{\beta} \cdot \Delta\dot{\beta} + \Delta\beta \cdot \dot{\bar{\beta}}.$$

At first sight it might be tempting to mix mathematical and physical linearization. Thus one could start with a Taylor-expansion and apply ordering schemes whenever the jungle of formulas seems to become impenetrable. On the other hand, numerical stability of solution procedures is closely connected to symmetry properties of the equations of motion. These symmetry properties are conserved best when using a consistent, i.e. a unified linearization. The option taken by Vibrat is a rigorous Taylor expansion of first order.

3. Quasilinearization

To be profitable the advantages of linearity have to be augmented by some method to get the solutions of the linearized equations from the solutions of the nonlinear equations. To overcome this obstacle Vibrat uses quasilinearization, i.e. convergence of the "linearized" solutions to the "nonlinear" ones.

As an example consider Euler's equation for a physical pendulum

$$\tilde{\omega} I \omega + I\dot{\omega} - m \cdot \tilde{r}\, g = 0 \qquad (1)$$

where I represents inertia, ω angular velocity, m mass, g gravitation, and r locates the center of gravity.

This equation depends nonlinearly on the attitude angles θ_1, θ_2, θ_3 of orientation and their time derivatives. Quasilinearization now gives a sequence of linear equations

$$M(\bar{x}_k) \cdot \Delta\ddot{x}_k + C(\bar{x}_k) \cdot \Delta\dot{x}_k + K(\bar{x}_k) \cdot \Delta x_k = f(\bar{x}_k) \text{ , with} \qquad (2)$$

$$f(\bar{x}_k) = \tilde{\omega}_k I \omega_k + I\dot{\omega}_k - m \cdot \tilde{r}\, g \text{ .} \qquad (3)$$

Here the attitude angles were collected in a vector $x = (\theta_1\ \theta_2\ \theta_3)^t$ and expanded as $x = \bar{x} + \Delta x$.
From the theory of quasilinearization (Ref. [8]) it is known that both sides of (2) tend to zero as k increases. This implies the convergence of the ω_k to the true angular velocity ω.

4. Floquet-Liapunoff theory

As stated before, Vibrat was designed as a supplement to existing well proven simulation programs, developped at MBB. Although not confined to stationary flight conditions these are the typical application areas of Vibrat. Mathematically speaking, a linear boundary value problem with periodic coefficients and excitation is solved by the Floquet-Liapunoff theory. In contrast to simulation procedures where in general many revolutions of the rotor have to be considered, the time domain for the boundary value problem consists of only one revolution. For a symmetric rotor with m identical blades it even reduces the time to the m-th part of a revolution through multi-blade coordinates ([9]). Moreover, in many practical cases the use of multi-blade coordinates transforms the periodic coefficients to constant ones (exactly or approximately) thus reducing also complexity of the problem considerably.

5. Multi body system (MBS)

A further requirement of Vibrat was its easy manageability. In particular, data input should be small compared to large FEM-programs as NASTRAN. In addition the Vibrat input data had to be adapted to the usual output forms of measured and computed data in use at MBB. This was achieved by choosing an MBS model with hybrid coordinates (Ref. [10,11]). Its motion is completely determined by the relative motion of contiguous bodies against each other. Figure 2 illustrates how in a system of n bodies $B_1, \ldots, B_n$ an arbitrary body B_i is connected with and moves relative to its predecessor B_{i-1}. Together with the legend the figure determinates the chosen MBS in a very simple manner.

6. Block-diagram

In addition to the stated characteristics Vibrat includes the following more or less common features of MBS (Ref.[11], [12], [13]):

- transformation to modal coordinates for elastic bodies
- transformation from rotating to nonrotating coordinates or v.v. (multi-blade transformation)
- reduction of degrees of freedom by adequate transformation (e.g. condensation)
- reduction to a first-order problem by transformation into state space.

The block diagram of Fig. 3 roughly sketches how Vibrat works.

The following paragraphs will be devoted to a more detailed description of the underlying algorithm.

7. Equations of motion

Vibrat starts from d' Alembert's principle in Lagrange's form ([12])

$$(4) \quad \sum_{j=1}^{n} (l_j^t \cdot \frac{\partial v_j}{\partial \dot{q}_i} \cdot \delta q_i + \delta W_j) = 0 \text{ for } i = 1, \ldots, n .$$

Here the 6-vector $l_j = \{\frac{f_j}{m_j}\}$ represents inertia, aerodynamical and external forces and moments at body j in their local frame. The 6-vector $v_j = \{\frac{u_j}{\omega_j}\}$ represents translational and angular velocity. The virtual internal work δW_j will be omitted in the sequel not to overburden the text. The l_j, v_j are regarded as functions of the generalized coordinates q_i and their time derivatives. The equations of motion (4) will be linearized in three steps. In the first step a Taylor expansion of the physical equations of motion will be given, i.e. an expansion of (4) with the generalized coordinates q_i replaced by physical coordinates x_i. In the second step the physical coordinates x_i will be transformed to generalized coordinates q_i. In the third step this generalization transformation will result in a Taylor expansion of the generalized e.o.m. (4).

8. Taylor expansion of the physical equations

If the l_j, v_j in (4) are given as functions of physical (generalized) coordinates the resulting equations of motion will be termed physical. Suppose the following decompositions into a noninfinitesimal reference point and an infinitesimal variation (Ref. [13]):

$$(5a) \quad r_k = \bar{r}_k + \Delta r_k \qquad (5b) \quad \theta_k = \bar{\theta}_k + \Delta\theta_k$$

$$(5c) \quad \omega_k = \bar{\omega}_k + \Delta\omega_k, \quad \text{where} \quad \Delta\omega_k = \Delta\dot{\theta}_k + \tilde{\bar{\omega}}_k \cdot \Delta\theta_k$$

$$(5d) \quad A_k = \Delta A_k \cdot \bar{A}_k, \quad \text{where} \quad \Delta A_k = Id - \Delta\tilde{\theta}_k .$$

Collect r_k, θ_k in the 6-vector $x_k = (r_k, \theta_k)^t$, and the local vectors x_k in the global vector $x = (x_1, \ldots, x_n)^t$. The decompositions above induce corresponding decompositions $x_k = \bar{x}_k + \Delta x_k$ and $x = \bar{x} + \Delta x$. Now suppose the index j fixed and suppress it.
Then the decompositions (5) admit a first order Taylor expansion

$$(6a) \quad l(x) = l(\bar{x}) + \sum_k (M_k(\bar{x}) \cdot \Delta\ddot{x}_k + C_k(\bar{x}) \cdot \Delta\dot{x} + K_k(\bar{x}) \cdot \Delta x_k)$$

and a second order Taylor expansion

$$(6b) \quad v(x) = v(\bar{x}) + \sum_k (\bar{V}_k(\bar{x}) \cdot \Delta\dot{x} + V_k(\bar{x}, \Delta x) \cdot \Delta\dot{x})$$

In the last formula the V_k are linear functions of Δx. The second order terms $V_k\,(\bar{x},\ \Delta x) \cdot \Delta\dot{x}$ are necessary to get the first order expansion

$$\text{(6c)} \quad \frac{\partial v}{\partial \Delta\dot{x}_i} = \bar{V}_i(\bar{x}) + V_i(\bar{x},\ \Delta x).$$

The first order expansion of $w_i = 1^t\ \dfrac{\partial v}{\partial \Delta\dot{x}_i}$ is obtained by multiplication of (6a) and (6c):

$$\text{(6d)} \quad w_i(x) = \bar{w}_i + \sum_k (M_{ik} \cdot \Delta\ddot{x}_k + C_{ik} \cdot \Delta\dot{x}_k + K_{ik} \cdot \Delta x_k).$$

The "physical" coefficients of (6d) are given by

$$\text{(7)} \quad \bar{w}_i = \bar{V}_i\ \bar{v}_i,\quad M_{ik} = \bar{V}_i\ M_k,\ C_{ik} = \bar{V}_i\ C_k,\ K_{ik} = \bar{V}_i\ K_k - \tilde{\bar{w}}_i \cdot V_k.$$

They were calculated by an extended and meticulous elaboration. Its presentation is far beyond the scope of this paper. As an example their inertia terms are summarized in the appendix. It should be noted that the determination of the coefficients in (7) is the crucial part of the work to be done for the Taylor expansion of the equations of motion. For the simple coupled rotor-body system of Fig. 4a (taken from Ref. [14]) the physical matrices are given in Fig. 4b.

9. Introduction of generalized (time) coordinates by separation

Various reasons for a coordinate transformation have already been mentioned. Suppose that the matrix Φ_i maps the vector q_i of local generalized coordinates onto the vector Δx_i of local physical coordinates (variations):

$$\text{(8a)} \quad \Delta x_i(s;\ t) = \Phi_i(s;\ t) \cdot q_i(t),$$

where s and t are the independent spatial and time variables respectively. Consequently the time derivatives of the Δx_i are

$$\text{(8b)} \quad \Delta\dot{x}_i = \dot{\Phi}_i\ q_i + \Phi_i\ \dot{q}_i$$

$$\text{(8c)} \quad \Delta\ddot{x}_i = \ddot{\Phi}_i\ q_i + 2\dot{\Phi}_i\ \dot{q}_i + \Phi_i\ \ddot{q}_i\ .$$

After replacing Δx_i by $\Phi_i\ q_i$ the generalized force component w_i becomes a function w_i' of the generalized coordinates $q_i = q_i(\psi)$ with $\psi = \Omega t$, provided the reference point $\bar{x}$ is fixed:

$$\text{(9)} \quad w_i'(\bar{x},\ q) = \bar{w}_i' + \sum_k (M_{ik}' \cdot \ddot{q}_k + C_{ik}' \cdot \dot{q}_k + K_{ik}' \cdot q_k).$$

The "generalized" coefficients of (9) are obtained from the "physical" ones of (7) as

$$(10)\quad \bar{w}_i' = \Phi_i^t \bar{w}_i \ , \quad M_{ik}' = \Phi_i^t M_{ik} \Phi_k \ , \quad C_{ik}' = \Phi_i^t (2M_{ik}\dot{\Phi}_k + C_{ik}\Phi_k)$$

$$K_{ik}' = \Phi_i^t (M_{ik}\ddot{\Phi}_k + C_{ik}\dot{\Phi}_k + K_{ik}\Phi_k) \ .$$

If v is regarded as a function of q_i, $\dot{q}_i$ instead of Δx_i, $\Delta\dot{x}_i$ it is denoted by v'. An inspection of (6b) and (8b) shows that v and v' are related by

$$(11)\quad \frac{\partial v'}{\partial \dot{q}_i} = (\bar{V}_i + V_i) \cdot \Phi_i = \frac{\partial v}{\partial \Delta\dot{x}_i} \cdot \Phi_i .$$

This result justifies the interchange of generalization and partial differentiation performed implicitly in the last two paragraphs.

One example for the generalization matrices is to transform the rotating frame of the rotor in Fig. 4a into the nonrotating frame of the fuselage. Instead we use $\Phi = \Phi_1$ for the inverse transformation, i.e. from the nonrotating to the rotating frame. This choice may be preferable for rotors with at most two blades (Ref.[13]). It may be expressed by $\Phi\dot{\Phi}^t = -\tilde{\omega}$ (Ref.[13]). This identity implies $\dot{\Phi} = \tilde{\omega}\Phi$ and $\ddot{\Phi} = \tilde{\omega}\tilde{\omega}\Phi$ where $\dot{\omega} = 0$ is supposed.

10. The global linear state equation

In the sequel it will be supposed that the generalization transformations are chosen such that the virtual displacements δq_i in (4) are varying independently. Then, still neglecting the deformation energy, the equations of motion of the MBS are equivalent to the equations

$$(12)\quad 1_j^t \ \frac{\partial v_j}{\partial \dot{q}_i} = 0 \text{ for } i, j = 1, \ldots, n .$$

From (9) the linearization of (12) is obtained by

$$(13)\quad \bar{w}_i^{j\prime} + \sum_{k=1}^{n} (M_{ik}^{j\prime} \ddot{q}_k + C_{ik}^{j\prime} \cdot \dot{q}_k + K_{ik}^{j\prime} \cdot q_k) = 0 \qquad \text{for } i, j=1, \ldots, n.$$

Recall that the upper index j refers to the j-th body and was suppressed up to now.
Now using the local matrices $\sum_k M_{ik}^{j\prime}$, $\sum_k C_{ik}^{j\prime}$, $\sum_k K_{ik}^{j\prime}$, $\bar{w}_i^{j\prime}$ as submatrices global matrices A, B, C, f are composed. In terms of them the equations (13) are collected in the single equation

$$(14)\quad A \cdot \ddot{q} + B \cdot \dot{q} + C \cdot q = f .$$

The coefficients A, B, C, f for the coupled rotor-body system are given in 4c.

This is the linearized e.o.m. Its coefficients A, B, C, f depend on a reference point $\bar{x}$ and its first and second time derivative. For stationary cases the dependence is periodic with time, i.e. (14) represents a <u>second order linear two point boundary value problem with periodic coefficients.</u>

By introduction of the state vector defined as $y = \left\{\begin{array}{c} q \\ \hline \dot{q} \end{array}\right\}$, the matrix

$S = \left[\begin{array}{c|c} 0 & Id \\ \hline -A^{-1}C & -A^{-1}B \end{array}\right]$ and the vector $g = \left\{\begin{array}{c} 0 \\ \hline A^{-1}f \end{array}\right\}$ it reduces to the first order

matrix differential equation

$$(15) \quad \dot{y} = Sy + g, \qquad S = S(\bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})$$

To this final form of the linearized equations of motion the Floquet-Liapunoff theory is repeatedly applied. The solution q_k (Δx_k) obtained in the k-th step is used to replace the reference point $\bar{x}_{k-1}$ of the previous step by the new reference point $\bar{x}_k = \bar{x}_{k-1} + \Delta x_k$. It is known from the theory of quasilinearization (Ref. [8]) that this iteration procedure amounts to a Newton-Raphson method in function spaces. In particular, a theorem of Kantorovich assures convergence of the $\bar{x}_k$ to the true solution of the full nonlinear equations of motion (4), provided the initialization $\bar{x}_0$ of the point was properly chosen.

11. Concluding remarks

In the past several powerful algorithms for multi body systems (MBS) were developed and applied to rotary wing aeroelastic problems. The resulting equations of motion are generally nonlinear with time-varying periodic coefficients. The solution of these equations is a formidable task.

The dynamic response and stability characteristics of a helicopter or wind turbine under steady state conditions are of special interest. In this case - provided the system equations would be linear - the stability characteristics are efficiently calculated by an eigenvalue analysis and the dynamic response by the solution of a linear two-point boundary problem. This solution technique is well established for rotary wing problems. It is also known that for nonlinear periodic systems the aforementioned solution method is applicable in combination with the method of quasilinearization.

In this paper an efficient algorithm is presented for a consistent linearization of the inertial terms in a MBS with tree structure. The algorithm is general enough to include both rigid and elastic bodies. The same linearization procedure (Taylor expansion) is applicable for the derivation of the structural, aerodynamical and gravitational terms in the equations of motion.

The new algorithm is used and programed in the aeroelastic computer program Vibrat which is under development at MBB. The new program includes both rigid and elastic components by the use of hybrid coordinates.

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13. Notations and dimensions

A_j	transformation matrix from frame e_{j-1} to e_j, 3 x 3
A_k^i	$= A_k \dots A_{i+1}$, 3 x 3
A^t	transpose of a matrix A
$\dot{\beta}_j$	angular velocity (transformed), 3 x 1
c_j	representation of $\vec{c}_j$ in e_j, 3 x 1
θ_j	vector of attitude angles, 3 x 1
e_j	right-handed orthogonal frame, fixed in j-th body
Φ_j	generalization matrix, 6 x n_j
I_j	inertia matrix (normalized by mass), 3 x 3
$\bar{I}_j$	$=$ trace $(I_j) \cdot Id - 2I_j$, 3 x 3
Id	identity matrix, 3 x 3
j	index of j-th body (j dropped in 4a, 4b)
l_j	$= \left\{\frac{f_j}{m_j}\right\}$ load-vector (normalized by mass), 6 x 1
n	number of bodies in MBS
n_j	number of generalized degrees of freedom of j-th body
q_j	vector of generalized coordinates, n_j x 1
r_j	representation of $\vec{r}_j$ in e_{j-1}, 3 x 1
ω_j	representation of $\vec{\omega}_j$ in e_j
$\tilde{\omega}$	$= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$
x_j	$= \left\{\frac{r_j}{\theta_j}\right\}$ vector of local coordinates, 6 x 1
v_j	$= \left\{\frac{u_j}{\omega_j}\right\}$ vector of $\left\{\frac{\text{translational}}{\text{angular}}\right\}$ velocity, 6 x 1

Conventions: The loads l_j are normalized by the mass of the j-th body. For vectors and matrices a more mathematical notation was used, often suppressing in most cases { }, [ ] to indicate vectors or matrices respectively.

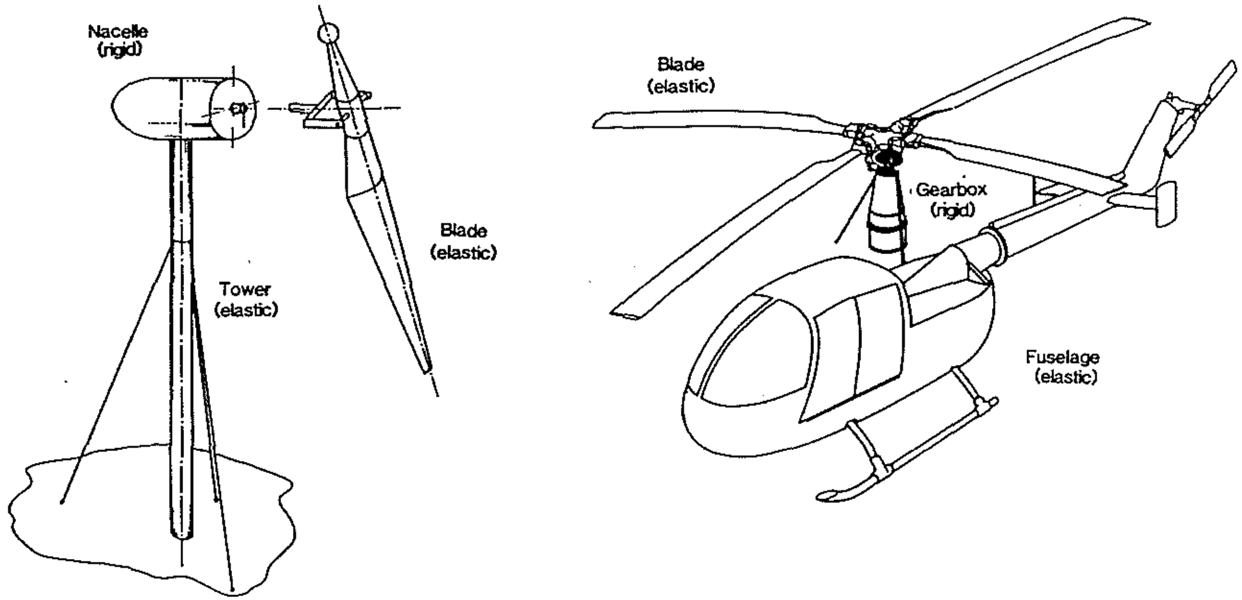
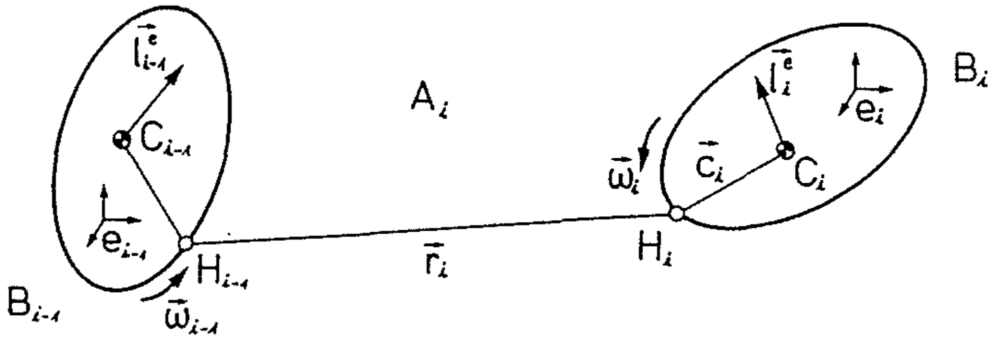


Figure 1: Main Components of Helicopters and Wind Turbines



Legend

e_0	inertial frame
H_0	reference point fixed in the inertial frame
H_i	hinge point of B_i
C_i	center of gravity of B_i
$\vec{r}_i$	locates H_i relative to H_{i-1}
$\vec{c}_i$	locates C_i relative to H_i
$\vec{l}_i^{e}$	external load at C_i
e_i	reference frame fixed in B_i
$\vec{\omega}_i$	angular velocity of e_i relative to e_{i-1}
ϑ_i	vector of Tate – Bryant attitude angles
A_i	maps frame e_{i-1} onto frame e_i
m_i	mass of B_i
$\mathbb{I}_i$	inertia dyad of B_i about H_i

Figure 2: Two Contiguous Bodies Characterizing the MBS

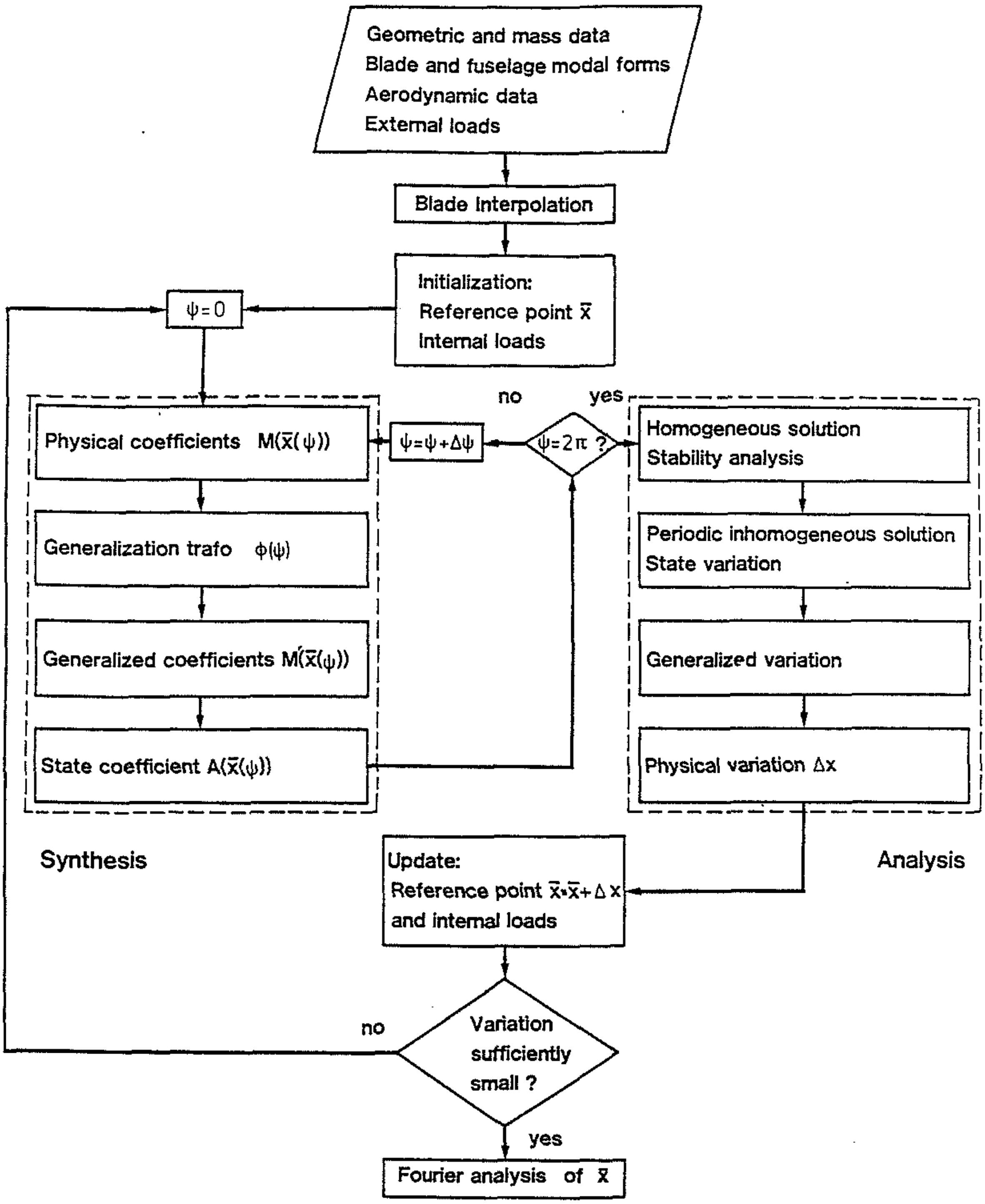
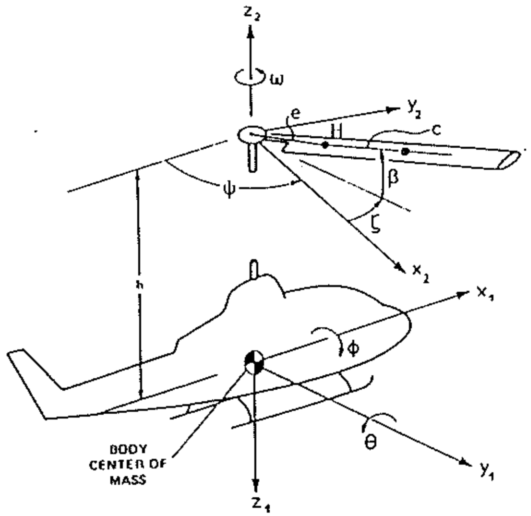


Figure 3: Block Diagram of the Computer Program VIBRAT

a) model



b) equations

$$\left\{\begin{array}{c}\tilde{\omega}\tilde{\omega}e + c \\ \hline \tilde{\omega}I\omega + \tilde{c}\tilde{\omega}\tilde{\omega}e\end{array}\right\} + \left[\begin{array}{c:c} Id & -\tilde{c} \\ \hdashline \tilde{c} & I \end{array}\right]\Delta\ddot{x} + 2\left[\begin{array}{c:c} \tilde{\omega} & \tilde{\omega}\tilde{c} \\ \hdashline \tilde{c}\ \tilde{\omega} & \frac{1}{2}\ \tilde{\tilde{I}}\omega \end{array}\right]\Delta\dot{x} + \left[\begin{array}{c:c} \tilde{\omega}\tilde{\omega} & \tilde{\omega}\tilde{\omega}e + c - \tilde{\omega}\tilde{\omega}\tilde{c} \\ \hdashline \tilde{c}\tilde{\omega}\tilde{\omega} & (I\tilde{\omega} - \tilde{\omega}\bar{\tilde{I}})\tilde{\omega} + \tilde{c}\tilde{\omega}I\omega \end{array}\right]\Delta x = 0$$

Figure 4a, b: Coupled Rotor-Body Model (Ref. 14) and Equations of Motion in the Rotating System (Inertial Terms)

Define the matrices $\Phi = \left[\begin{array}{c|c} \Phi_1 & 0 \\ \hline 0 & \Phi_2 \end{array}\right]$, $V^t = \left[\bar{V}_1 \,\middle|\, \bar{V}_2\right]$, $T = \Phi V$,

and the operator $\langle u,v \rangle = \frac{1}{2}(\tilde{u}\tilde{v} + \tilde{v}\tilde{u})$.

Then the contribution of the rotor to the global equation (14) can be written in the compact form $(T^t A' T)\ddot{q} + (T^t B' T)\dot{q} + (T^t C' T)q = T^t f'$,

where $f' = (\tilde{\omega}\tilde{\omega}c + e \; , \; \tilde{\omega}I\omega + \tilde{c}\tilde{\omega}\tilde{\omega}e)^t = (\varphi \; , \; \mu)^t$,

$$A' = \left[\begin{array}{c|c|c|c} Id & -\tilde{c} & Id & -\tilde{c} \\ \hline \tilde{c} & I & \tilde{c} & I \\ \hline \multicolumn{2}{c|}{} & Id & -\tilde{c} \\ \cline{3-4} \multicolumn{2}{c|}{\text{symm.}} & \tilde{c} & I \end{array}\right] , \quad B' = 2 \left[\begin{array}{c|c|c|c} \tilde{\omega} & -\tilde{\omega}\tilde{c} & \tilde{\omega} & -\tilde{\omega}\tilde{c} \\ \hline \tilde{c}\tilde{\omega} & \frac{1}{2}\widetilde{I\omega} & \tilde{c}\tilde{\omega} & \frac{1}{2}\widetilde{I\omega} \\ \hline \multicolumn{2}{c|}{} & \tilde{\omega} & -\tilde{\omega}\tilde{c} \\ \cline{3-4} \multicolumn{2}{c|}{\text{antim.}} & \tilde{c}\tilde{\omega} & \frac{1}{2}\widetilde{I\omega} \end{array}\right] ,$$

$$C' = \left[\begin{array}{c|c|c|c} 0 & 0 & \tilde{\omega}\tilde{\omega} & -\tilde{\omega}\tilde{\omega}\tilde{c} \\ \hline 0 & \langle \omega, I\omega \rangle & \tilde{c}\tilde{\omega}\tilde{\omega} - \tilde{\varphi} & \tilde{\omega}(I\tilde{\omega} - \widetilde{I\omega}) + \tilde{\omega}(\tilde{\omega}\widetilde{e+c} - 2\widetilde{e+c}\tilde{\omega})\tilde{c} \\ \hline \multicolumn{2}{c|}{} & \tilde{\omega}\tilde{\omega} & -\tilde{\omega}\tilde{\omega}\tilde{c} \\ \cline{3-4} \multicolumn{2}{c|}{\text{symm.}} & \tilde{c}\tilde{\omega}\tilde{\omega} & \tilde{\omega}I\tilde{\omega} + \langle \omega, I\omega \rangle + \langle c, \mu \rangle \end{array}\right]$$

Figure 4c: Global Generalized Coefficients for Rotor-Body Model

Appendix

Linearization of physical equations of body j = main part of linearization (inertia)

1) local differentiation: $\dot{r}_i = \frac{d}{dt} r_i \, , \quad \dot{\omega}_i = \frac{d}{dt} \omega_i$

2) transformation: $s_i = \bar{A}^{i-1}_j r_i \, , \quad \beta_i = \bar{A}^i_j \omega_i$, where $\bar{A}^i_k = \bar{A}_k = \bar{A}_k \cdot \ldots \cdot \bar{A}_{i+1}$

3) partial sums: $s^i_k = s_{i+1} + \ldots + s_k, \ \beta^i_k = \beta_{i+1} + \ldots + \beta_k, \ s^i = s^i_j + c_j, \ \beta^i = \beta^i_j$

The order of these operations must be obeyed strictly. Then expressions as $\dot{s}^i_k$ are unique, namely

$$\dot{\beta}^i_k = \sum_{l=i+1}^{k} \bar{A}^l_j \dot{\omega}_l$$

4) basic auxiliary terms: $\rho_i = \tilde{\beta}_i s^i \, , \ \sigma_i = 2\overset{o}{\beta}_{i-1} + \beta_i, \ \tau_i = \rho_i + 2\rho^i_j, \ \mu_i = \tilde{\beta}_i \beta^i$

5) composed auxiliary terms: $P_i = \tilde{\sigma}_i \tilde{\beta}_i + \tilde{\dot{\beta}}_i + \, , \quad Q_i = P_i + 2\tilde{\beta}_i \cdot \frac{d}{dt}, \ S_i = \tilde{\sigma}_i \tilde{s}^i + \tilde{\tau}_i + 2\tilde{\dot{s}}^i$

$$T_i = \sum_{l=1}^{i-1} (Q_l \tilde{s}^i - \widetilde{Q_l s^l}) + 2\tilde{\rho}^{i-1}_j \tilde{\overset{o}{\beta}}_{i-1} - \tilde{\dot{\beta}}_i \tilde{s}_i - \tilde{\ddot{s}}^o_i$$

$$U_i = I_j \tilde{\sigma}_i - 2\tilde{\beta}^o \bar{I}_j, \text{ where } 2\bar{I}_j = \mathrm{trace}\,(I_j) \cdot \mathrm{Id} - 2I_j$$

$$V_i = I_j (\overset{o}{\mu}_{i-2} + \dot{\overset{o}{\beta}}_{i-1})^{\sim} + (I_j \tilde{\overset{o}{\beta}}_{i-1} - 2\tilde{\beta}^o \bar{I}_j) \tilde{\overset{o}{\beta}}_{i-1}$$

Physical coefficients:

M_k splits into four (3,3)-matrices according to $M_k = \left[\begin{array}{c|c} M^1_k & M^2_k \\ \hline M^3_k & M^4_k \end{array}\right]$, $\bar{v}_i$ into two 3-vectors $\bar{v}_i = \left\{\begin{array}{c} \bar{v}^1_i \\ \hline \bar{v}^2_j \end{array}\right\}$.

The cup $\vee$ over a function of c_j means truncation, i.e. neglecting of c_j .

For shortness, we replace $\bar{A}^i_k$ by B^i_k .

$$\bar{v}^o_j = (\ddot{s}^o + \sum_{i=1}^{j} Q_i s_i \, , \ \tilde{c}_j \check{\bar{v}}^o_j + \tilde{\beta}^o I_j \beta^o + I_j \dot{\beta}^o + I\mu^o_j)^t$$

$$\bar{V}_i = \left[\begin{array}{c|c} B^{i-1} & -\tilde{s}^i_j B^i \\ \hline O & B^i \end{array}\right]^t , \quad M_k = \left[\begin{array}{c|c} B^{k-1} & -\tilde{s}^k B^k \\ \hline \tilde{c}_j \check{M}^1_k & \tilde{c}_j \check{M}^2_k + I_j B^k \end{array}\right] , \quad C_k = \left[\begin{array}{c|c} 2\tilde{\beta}^o_{k-1} B^{k-1} & -(\tilde{s}^k \tilde{\beta}_k + S_k) B^k \\ \hline \tilde{c}_j \check{C}^1_k & \tilde{c}_j \check{C}^2_k + (I_j \beta_k + U_k) B^k \end{array}\right]$$

$$K_k = \left[\begin{array}{c|c} \dot{P}^o_{k-1} B^{k-1} & -(\tilde{s}^k \tilde{\dot{\beta}}_k + S_k \tilde{\beta}_k + T_k) B^k \\ \hline \tilde{c}_j \check{K}^1_k & \tilde{c}_j \check{K}^2_k + (I_j \dot{\beta}_k + U_k \tilde{\beta}_k + V_k) B^k \end{array}\right] , \quad \tilde{\bar{w}}_i V_k = \left[\begin{array}{c|c} O & (B^{i-1})^t \tilde{f}_j B^k \\ \hline B^{it} \tilde{f}_j B^{k-1} & B^{it} (\tilde{f}_j \tilde{s}^i_k + \widetilde{\tilde{s}^i_j f_j + m_j}) B^k \end{array}\right] \quad \text{for } i<k \, .$$

If i=k the bottom half of $\tilde{\bar{w}}_i V_k$ has to be replaced by $[O \ \ \mathrm{Sym}_k (B^{kt} (\tilde{s}^k_j f_j + m_j))]$.

Sym_k operates on an arbitrary 3-vector u by $\mathrm{Sym}_k(u) = \frac{1}{2} \begin{bmatrix} 0 & u_{12} & u_{13} \\ -u_{12} & 0 & u_{23} \\ -u_{13} & -u_{23} & 0 \end{bmatrix} u - \frac{1}{2}\tilde{u}$.

The matrix of the right hand side depends on the order of rotations in A_k .

If the chosen order is represented by $(\xi\eta\zeta)$ then $u_{\xi\eta} = u_{\eta\zeta} = u_{\xi\zeta} = +1$.